Physics Lectures

PH 121 Fundamentals of Physics I

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Preface

Preface Head

This set of notes is intended to be an aid to the student. There are likely errors and mistakes, so use the text, but don't expect perfection. If there are things that are confusing, please talk to me or ask questions in class.

Acknowledgments

I would like to thank the many students who suffered through classes giving feedback to make these lectures better.

BYU-I University *R. Todd Lines*.

Contents

1 What is Physics

This is sort of an introduction. All the regularly structured chapters will start in a particular way. I will give you the main topic in the title, and then give what I think are the four most significant concepts in the chapter. That will give you a warning about what is important in what you will read. You will read a chapter before each lecture and the list of important topics will act as a study guide both for the reading and for the lecture. You should test your reading after you read by reviewing this list of important topics and making sure you can explain each of them to someone else. Find a class mate or roommate, or spouse or dog or whatever you have, and explain these concepts to them. By teaching, you will learn more completely. But for this first chapter, you will not have bought your book nor will you have read in advance. So I structured this chapter differently. We will talk about this together, then you will have this printed text to refer to later. Think of this chapter as a set of instructions for the rest of the course.

Most of you will have taken some science classes in high school. Before we start learning how to solve detailed problems in physics, we should pause to discuss why this science called physics is different from what you already know. Let's start with the scientific method–something that you do know–and show how it is applied differently in this new science.

The scientific method

Most of us in this class are not science majors. So we should talk for a moment about what science is, and in particular, what physics is. Let's start with, the scientific method.

The scientific method

In your high school Earth science or biology class (or even a high school physics class) you probably learned the scientific method. It is usually presented as shown in figure (1) on the left hand side (green bubbles).

1.Scientific_Method

We remember that first you must have an idea about how some part of the universe works. After studying the idea, noting what others have said on the subject we form a mental model that incorporates this new idea. This idea is formally called a *hypothesis*. To be science, we will have to test this hypothesis through experiment. That means you have to think of a consequence of the new idea of how the universe works, and try it out. You have to see if the universe really works according to your idea. If the experiment can be interpreted as a positive result, then we declare that we have learned something through the scientific method. We call our hypothesis that is now tested a *theory*. We publish our result so the rest of the scientific world can learn of our results. The details of doing this are taught in our program in PH150, our first lab class.

Although this is a good statement of the scientific method, it is really not the end of the story in physics (or other sciences, for that matter). The story continues with the experiment that you did being repeated by other researchers.

An odd but important belief that rests at the heart of the scientific method is that we cannot prove something to be true through scientific induction[1]. We can only prove things to be false! Thus every time you hear a news reporter say that "scientists have proven..." you should be sceptical! After you have tested your hypothesis, and declared it to be a theory, the theory must go through a rigorous set of attempts to prove it false. We call this process falsification. Many researchers over some time test the predictive power and consistency of the theory with new results from their experiments and the results of previous theories.

If these researchers agree with your conclusions, the idea you tested may become an accepted *model*. I have used the word "model" to describe a theory of limited applicability or a step on the way to becoming a general theory. Not all scientists use this term. Sometimes scientists use "model" to describe their mental image of how things work. I will use the word "model" as a limited theory, but I may slip occasionally and call my mental picture a "model."

If the theory is able to predict correct behavior beyond the experiment you used to test your hypothesis, and it's results are all positive, then the theory may become accepted as a general theory. In earlier times this would have been called a scientific law. Now the use of the words "scientific law" usually means a well established empirical relationship. This means a well tested equation which concisely embodies the ideas contained in our theory. Notice that if there are any negative results, it is back to the drawing board! The whole scientific process must start over.

[1] The process of inferring a general law or principle from the observation of particular instances (OED)

Also notice that no theory is beyond doubt! New experiments may bring new results that will cause new hypothesis and new theories to be formed that will take the place of the old, established theories.

This makes the scientific method look a little complicated, but very ordered. In reality, the steps are not always done in order. Worse, the different branches of science disagree on the method of doing science a little. For example, in biology and psychology predictive experiments are harder to do! There is less emphasis on prediction in these sciences. But the green bubble steps are usually agreed on in introductory descriptions of science.

There are also differences in what the steps of the scientific process mean in different branches of science. Most are the same, but in physics the hypothesis is generally reduced to a *mathematical equation* that predicts an outcome for the experiment. In physics concepts are transformed into equations that give predicted results. Even when the idea becomes accepted theory, with an accepted scientific law like

$$E = mc^2$$

Einstein's famous equation, you can see that complex ideas (the ideal that energy and mass are equivalent) is expressed as an equation that can be used to test the idea. In physics we talk of Einstein's theory, but physicists talk more about solutions to Einstein's equations–that is the real test of an idea in physics because solutions to the equations are predictions of what will actually happen. And if the predictions don't come true, we have falsified the theory!.

In our class, you won't be required to invent new ideas about how the universe works. We will be learning about the collection of ideas that others have already tested for the motion of single objects. But in our homework and tests, we will use this difference between physics and other branches of science. We will always try to find an equation as a solution. This will be our standard way to do physics. This might seem a little scary at first, but we will get used to it quickly. So in this class we will use two parts of the scientific method in each problem. We will make a mathematical expression of our hypothesis. We will call this a symbolic answer. It is the embodiment of our scientific theories that we will study in each chapter or lecture. The numerical answer is a numerical prediction from our hypothesis. When the problem is graded, you will know if your if your prediction is good!

A good theory is able to predict things that will happen. It must not predict anything that does not happen. When a theory's predictions are wrong, it must be changed or even abandoned.

Notice that this expression of the scientific method does give theories that *explain* many things. But explaining is not enough to be accepted in physics. A theory must be able to predict in order to become named a scientific law or a general theory.

Identify several scientific laws, theories

Scope of Physics

Scope is a word which here means a limitation in topics (not a mouthwash). There must be some limitation in topics, or physics would be synonymous with biology or chemistry (or both). In practical usage, physics is *not* the study of living things and is generally *not* the study of how atoms form compounds. So what is physics?

It is the study of how things move. This may seem very restrictive, but it really is not. Physics incorporates everything from the motion of a ball at a play ground to the motion of atoms in a star to the motion of ions through a cell lining.

Our approach to physics will divide the world of physics into two main parts. The first part is called "classical" physics. It contains the theories that explain and predict motion of large objects at fairly slow speeds. It takes us three courses to cover all of classical physics, PH121, PH123, and PH220. The second part is called "modern" physics. It contains the theories that explain and predict motion of small objects or very fast objects. We start studying modern physics in PH279. In our PH121 class we will start with the part of classical physics called "mechanics." Mechanics deals with the motion of every-day objects like cars, or balls, or people. We will start with a one to a few objects. We will let those few objects travel in straight lines at first, but soon we will allow the objects to follow more complex paths or even to rotate.

But how does physics relate to biology, or chemistry, or any other branch of science? Purists think that everything is physics. That is, if you knew all the particles that make up something, and knew how they moved, you could apply the laws of physics, and predict the behavior of any object. When it comes to chemistry, this is true. We can explain what we see in chemistry by applying what we know from physics to the atoms and particles of mater. We can explain much of biology this way as well. We would like to explain psychology with basic physical laws, but this has yet to be done. I believe, in part, this is because there are actions by our spirits that physics does not yet recognize that affect the science of psychology. Maybe in your life this will be shown. But for now this is just speculation. What we will say for the purposes of PH121 is that physics is the basic underlying rules behind the other sciences, but at some point the mathematical book-keeping to use the basic laws becomes too difficult, so we tend to study compounds or whole animals rather than keep track of all the motions of their individual atoms.

Course Structure

This class is designed into several parts:

1. How we describe motion
2. Motion in one dimension
3. Motion in more than one dimension
4. Forces and motion
5. How objects interact
6. Impulse and momentum
7. Energy
8. Rotation
9. Gravitation

These topics will let you know everything you need to know to understand how things move and to solve motion problems.

By the time we are done, you can explain how bungee jumping, skydiving, scuba diving, and other fun things work. Let's take some examples

Four Ball Toy Experiment

Rotating Wheel Experiment

Just a note on testing: all tests are cumulative, in that the material builds. If you learn one chapter, and then figure you can forget all you learned because we are on the next chapter, you will be less pleased with your performance.

There is less emphasis on memorizing in this class. In Biology, and even in Chemistry much of what you need to know is memorized. In Physics, there are an infinite number of physics questions I could ask you. The only hope of getting any one of them right is to understand the concepts behind how things move (that is what we will learn!) and to have a systematic approach to solving the problems. So that is just what we will do. And I hope we will have fun along the way!

2 What is Motion?

We have said that we will study motion in this class. But what is motion?

The answer will take the rest of this class, most of PH 123, PH220, and PH 279 to answer. But we will have to start somewhere. Let's make a provisional definition of motion, one that we can refine as we become more knowledgeable.

Definition 2.1 *Motion is the change of an object's position in an amount of time*

How do we depict motion?

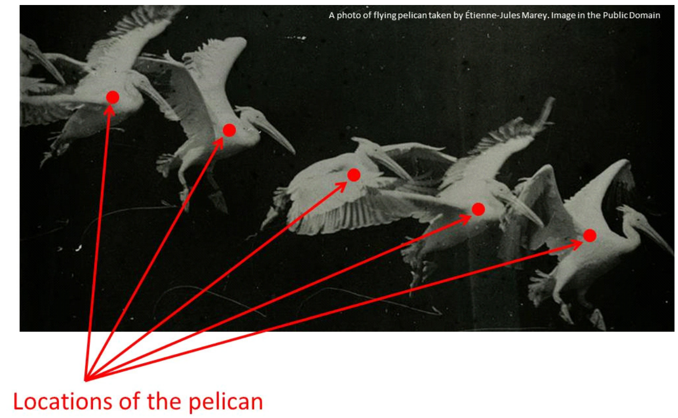
Question 121.1.1

Question 121.1.2

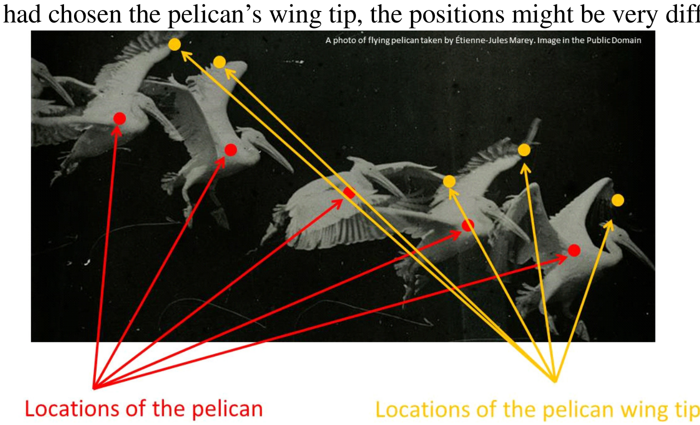
An early photographer named Étienne-Jules Marey developed a way of showing how an object's position changed in time. He made several photographs of the object on the same piece of photographic film, each exposure being a set time later than the previous. Here is his photograph of a pelican.

Photograph of flying pelican taken by Étienne-Jules Marey c. 1992. (Image in the Public Domain)

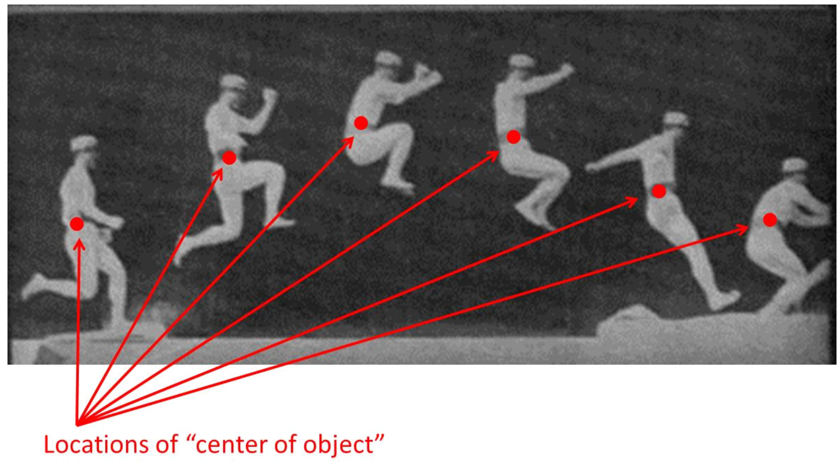
In this case, the object is the pelican, and we see that the pelican has changed position in the time between exposures. But this photograph also shows the problems with our simple definition. So far, we have considered the pelican as the object. So we could define a single position as the location of the pelican. In the next figure, a red dot represents the “location of the pelican” and we see that the position of this dot changes in time.



But if we had chosen the pelican’s wing tip, the positions might be very different.

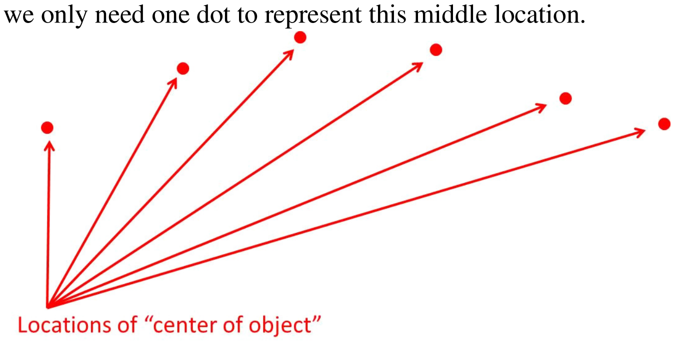


So when we say “an object” is in motion, just what do we mean? In the next figure, a man is jumping. But once he has left the ground, the middle of the man will follow the same path a ball would follow if it had the same mass and was launched at the same speed.



Man in White Jumping, Étienne-Jules Marey. (Image in the Public Domain)

The man could wave his arms or do a summersault in the air. But his middle would fallow the same path. Then, we could start our study of motion by studying how the middle of things move. This is a good representation of the motion of the object as a whole. And we only need one dot to represent this middle location.



Of course this is an approximation. We could think of a case where the man was holding a ball and threw the ball part way through his jump. Then the “middle” of the man-ball system might move in a different way than the man. But that is a complication we will take on later. For now, we will deal with the motion of objects that can be described as begin one whole object that won’t come apart during the motion we observe.

The photos we have used are tricky to take, so we will switch to diagrams that we can draw, but that represent the same thing. An object is shown at several positions with each new drawing representing the position of the object a set time Δt later. Here is such a diagram

The object is a ball. The motion starts from the left. Note that the position changes between the first two drawings of the ball. So the ball is moving. But also notice that the position change is less between the second and third drawings of the ball. We can tell from this that the ball is slowing down.

Let's give some names to what we have learned so far:

Definition 2.2 *Particle model: Using a "middle" point on an object to represent the location of the object*

Definition 2.3 *Motion diagram: A diagram that shows drawings of an object where each drawing shows the object a set time later.*

How do we describe when and where something happened?

Motion is the change of an object's position in an amount of time. Then we are going to have to have a way to express the time at which something happened and we are going to have to be able to express the position where it happened. Let's see how we do this in physics.

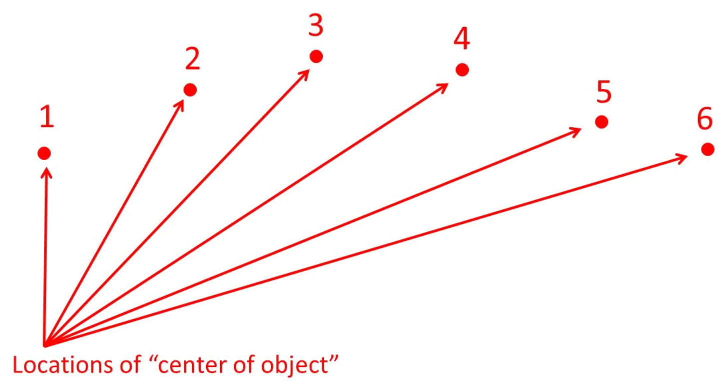
Duration: Experimental time

Let's consider measuring time in an experiment. Ideally, we would start timing our experiment right when the experiment begins. So we start with an initial time of $t_i = 0$. We could, say, start a stop watch right when the experiment starts, and stop the stop watch when we are done. Then the reading on the watch is the duration of the experiment. But this ideal situation is not always possible. For one thing, you may be working in a team, and your lab partner may be slow (or fast) in starting the stop watch. But even more importantly, we may need to consider *parts* of an experiment. Think about our motion diagrams. We could consider the entire motion of an object as our experiment. Then each new position in our motion diagram represents part of the entire motion. And there is a beginning time and ending time for each part of the motion. We need a way to express both the starting and stopping times for parts of experiments and the duration of the part.

We can use a math symbol to represent our set time between two drawings in a motion diagram. That symbol is Δt. The first part is the Greek letter Δ pronounced "delta." We will use this symbol to mean a difference between two times.

$$\Delta t = t_f - t_i$$

where t_f is the final time we are considering (for either the whole experiment or for the part of the experiment on which we are concentrating) and t_i is the initial time. For example, we could go back to our jumping man. We could number each of the positions where the middle of the man shows up in the photo.



Then the entire duration of the whole experiment would be

$$\Delta t_{total} = t_6 - t_1$$

and the duration of the first part of the experiment, the time between when the man was at position 1 and when he was at position 2, is given by

$$\Delta t_{\text{part 1}} = t_2 - t_1$$

When we were making our motion diagrams we said we would make series of pictures with each picture "being a set time later than the previous." We can use our new notation to see how to write our "set time." By "set time" we mean that Δt is not changing as we go from one part of the experiment to another. So for the first part of the diagram we can write the duration as just

$$\Delta t = t_2 - t_1$$

and since the duration of each part of the experiment is the same we could write

$$\begin{aligned} \Delta t &= t_3 - t_2 \\ \Delta t &= t_4 - t_3 \\ &\vdots \end{aligned}$$

and so forth.

This is not really too strange. If I ask you how long your physics class lasts, you would tell me "an hour." This is correct. It really does not matter if class starts at 9:00AM or 3:15PM. The class is still an hour. But we could say that the duration of the class is a Δt so that

$$\Delta t_{class} = \text{10:00am} - \text{ 9:00am}$$

or

$$\Delta t_{class} = \text{4:15am} - \text{ 3:15am}$$

The duration is the same. And that hour is only part of your day.

We will use the symbol Δ to mean a difference between two quantities, like time, often in this class.

Note that writing duration this way solves a lot of our experimental problems. If your lab partner starts the stop watch too late, then you just record the beginning and ending times and you can still find the duration. Beginning times can even be negative! Think of a rocket launch. The NASA official always starts the countdown by saying "$t - 10\,\mathrm{s}$." The time is measured from the launch time. If the launch time is $t = 0$, then before the launch time is negative time. But our Δt notation for duration handles this just fine. Suppose you try this. Suppose you are going out on a date. You tell yourself that $t = 0$ is the start of the date. But you have to get ready for the date and you start getting ready an hour before the date starts. Then the beginning of the date experience is at $t_i = -1\,\mathrm{h}$. Further suppose you go for a movie and ice cream and arrive home three hours after the start of the date. Then you could write

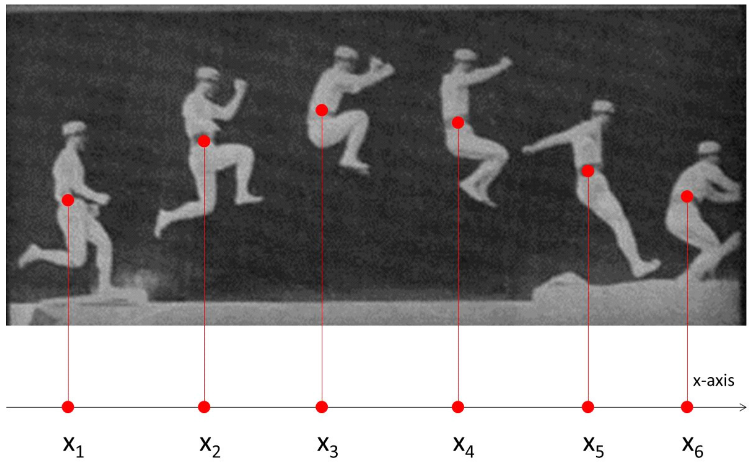
$$t_f = 3\,\mathrm{h}$$

and the duration of your date would be

$$\begin{aligned}\Delta t &= t_f - t_i \\ &= 3\,\mathrm{h} - (-1\,\mathrm{h}) \\ &= 4\,\mathrm{h}\end{aligned}$$

Position and Displacement

Now that we have a way to write the time involved with an experiment, let's find a way to describe how the object's position changes. Happily, we can use the same notation!



We can mark the position of the man at each part of his jump. In each segment of the jump he will arrive at a different position. You can see this as red dots plotted on an axis under the picture of the man. Each of the positions (red dots) is labeled with a position label (x_1, x_2, $x_{3,}$ etc.). Then we can write how far he traveled in the horizontal direction as

$$\Delta x = x_f - x_i$$

We can write a displacement for the entire jump as

$$\Delta x = x_6 - x_1$$

we are using the Δ in the same way we did for time to mean a difference between two quantities, this time two positions.

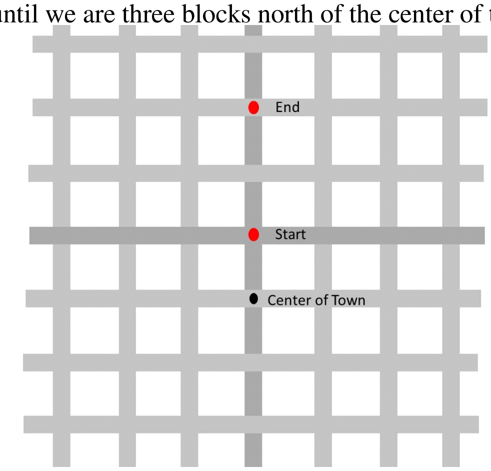
Our jump picture has images of the man at equal time intervals. But the distance the man travels in each time interval is not necessarily the same. We could write the displacements for each part of the jump as

$$\begin{aligned}
\Delta x_{21} &= x_2 - x_1 \\
\Delta x_{32} &= x_3 - x_2 \\
\Delta x_{43} &= x_4 - x_3 \\
&\vdots
\end{aligned}$$

In each case, there are two subscripts. For example Δx_{21} has the subscripts "2" and "1." It may look like a single subscript of "21," but it's not usual that we will mark up positions above 9. and then we use commas. For example

$$\Delta x_{43,42} = x_{43} - x_{42}$$

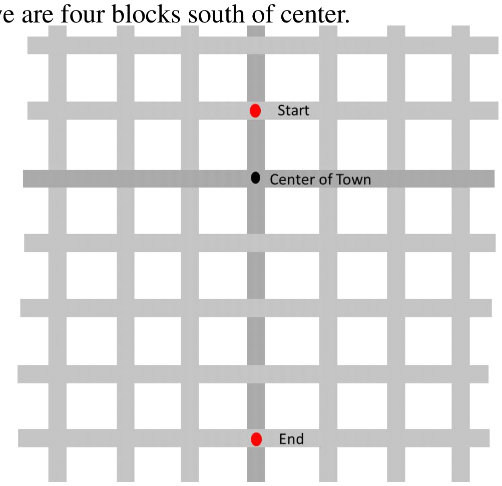
Note that displacement is a little different than distance. Displacements can be negative. If I am going to the right, my displacement is positive, but if I am going to the left my displacement is negative. For example, suppose we pick an origin for motion to be the center of Rexburg. And suppose we start our experiment a block north of the center of town and we walk until we are three blocks north of the center of town.



Then our displacement is

$$\begin{aligned}\Delta x &= 3\text{blocks} - 1\text{block} \\ &= 2\text{blocks}\end{aligned}$$

We could also go south. So suppose we start again one block north of the town center, but we walk until we are four blocks south of center.

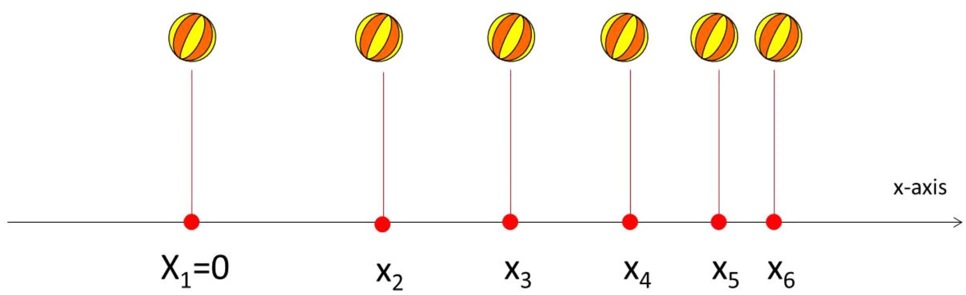


The our displacement is

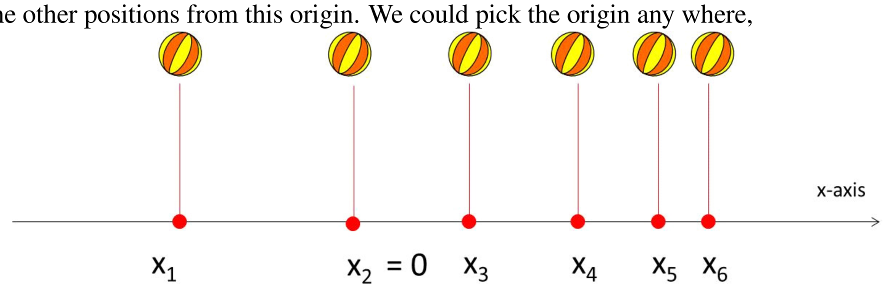
$$\begin{aligned}\Delta x &= -4\text{blocks} - 1\text{block} \\ &= -5\text{blocks}\end{aligned}$$

In effect, displacement gives not just the distance we traveled, but also tells us the direction we went. Directions will be very important in our study of motion, so we will prefer to use displacement rather than just distance often in this course. In our walking example, negative means "going South" and positive means "going north."

In the case of the man jumping the $\Delta x's$ do look like they are all equal. But in the ball diagram we can see that this is not the case.



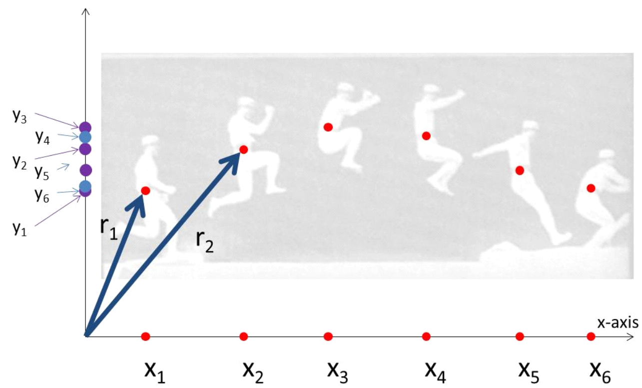
Note that in this figure we picked an *origin* as the starting point for measuring position. We label that $x = 0$ (in our case $x_1 = 0$, the first position of the ball). And we measure the other positions from this origin. We could pick the origin any where,



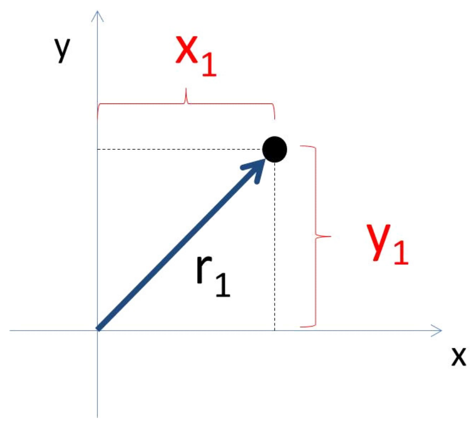
but often it is nice to pick the origin at the starting position of the object in our experiment.

Two dimensional displacement

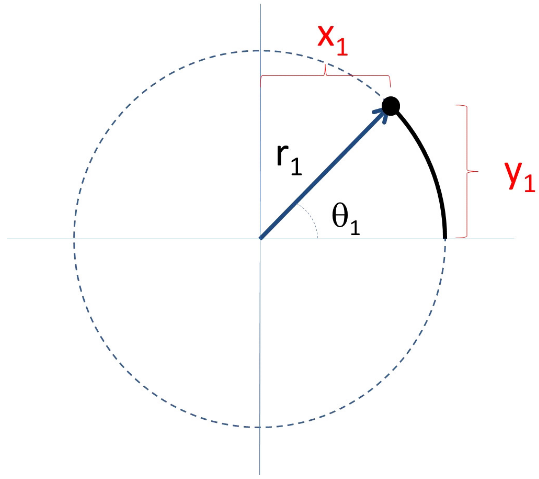
So far our displacement is only been in the x-direction, but our man's jump was not just horizontal motion, nor was it just vertical motion. It is really both at once. We should find a way to describe a motion that is part horizontal and part vertical. To do this let's step back and look at our jump again.



We could represent each position of our jumping man with new quantity. This new quantity must have a vertical and a horizontal part to it.



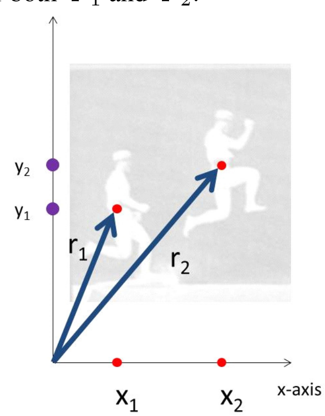
Let's take the point we labeled r_1 in the last figure and look at it carefully. We can see that r_1 must be made from x_1 and y_1 in some way.



If we think about it for a moment, this reminds us of polar coordinates. The distance r_1 at the angle θ_1 is related to the distances x_1 and y_1. So it really is quite natural to describe our position of point 1 by the line from the origin, r_1.

Notice in the figure we have drawn this new quantity like an arrow. And notice that this new quantity has not only a length, but also a direction, θ_1. We will call this new type of quantity a *vector*.

Specifically, the vector $\overrightarrow{\mathbf{r}}_1$ is a *position vector*. It gives the location of one of our points (x_1, y_1). Now let's consider both $\overrightarrow{\mathbf{r}}_1$ and $\overrightarrow{\mathbf{r}}_2$.



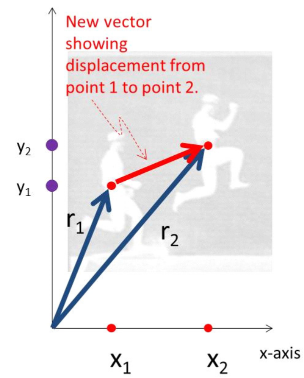
If these vectors represent our first and second positions, then it should be true that in some way

$$\Delta\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}_2 - \overrightarrow{\mathbf{r}}_1$$

must represent the displacement from point 1 to point 2. But what does it mean to subtract and add vectors?

Vector Addition (and Subtraction)

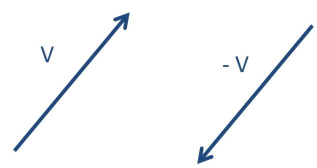
In our example of the jumping man, the displacement from point 1 to point 2 would be given by another arrow. After all, a displacement tells us how far we got and in what direction. The man jumping went a distance and in a particular direction. We use arrows to show how far and in what direction.



So this new arrow is also a vector, it has a length and a direction. So somehow we need to take $\overrightarrow{\mathbf{r}}_1$ and $\overrightarrow{\mathbf{r}}_2$, subtract them, and end up with the new vector.

Let's make a helpful definition: The negative of a vector is a vector of the same size going the opposite direction.

So if I have a vector, $\overrightarrow{V}$, as shown in the next figure



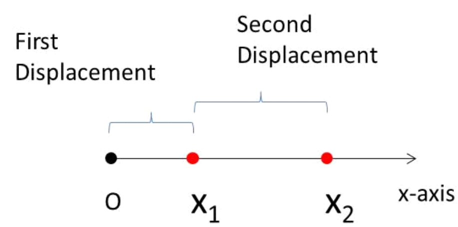
then $-\overrightarrow{V}$ will be the vector shown. Think, a negative sign gives us the opposite of a number (at least additively). The opposite of North is South, so with directions the negative of a direction should be "the other way."

The negative of $\overrightarrow{V}$ is just as long as $\overrightarrow{V}$, and goes the opposite direction. Then in our jumping man case we can see that

$$\begin{aligned}\Delta\overrightarrow{\mathbf{r}} &= \overrightarrow{\mathbf{r}}_2 - \overrightarrow{\mathbf{r}}_1 \\ &= \overrightarrow{\mathbf{r}}_2 + \left(-\overrightarrow{\mathbf{r}}_1\right)\end{aligned}$$

must mean to take $\overrightarrow{\mathbf{r}}_2$ and add to it a vector that has the length r_1 but goes the opposite direction of $\overrightarrow{\mathbf{r}}_1$. Let's draw $-\overrightarrow{\mathbf{r}}_1$ first

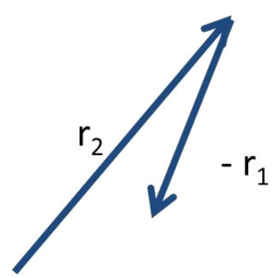
Now let's think about what it means to add displacements. In just the x-direction to add displacements we would go the first displacement, the go the second displacement starting at the end of the first displacement.



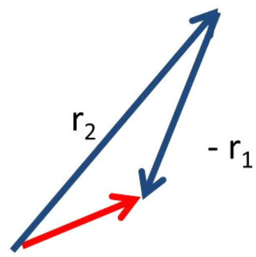
This works the same way for a set of displacements in the y-direction. So we might guess that this will work for our vector displacements. Our displacement is

$$\Delta\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}_2 + \left(-\overrightarrow{\mathbf{r}}_1\right)$$

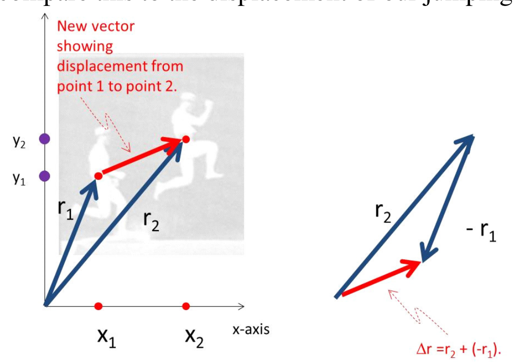
We will travel $\overrightarrow{\mathbf{r}}_2$ and then travel $-\overrightarrow{\mathbf{r}}_1$.



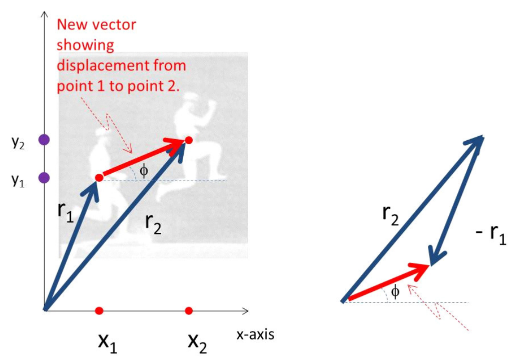
So after going the distance r_2 in the direction of r_2 we then turn into the direction of $-r_1$ and travel the distance r_1. The result is our red vector shown in the next figure



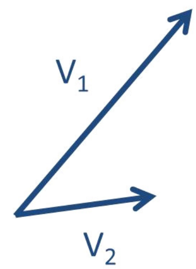
Notice that if we compare this to the displacement of our jumping man



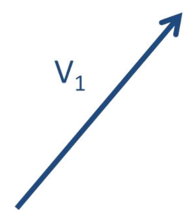
that the two red vectors are exactly the same length and that they point at exactly the same angle! This process seems to have worked!



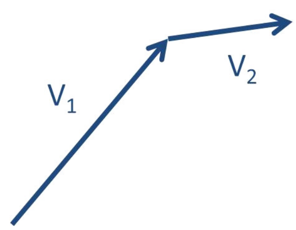
So we have a way to find the sum (and the difference) of two vectors. Let's summarize our steps. For a sum of two vectors, $\overrightarrow{\mathbf{V}}_1 + \overrightarrow{\mathbf{V}}_2$



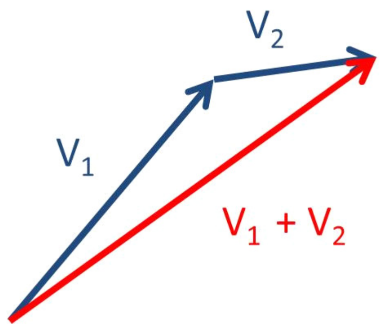
1. Draw $\overrightarrow{\mathbf{V}}_1$



2. Draw $\overrightarrow{\mathbf{V}}_2$ but start $\overrightarrow{\mathbf{V}}_2$ at the place $\overrightarrow{\mathbf{V}}_1$ stopped. We sometimes call this drawing the vectors "tip-to-tail." It's like two directions in a compass course for those of you who were Boy Scouts.



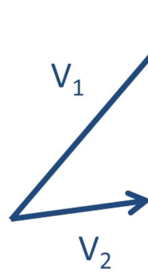
3. Finally draw a new vector from the tail of $\overrightarrow{\mathbf{V}}_1$ to the tip of $\overrightarrow{\mathbf{V}}_2$.



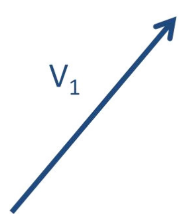
this new vector is $\overrightarrow{\mathbf{V}}_1 + \overrightarrow{\mathbf{V}}_2$.

Let's think about if this makes sense. If I tell you to walk in the $\overrightarrow{\mathbf{V}}_1$, a distance V_1 and then to stop and turn into the $\overrightarrow{\mathbf{V}}_2$ direction and walk a distance V_2. You would get to the same place as if you walked in the direction of the red arrow marked $\overrightarrow{\mathbf{V}}_1 + \overrightarrow{\mathbf{V}}_2$. So this does seem to be the sum of two vector displacements.

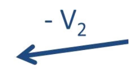
For subtracting two vectors, we just add one additional step. We have to reverse one of the vectors because we are adding a negative displacement For a difference of two vectors, $\overrightarrow{\mathbf{V}}_1 - \overrightarrow{\mathbf{V}}_2$



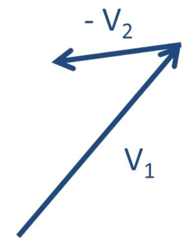
1. Draw $\overrightarrow{\mathbf{V}}_1$



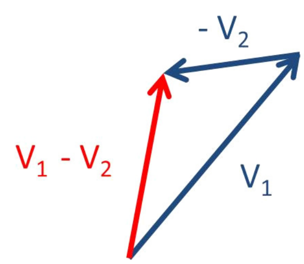
2. Draw $-\overrightarrow{\mathbf{V}}_2$ the inverse of $\overrightarrow{\mathbf{V}}_2$



3. Now move $-\overrightarrow{\mathbf{V}}_2$ so that it starts at the place $\overrightarrow{\mathbf{V}}_1$ stopped. We are adding $-\overrightarrow{\mathbf{V}}_2$ to $\overrightarrow{\mathbf{V}}_1$



4. Finally draw a new vector from the tail of $\overrightarrow{\mathbf{V}}_1$ to the tip of $-\overrightarrow{\mathbf{V}}_2$.



this new vector is $\overrightarrow{\mathbf{V}}_1 - \overrightarrow{\mathbf{V}}_2$.

We will use vectors for the rest of this class, for much of PH 123 and all of PH220 (or PH223). If you are a physics major or a mechanical engineering major, you will use vectors for the rest of your career. So it is worth getting used vectors and how to use them.

Now that we can describe how the time in an experiment changes, and we can describe how the position in an experiment changes, we can mathematically describe motion. In our next lecture, we will find that the velocity of an object is a combination of position (using displacement) and time

$$\overrightarrow{\mathbf{v}} = \frac{\Delta \overrightarrow{\mathbf{r}}}{\Delta t}$$

but this is not too much of a surprise. After all displacement is measured in miles sometimes, and time in hours, and we have been measuring velocities in miles per hour for years now. So we will be on familiar ground!

3 Velocity and Acceleration

Now that we have a way to describe the time and the displacement of an object, we are ready to describe how the object moves using mathematics. For example, how fast is our jumping man going? If we know how far he has gone $\Delta x_{total} = x_f - x_i$ and how long it took for him to travel that displacement, $\Delta t_{total} = t_f - t_i$, can we find his speed?

From everyday experience with motion, we know speed is how far we travel divided by how long it took to travel. Say you go From Rexburg to Idaho Falls (IF). The displacement would be

$$\Delta x = 30\,\mathrm{mi}$$

and let's say it takes you a half hour to get to IF. Then the time would be

$$\Delta t = \frac{1}{2}\,\mathrm{h}$$

So the speed we travel to IF must be

$$v = \frac{\Delta x}{\Delta t} = \frac{30\,\mathrm{mi}}{\frac{1}{2}\,\mathrm{h}} = 60\frac{\mathrm{mi}}{\mathrm{h}}$$

This is a general formula. Speed is always how far we go divided by how long it took to get there.

Of course we don't want to use English units in this class, so we would say that

$$\Delta x = 48.28\,\mathrm{km}$$

and

$$\Delta t = \frac{1}{2}\,\mathrm{h} = 1800.0\,\mathrm{s}$$

so we would usually say

$$v = \frac{48.28\,\mathrm{km}}{1800.0\,\mathrm{s}} = 26.822\frac{\mathrm{m}}{\mathrm{s}}$$

In our jumping man case we can see that the man travels about $1.5\,\mathrm{m}$ in each part of the jump. Then the total displacement is

$$\Delta x_{total} = 5\Delta x_{21} = 5 \times 1.5\,\mathrm{m} = 7.5\,\mathrm{m}$$

and suppose

$$\Delta t = 1.07\,\mathrm{s}$$

then the man's speed would be

$$v = \frac{\Delta x}{\Delta t} = \frac{7.5\,\mathrm{m}}{1.07\,\mathrm{s}} = 7.01\frac{\mathrm{m}}{\mathrm{s}}$$

Our physics definition of speed is a little different than our every-day definition of speed. For example, if the jumped as in the photograph, but then jumped back in the same amount of time. His total Δx would then be zero. What would his speed be for the two jumps together?

$$v = \frac{\Delta x_{\text{two jumps}}}{\Delta t_{\text{two jumps}}} = \frac{0}{2.14\,\mathrm{s}} = 0$$

which might not be what you expected. The total distance traveled by the man would be

$$d = 2 \times 7.5\,\mathrm{m} = 15.0\,\mathrm{m}$$

and the time of the two jump experiment would be $1.14\,\mathrm{s}$. So it seems that the man's speed should be

$$speed = \frac{d}{t} = \frac{15.0\,\mathrm{m}}{2.14\,\mathrm{s}} = 7\frac{\mathrm{m}}{\mathrm{s}}$$

but that is not what we got with our new formula. Our new equation has a displacement in it. And displacement tells us how much progress we have made from where we started. So if our man jumps back to the start, he has made no progress, and his speed in this new sense is zero! This is a very specific definition of speed! you might be content to use our ever-day definition of speed while driving your car. In fact, that is exactly what a police man would use to determine if you need a speeding ticket. But if ordered a pizza from a restaurant that delivered pizza, you might only care about how fast a pizza made it from the restaurant location to your apartment. For the pizza delivery, our new definition of speed might make a lot of sense. If the delivery person crisscrossed town before making the delivery, the extra travel wouldn't be progress as far as you are concerned, and therefore wouldn't be counted when calculating the speediness of the delivery.

We found before that displacement can be negative. Our new definition of speed uses displacement. That means the new speed could end up being negative. But negative speed might not seem to make much sense. What do we mean when we say speed is negative?

Velocity

Remember our example of walking through town. We found that if we started a block north of the center of town and we walk until we are three blocks north of the center of town. Then our displacement was

$$\begin{aligned} \Delta x &= 3\text{blocks} - 1\text{block} \\ &= 2\text{blocks} \end{aligned}$$

If we started again one block north of the town center, but we walk until we are four blocks south of center. The our displacement would be

$$\begin{aligned} \Delta x &= -4\text{blocks} - 1\text{block} \\ &= -5\text{blocks} \end{aligned}$$

One displacement is positive and one is negative. What makes the difference? *Direction!* In our example, walking south gave negative displacement and going north gave positive displacement. So our speed quantity that we got for the jumping man is really more than speed. Using the displacement to find the speed gives us the speed we travel *and* the direction we travel. We need a new name for our speed term so we know it has both how fast we travel and which direction we travel. We will call this *velocity*.

Note that in physics speed and velocity are similar, but not the same. We will call how fast we are going the speed. It won't be positive or negative. It is just a value that says how fast we go. But velocity will mean both the speed and the direction. Velocity is the speed, with a plus or minus sign added to tell us which way we are going. So properly we should say that

$$v = \frac{\Delta x}{\Delta t}$$

is a velocity, not just a speed. But

$$|v| = \left|\frac{\Delta x}{\Delta t}\right|$$

is just a speed.

The absolute value signs are awkward, so lets make a new symbol that tells us we have not just a value for the speed, but also a direction. Let's put an arrow over any term that has direction in it.

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta\mathbf{x}}}{\Delta t}$$

then $\overrightarrow{\mathbf{v}}$ means velocity.

Without the arrow, v, means just the speed. Note that v won't ever be negative.

As an example, suppose it takes us 20 minutes to travel our 5 blocks. We say our walking velocity is

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta\mathbf{x}}}{\Delta t} = \frac{5\text{blocks}}{20\,\text{min}} = -0.24\frac{\text{blocks}}{\text{min}}$$

or about a quarter of a block a minute. Our speed would be

$$v = 0.24\frac{\text{blocks}}{\text{min}}$$

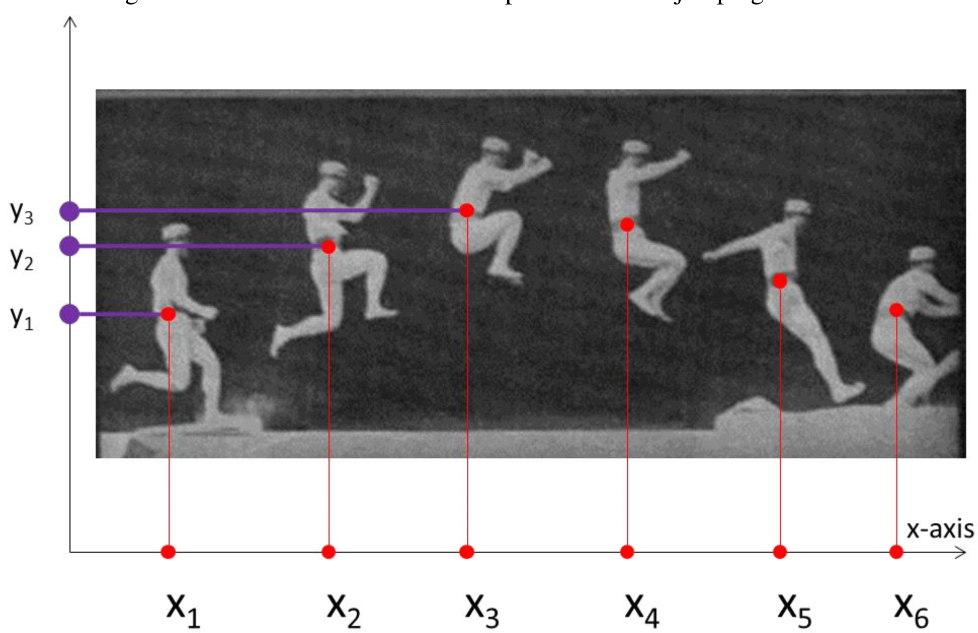
and our direction is given by the negative sign, where we have collectively agreed that South is the negative direction for our experiment.

Of course we know a name for a quantity that has both an amount and a direction. This is like our position vectors which had a length and a direction. Only now the length is a speed, and that is kind of strange. That length or speed part is the "amount" part of our quantity. Let's call the amount-part of a vector the *magnitude* of the vector and the direction part we will just keep calling "direction." So for a velocity vector the magnitude of the vector is the speed and the direction of the vector is which way the object is going. For position the magnitude of the position vector is how far away from the origin your position is, and the direction is the angle measured from the x-axis. Both position vectors and velocities are vectors because they have magnitudes and directions.

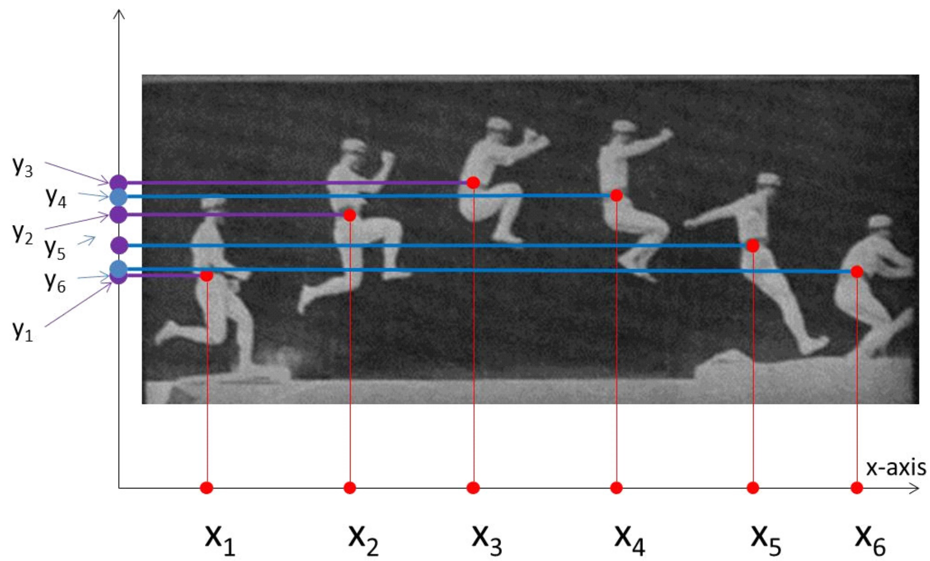
Vertical motion and equations

You might have objected to our analysis so far. It is true that our man travels horizontally, but he also travels vertically. How do we express displacement that is upward?

In the next figure we have marked the first three positions of the jumping man.



We can't use the variable x to describe how high up he went, because we have already used that variable for the horizontal displacement. So we picked another variable, y. And now we do just the same thing we did with horizontal displacement. We label the location of the center of the man at the first position as y_1 and the position of the center of the man at the second position as y_2 and so on until we have labeled each of the vertical positions of the center of the man.



Now we can define a vertical displacement

$$\Delta y = y_f - y_i$$

so that the total vertical displacement would be

$$\Delta y_{total} = y_6 - y_1$$

and we could find the displacement for each part of the man's jump

$$\begin{aligned} \Delta y_{21} &= y_2 - y_1 \\ \Delta y_{32} &= y_3 - y_2 \\ &\vdots \end{aligned}$$

We could even define a vertical velocity

$$\overrightarrow{\mathbf{v}}_y = \frac{\overrightarrow{\Delta \mathbf{y}}}{\Delta t}$$

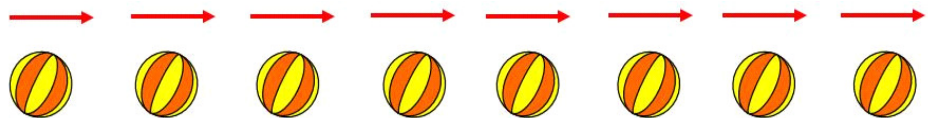
that would tell us how fast the man is going vertically and which direction he is going (up or down).

Of course, if we now label the vertical motion with the variable we chose for displacement in the vertical direction, we could also label the horizontal motion with the variable we chose for horizontal motion

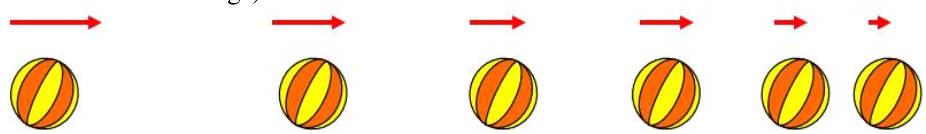
$$\overrightarrow{\mathbf{v}}_x = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$

and this is just what we normally do.

It is convenient to add velocity vectors to our motion diagrams. Then we can know for sure which direction our object is going (we can even dispense with the numbers!). Here we have a ball traveling at a constant velocity (both speed and direction are constant!).



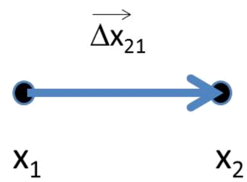
and here we have a ball changing speed so it is changing velocity (even though the direction does not change)



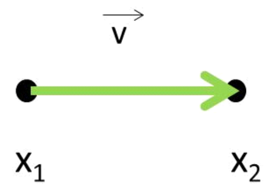
Often you will see motion diagrams where we have represented the object as just a dot (particle model) with the velocity vector attached to the dot.

It is important to realize that the vector is the average velocity of the object as it goes from x_1 to x_2, so it is might not be the velocity at x_1 or at x_2. This is like driving to IF with an average velocity of $26.8\frac{\mathrm{m}}{\mathrm{s}}$. Your speedometer right when you start and right when you end your trip might not be exactly $26.8\frac{\mathrm{m}}{\mathrm{s}}$. In fact, you will find that your speed changes a little along the way due to traffic conditions. What we have calculated is only an average. So it might be better to think of the average velocity vector as being somewhere in between x_1 and x_2. Soon we will calculate instantaneous velocity vectors, the kind of measurement made by your speedometer (plus a direction). Then the value for the velocity will be the speed and direction exactly at a point, say, x_1.

It is also good to realize that the artist that drew the last figure was being a little lazy. Here is a displacement vector Δx_{21}.



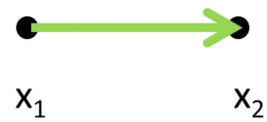
The magnitude of Δx_{21} should be exactly the distance from x_1 to $x_{2.}$ So the arrow is drawn correctly. It's length is just the distance from x_2 to x_2. But in the previous figure the artist drew the velocity vector as filling the space between x_1 and x_2.

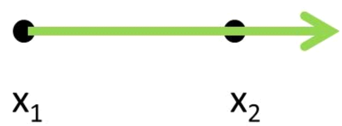


Should the velocity vector be the same length as the displacement vector? The answer is likely "no." Consider that

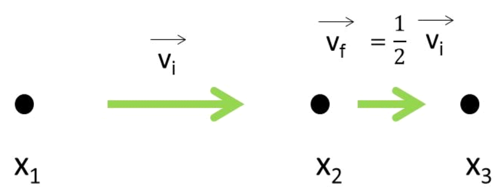
$$v = \frac{\Delta x}{\Delta t}$$

Unless we are very lucky, Δt would not likely be exactly $1\,\mathrm{s}$. So the numerical value for the speed won't be the same as the numerical value for the displacement. Unless we know the time, Δt, we don't even know how how long to make the vector $\overrightarrow{v}$! So we could draw the velocity vector in any of the following ways

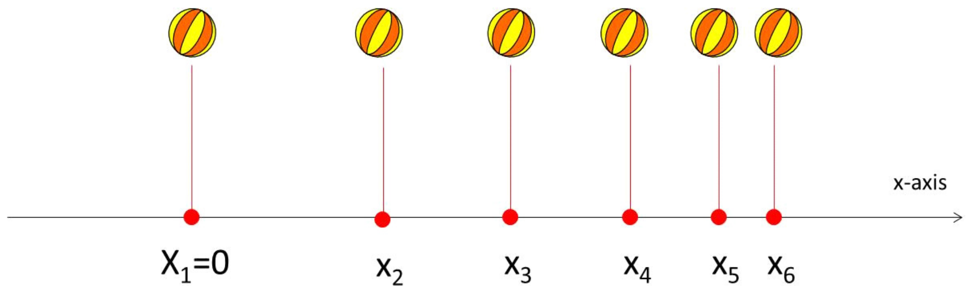




I think it is confusing to make the velocity vector fill the space between x_1 and $x_{2,}$ so I will try to not draw velocity vectors that way. Velocity is a different thing that displacement. So it is hard to compare their "lengths." So I recommend not trying to do so. If you don't know the time, Δt, but need to draw a velocity vector on your motion diagram (and we often do), just pick a length for $\overrightarrow{v}$. Make any changes to the velocity proportional to the original length you chose, so you can see how the velocity changes. For example, once you have picked a length for a velocity, if the velocity is slower by half later in your diagram, choose a velocity vector for the new point that is half the size of the original.



Acceleration



Let's look again at our moving ball case. Notice that at the first part of the motion the ball moves farther in Δt_1 than it does in Δt_5. Since we draw motion diagrams with equal time increments, and the displacements are different, the ball's velocity must be changing. We need a way to express the change in velocity mathematically if we are going to be able to understand this motion. And our example of velocity can show us a way to express this. In finding velocity we found a change in displacement in an amount of time

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t} = \frac{x_f - x_i}{\Delta t}$$

Now we want a change in velocity in an amount of time

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\Delta \mathbf{v}}}{\Delta t} = \frac{v_f - v_i}{\Delta t}$$

To show the change in the velocity we compare the velocity two points $v_f - v_i$ and divide by the time it took to make the change. For example, our ball may have moved $3\,\mathrm{m}$ between points x_1 and x_2. Suppose $\Delta t_1 = 1\,\mathrm{s}$. Then between points x_1 and x_2 the velocity would be

$$\overrightarrow{\mathbf{v}}_{12} = \frac{3\,\mathrm{m} - 0\,\mathrm{m}}{1\,\mathrm{s}} = 3\frac{\mathrm{m}}{\mathrm{s}}$$

but by the time we get to points x_5 and x_6 the ball has moved from $x_5 = 9.75\,\mathrm{m}$ to $x_6 = 11\,\mathrm{m}$ so

$$\overrightarrow{\mathbf{v}}_{56} = \frac{11\,\mathrm{m} - 9.75\,\mathrm{m}}{1\,\mathrm{s}} = 1.25\frac{\mathrm{m}}{\mathrm{s}}$$

Note that the velocity $\overrightarrow{\mathbf{v}}_{12}$ happens at some time in between t_2 and t_1. We are not exactly sure when. Let's say it is close to half way in between the two times, so $t_{12} \approx 1.5\,\mathrm{s}$. Likewise $t_{56} \approx 5.5\,\mathrm{s}$

Then the change in the velocity is

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\Delta \mathbf{v}}}{\Delta t} = \frac{1.25\frac{\mathrm{m}}{\mathrm{s}} - 3\frac{\mathrm{m}}{\mathrm{s}}}{5.5\,\mathrm{s} - 1.5\,\mathrm{s}} = -0.4375\frac{\mathrm{m}}{\mathrm{s}^2}$$

Notice that the units of our answer are m/s^2 or $(\mathrm{m}/\mathrm{s})/\mathrm{s}$. This is vaguely familiar.

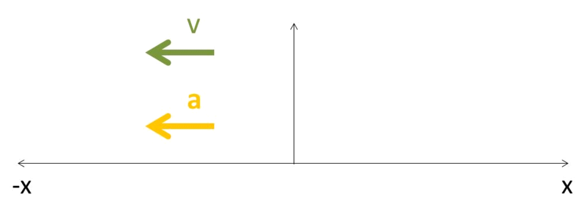
Think of how your car accelerates. We could go from 0 to $100\,\mathrm{km/h}$ in $3.6\,\mathrm{s}$. It is a change in our velocity in a given amount of time.

Also notice that the answer is negative! But what does this mean?

Like everything else, if the acceleration points to the left we call it negative and if it points to the right we all it positive. But really what is more important is that if the velocity and acceleration are in the same direction the mover is *speeding up*! likewise if the velocity and acceleration are in opposite directions the mover is *slowing down*. It matters whether the acceleration is in the same direction as the velocity or not.

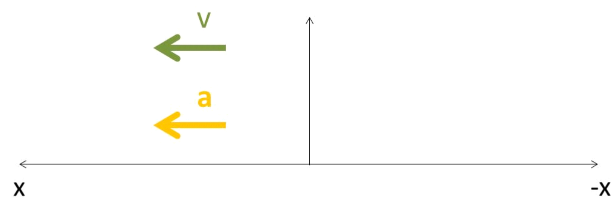
We can represent all this graphically by drawing the arrows. In the figure, we see a green arrow representing velocity and a yellow arrow representing acceleration. If they point the say way, the object is speeding up. If they point different directions, the object is slowing down.

There is a curious thing about acceleration in physics. We don't use the work "decelerate." We only use the word "accelerate." If we are slowing down, we are accelerating in the opposite direction we are going. It is important to notice that a negative acceleration is not necessarily slowing down. To see this, think about the following situation.



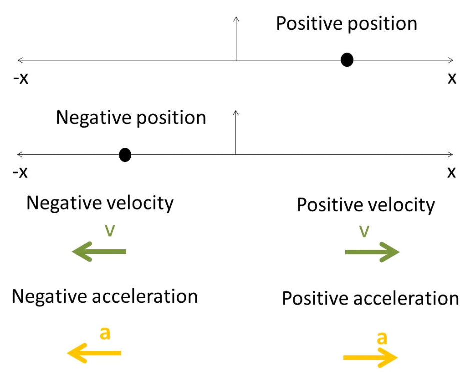
We have a velocity vector and an acceleration vector. We also have a coordinate system. The velocity would be negative simply because points in the negative direction (toward more negative numbers on the x-axis). The acceleration is also negative because it points in the negative direction. But if we had an object going in the negative x-direction an the acceleration was negative, the object would speed up. The acceleration and velocity are pointing the same way, so this is speeding up.

The coordinate system makes our velocity or acceleration negative or positive. We could define our coordinate system backwards so positive values were toward the left.

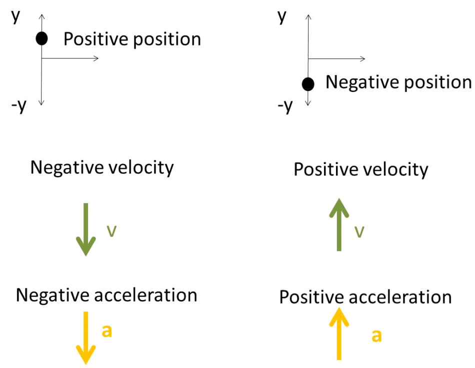


Then both velocity and acceleration would be positive, but still the object would be speeding up.

Since we are talking about coordinate systems, let's set up some standards for thinking about motion. We could define a coordinate system so that positive values are to the right, but usually we won't do that. Usually we will say that to the left in our coordinate system is negative and to the right is positive.



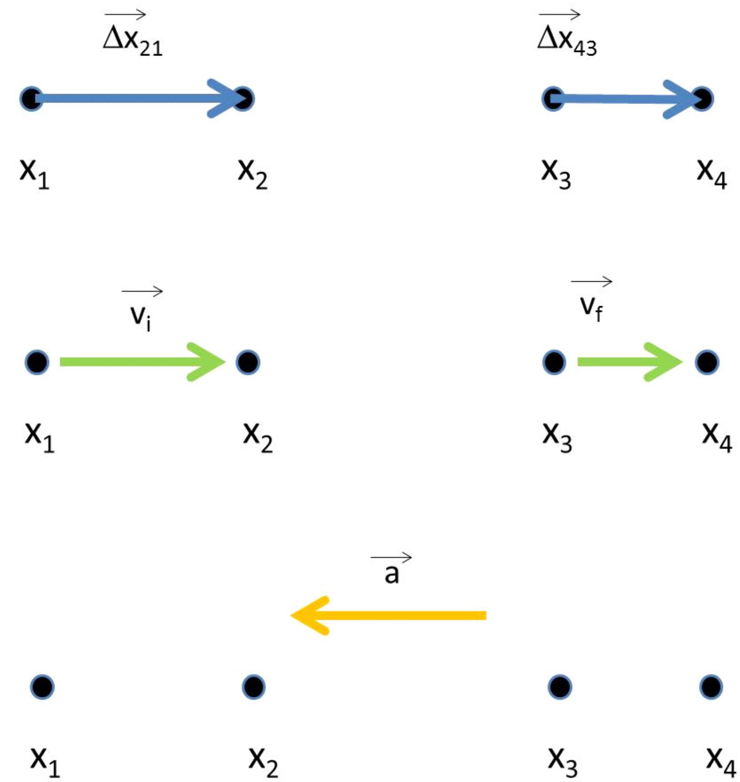
For the y-direction, we choose the upward direction to be positive and the downward direction to be negative.



Sometimes we will choose to break these standards, but for the most part these choices represent how we will define positive and negative directions in our study of motion.

Let's take an example. Suppose we have photos of an object at four locations, x_1 through x_4 that form a motion diagram.

We want to use these to draw the displacement, average velocity, and average acceleration. Here is the figure



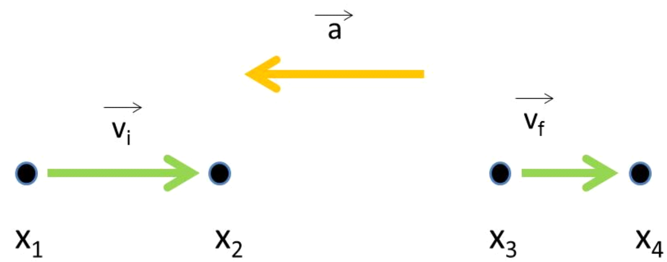
The magnitude of the displacement vectors Δx_{21} and Δx_{43} *must* fill the space between points 1 and 2 and points 3 and 4 respectively because the magnitude of the displacement if the distance between the points. The magnitude of the velocity vectors have to be in the same proportions as the displacements

$$\frac{v_i}{v_f} = \frac{\frac{\Delta x_{21}}{\Delta t}}{\frac{\Delta x_{43}}{\Delta t}} = \frac{\Delta x_{21}}{\Delta x_{43}}$$

so if $\Delta x_{21}/\Delta x_{43} = 3/4$ then so does $v_i/v_f = 3/4$. But the lengths of v_i and Δx_{21} do not have to be the same and neither do the lengths of v_f and Δx_{43}. They are just proportional. The length or magnitude of a is

$$a = \frac{v_f - v_i}{\Delta t_{total}}$$

and again we don't know Δt_{total} since we just have the pictures of the ball so we don't know how long to make the vector $\overrightarrow{a}$ but we do know that it must face to the left since v_f is less than v_i. Usually we don't draw the displacement vectors (because it is easy to see the displacement from the dot positions. So we might draw this situation with one motion diagram that looks like this



Vector Displacement and Velocity

So far we have our velocity as given by

$$\vec{\mathbf{v}}_x = \frac{\overrightarrow{\Delta\mathbf{x}}}{\Delta t}$$

$$\vec{\mathbf{v}}_y = \frac{\overrightarrow{\Delta\mathbf{y}}}{\Delta t}$$

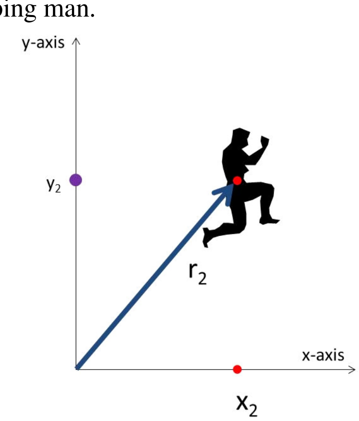
but now we realize that Δx and Δy are just parts of our vector displacement Δr. So $\vec{\mathbf{v}}_x$ and $\vec{\mathbf{v}}_y$ must be parts of a vector velocity as well! We can write our velocity as

$$\vec{\mathbf{v}} = \frac{\overrightarrow{\Delta\mathbf{r}}}{\Delta t}$$

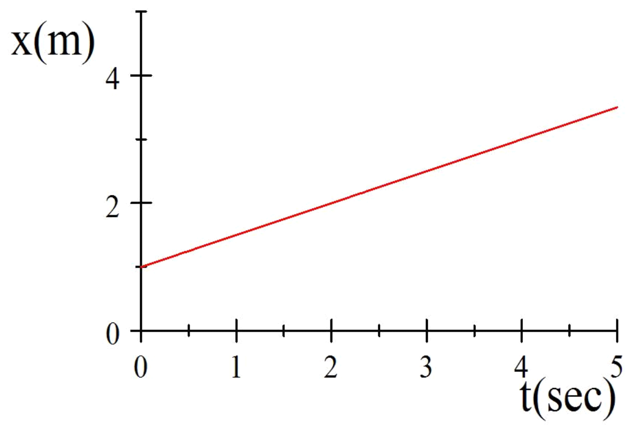
This works in any direction or combination of directions.

Position vs. time graphs

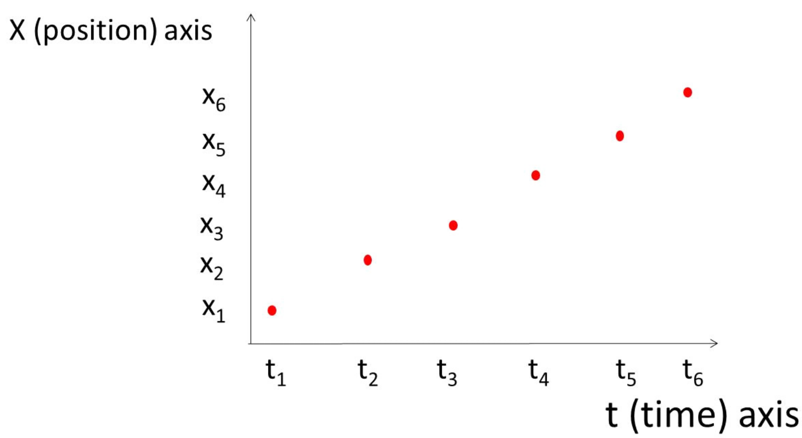
We have already used graphs in this class. But so far our graphs have given the position of an object, say, our jumping man.



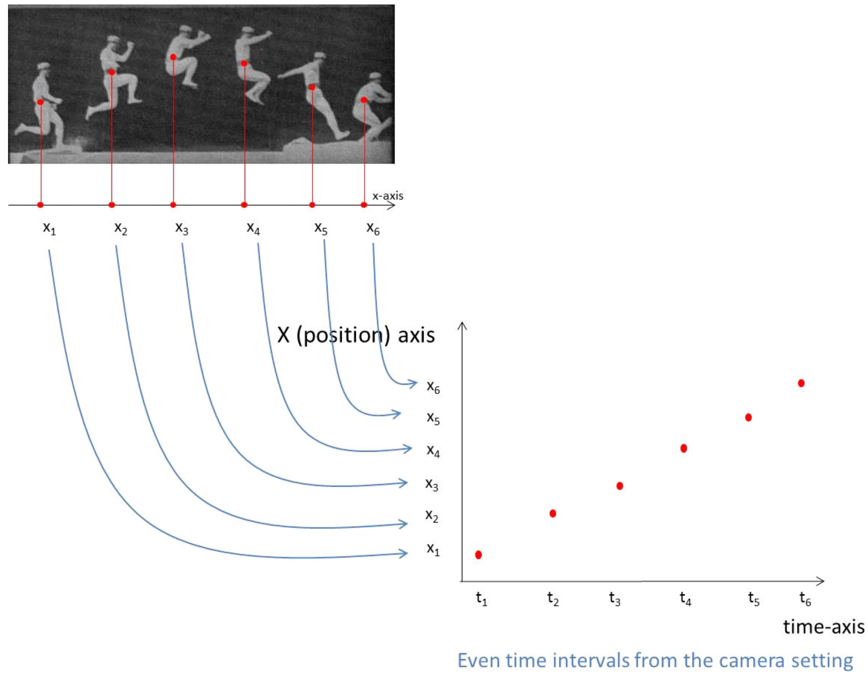
There is another kind of graph that is useful in describing motion. Since motion requires position and time, a graph of the position and time of the moving object helps us to know how the object is moving. We choose one of the axis of our graph to be time, usually the horizontal axis. Then the vertical axis will be position.



Here is an example for the jumping man.

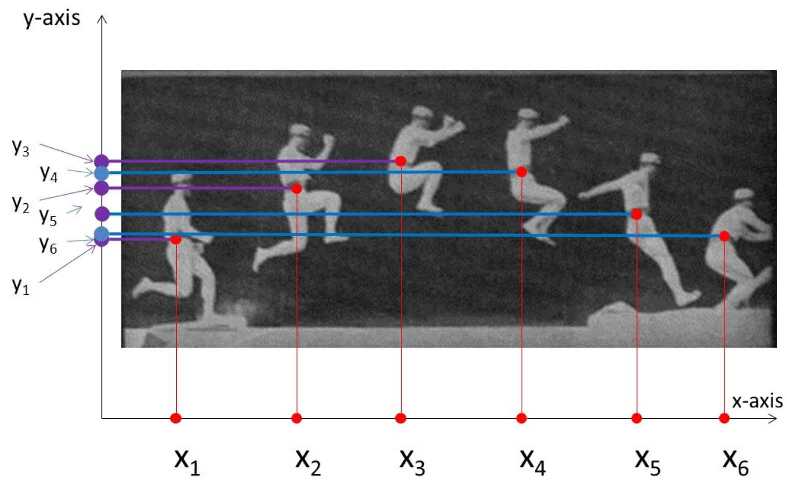


Let's take a moment and see how we formed this graph. The horizontal positions are placed on the vertical axis.

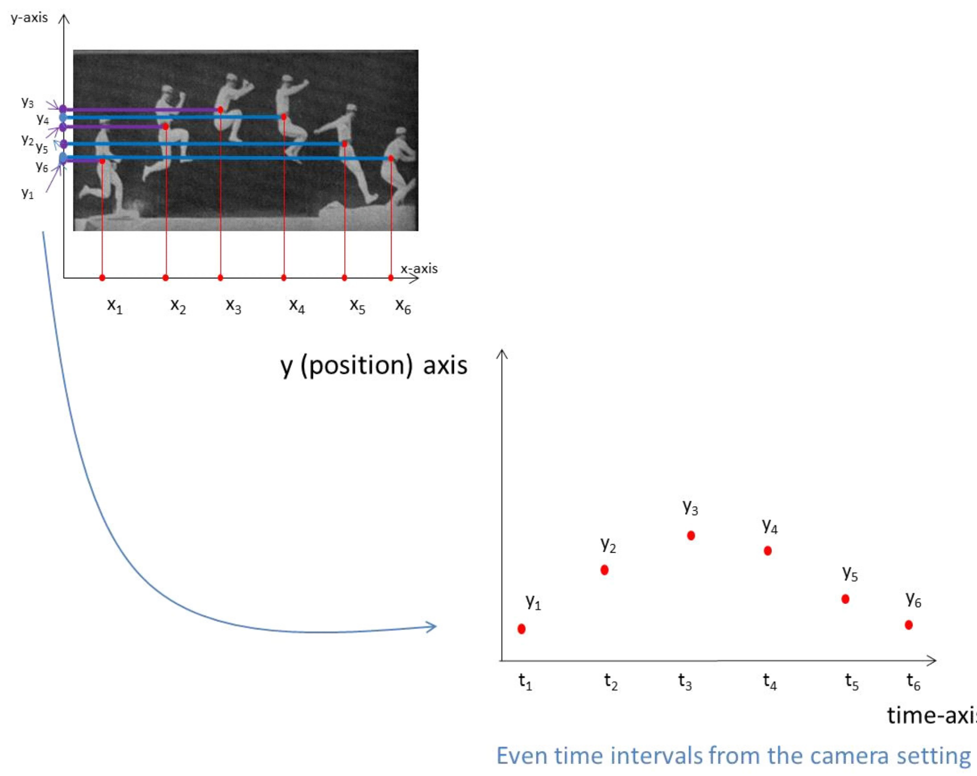


and we have assumed that the camera took pictures in even time increments. So each position is plotted a Δt away from the last along the t-axis. Since cameras usually operate in even time increments, but it is not required that the time steps be equal. But it is often convenient to use even time steps because it allows us to easily see changes in motion by noting the changes in position.

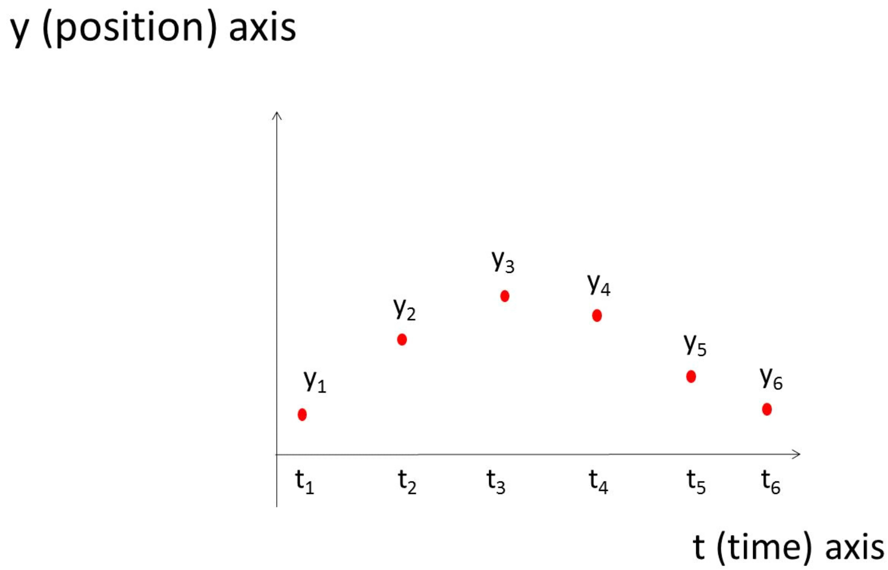
We will often use position vs. time graphs. Note that there are some difficulties with this type of graph. One is that we only plotted the x-position. But we know that we also have changes in the y-position for the jumping man. We could make another position vs. time graph for the jumping man for the horizontal (y) position. Immediately we realize that this will be more difficult because the y-positions don't fall in order on the y-axis. This is because the man goes up, and then comes back down.



But that is no problem. We can still make the graph, but this time we just have to be more careful about making the y-position axis.

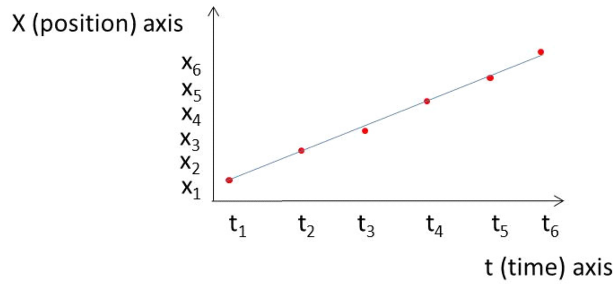


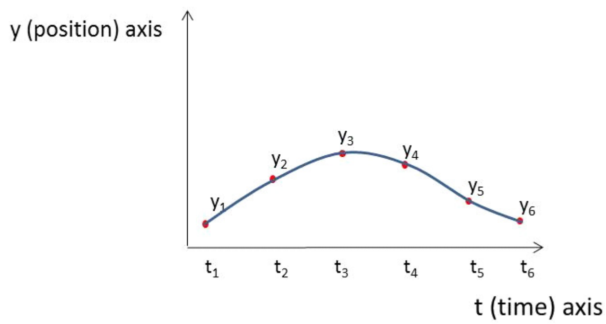
We are left with a graph that shows the vertical motion of the jumping man.



It is probably not to difficult that to see that we could combine these two graphs and plot r vs. time. But we will save that for another lecture.

It should also be clear that we have just plotted part of the jumping man's motion. We only have points for the positions where a photo was taken. But the man exists and is moving between the points. Sometimes we connect the dots with a line to show this missing motion





but it is important to realize that this is an estimate. We don't have data (photographs) to show us for sure where the man was in between our known points.

Still, a position vs. time graph is a powerful visualization tool that helps us understand the motion of an object.

4 Instantaneous Velocity

Physicists seem to like old words. The greek work $\kappa\iota\nu\varepsilon\omega$ (kineo) means to move. So the study of motion is often called *kinematics.* Today we tend to use this word to describe motion of objects near the Earth's surface or in situations like being near the Earth's surface. Since we live near the Earth's surface, we will start our mathematical study of motion with kinematics.

New Math

Likely your FDMAT 112 class is still teaching you how to take limits. And that is fine because we are going to use one today. Within the next three weeks, however, your calculus class will teach you...calculus! and calculus was invented by Newton to describe motion. So calculus is the mathematics of motion. We need to know how to do a little calculus today. I am going to let your math professor teach you the "why" of this new math. I am only going to teach you a little bit of "how" to do the new math.

Our new math is called a *derivative*. and for polynomial terms there is a formula for taking a derivative.

If we have a function like

$$u = at^n$$

where a is a constant, then the derivative of this function is written as

$$\frac{du}{dt} = ant^{n-1}$$

where du/dy is the complicated symbol for the derivative of the function u, and the new function that results from taking the derivative of the function, u, is the constant a, times n, times t to the $n-1$ power.

Suppose we have a polynomial with two terms so a function w is given by

$$w = at^n + bt^m$$

the derivative of the sum of two functions is just the sum of the derivative of the two functions. We can treat at^n as a function (it's just our old function u) and we can treat

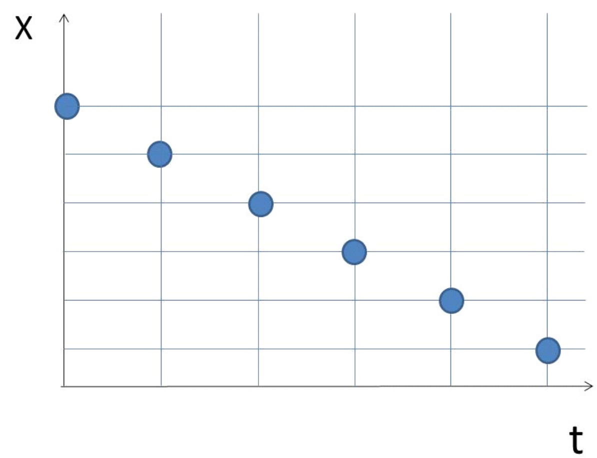
bt^m as another function. Then we use our derivative rule twice and add the results

$$\frac{dw}{dt} = ant^{n-1} + bmt^{m-1}$$

Notice that to use a derivative, we need a function. We will use this new math shortly.

Uniform motion

Uniform motion is a special case of motion that is useful, because we often have uniform motion. Think of driving your car. For long periods of time you may travel at $70\,\mathrm{mi/\,h}$ along the freeway. That is uniform motion. Here is a position vs. time diagram for uniform motion and a motion diagram for the same uniform motion.



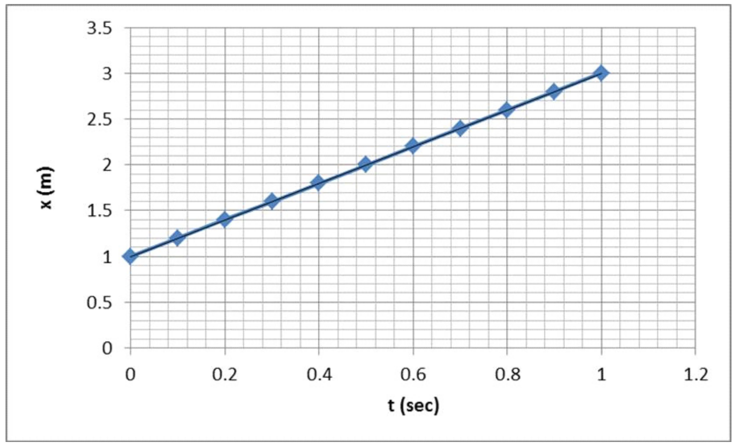
Notice that for this motion

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$

would be constant so

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\Delta \mathbf{v}}}{\Delta t} = 0$$

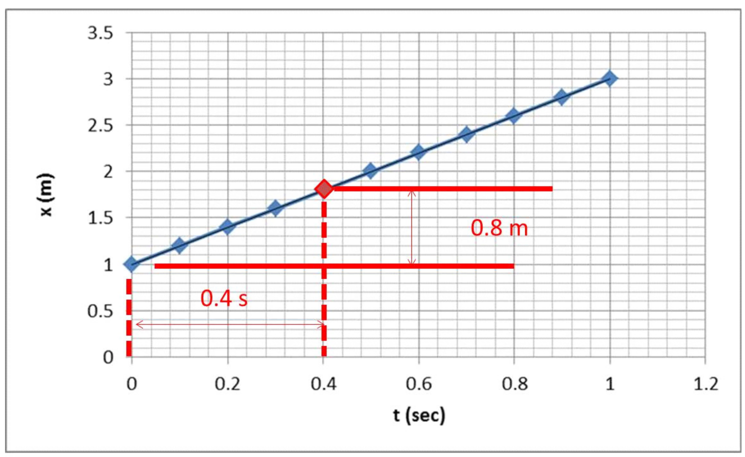
Let's take a specific position vs. time graph for a specific motion as an example. The following position vs. time graph describes a constant motion. What is the speed of the object?



The speed would be

$$v = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

where, because the motion is uniform, we can choose any two points for the final and initial points. Let's try x_0 and x_5 as our two points. Then we can see that



$$v = \frac{x_5 - x_0}{t_5 - t_0} = \frac{0.8m}{0.4s} = 2\frac{m}{s}$$

$$\begin{aligned} v &= \frac{x_5 - x_0}{t_5 - t_0} \\ &= \frac{0.8\,\mathrm{m}}{0.4\,\mathrm{s}} \\ &= 2\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Notice that this is just the “rise” over the “run” or the speed is just the slope of the line of a position vs. time graph!

Suppose that you know your speed, v, and your starting location, x_0 and you know how long you want to drive, could you find how far you will go? Of course! this just takes a little math.

Notice that we could rearrange our equation to solve for x_f.

$$v = \frac{x_f - x_i}{t_f - t_i}$$

$$v\,(t_f - t_i) = x_f - x_i$$

$$x_i + v\,(t_f - t_i) = x_f$$

so

$$x_f = x_i + v\Delta t$$

Let’s try such a problem. You wish to drive for half an hour, that is

$$\Delta t = \frac{1}{2}\,\mathrm{h} = 1800.0\,\mathrm{s}$$

and you wish to drive at

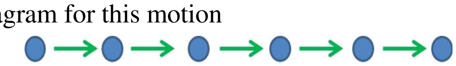
$$v = 70\,\mathrm{mi/\,h} = 31.293\frac{\mathrm{m}}{\mathrm{s}}$$

and let’s define our starting location as

$$x_0 = 0$$

how far will we go?

We should draw a diagram for this motion



and realize that it is uniform motion, then we can use the equation for x_f that we just found

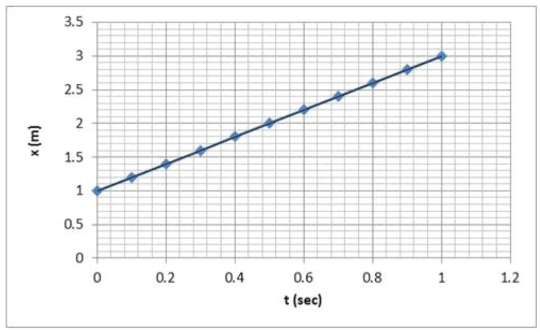
$$x_f = x_i + v\Delta t$$

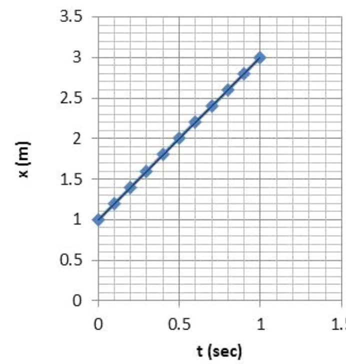
and fill in the pieces

$$
\begin{aligned}
x_f &= 0 + \left(31.293\frac{\mathrm{m}}{\mathrm{s}}\right)(1800.0\,\mathrm{s}) \\
&= 56327.\,\mathrm{m} \\
&= 56.327\,\mathrm{km} \\
&= 35.00\,\mathrm{mi}
\end{aligned}
$$

If you are taking this class in Rexburg, you might recognize this as a trip to Idaho Falls.

Let's look at two position vs. time graphs for uniform motion.





We know that the slope of the line in a position vs. time graph tells us the speed, but look at these two graphs carefully! Notice that they graph the same motion. One graph just has it's time graph compressed (kind of squished smaller) than the other. So although the second graph's slope looks steeper, this is only because of how we graphed the motion. If we had the same axes on both graphs, then they would look exactly the same. This means that we will have to be careful in how we interpret graphs of motion. We need to actually calculate the rise over the run or the numeric slope instead of just looking at the picture.

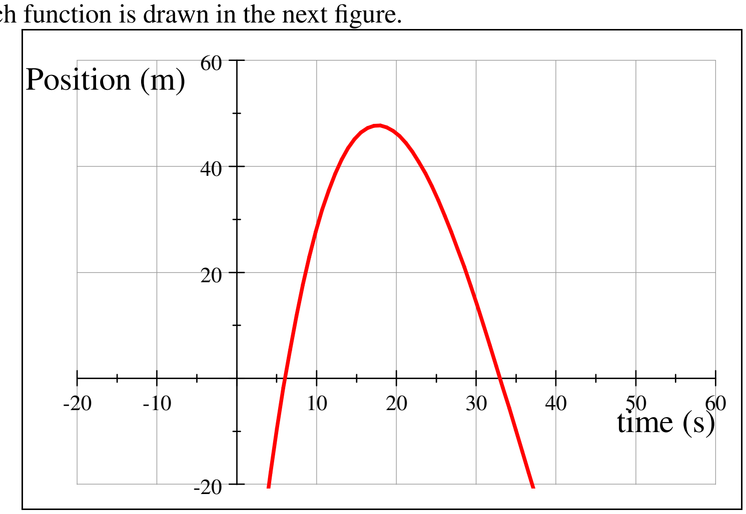
Armed with understanding how to interpret graphs for uniform motion, we can take on a new quantity.

Instantaneous Velocity

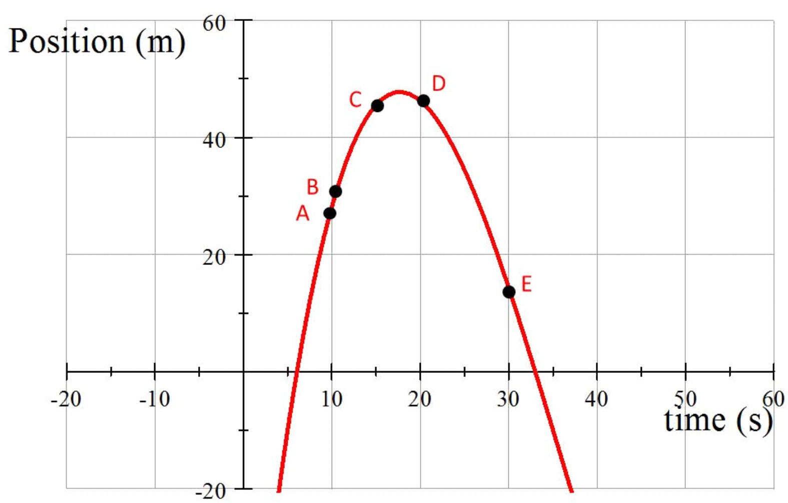
You may have thought as we have been talking that average velocity and average speed are nice, but generally when you think of speed, you are looking at a speedometer. The speedometer does not seem to be measuring average speed. It gives the speed you are traveling right now, an *instantaneous speed.* If we add in the direction we are going, we may also talk about an *instantaneous velocity*

For a constant velocity, the instantaneous velocity and the average velocity must be the same (no mater how small Δt, if v is constant, then it is constant). But what happens if we have a changing velocity so the velocity is a function of t?

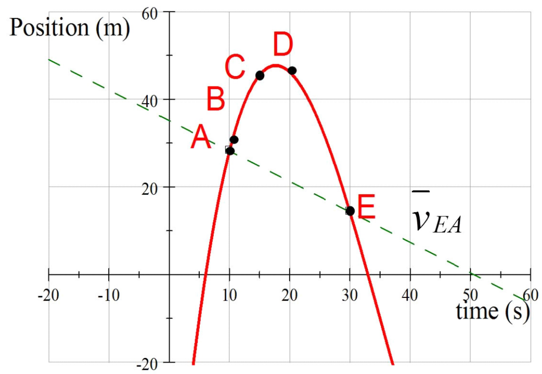
One such function is drawn in the next figure.



Let's start our experiment when the object is at position A as shown in the next figure.

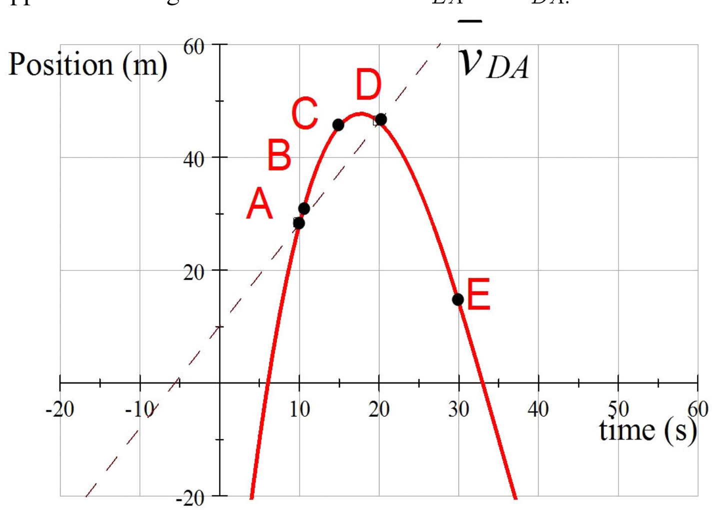


The position A we will call our initial position at our initial time, and position E as our final position and time, we have the average velocity given by the slope of the dashed line in the next figure that makes a chord-like cut across the function in the top part of the graph.



For this case the average speed v_{EA} measured over the time Δt_{EA} is really different that the instantaneous speed. In fact, between C and D the slope of our actual curve is zero. And the slope of the line through points A and E is really not zero. We can interpret that as meaning that the object stopped with zero speed between C and D! The time that the object stopped will affect the average speed. Think of going to Idaho Falls. You may travel at $70\,\mathrm{mi/\,h}$ on the freeway, but your average speed would be less if you stopped for lunch in Rigby!

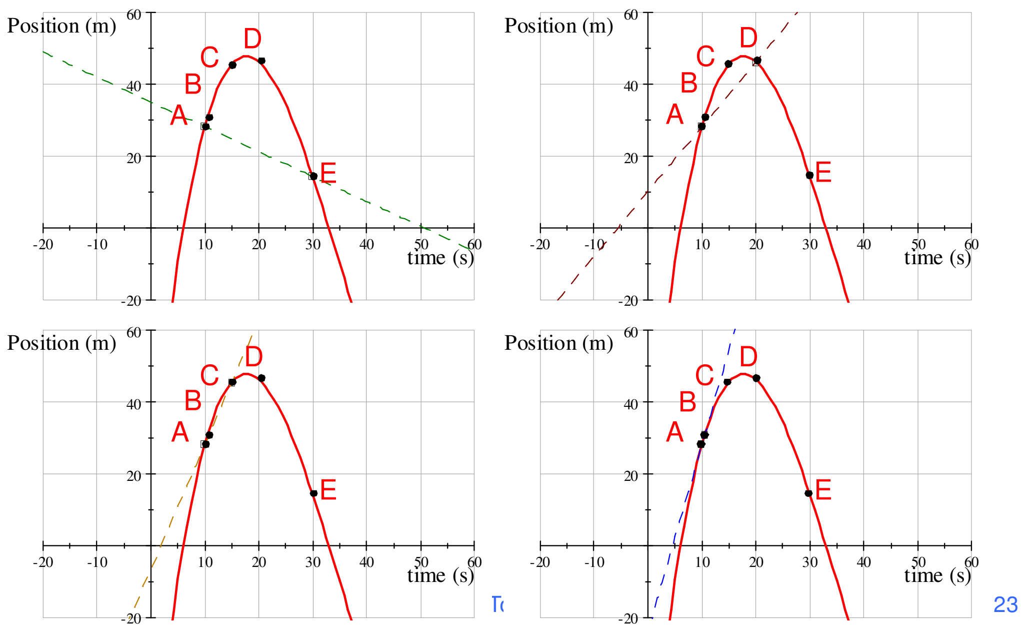
But suppose we changed our interval from Δt_{EA} to Δt_{DA}.



Notice that the slope of the dashed line is closer to the slope of the actual curve. And we believe from our uniform motion analysis that the slope of a curve in a position vs. time graph is the speed!

So by making Δt smaller, we got closer to what we have reason to believe is the instantaneous speed.

We can keep getting closer to A, The results of trying this are in the bottom two parts of the next figure. We see that if we take smaller and smaller time intervals ($\Delta t \rightarrow 0$), we see the dashed line becomes a tangent!



And now you will recognize what you have been studying in FDMAT 112! We are really taking a limit. We are taking the limit of the average velocity as $\Delta t \rightarrow 0$.

Definition 4.1 *Instantaneous velocity v is the limit of the average velocity as the time interval Δt becomes very small.*

Algebraically we can write this as

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \tag{4.1}$$

where the symbol

$$\lim_{\Delta t \rightarrow 0}$$

simply means that we let Δt become infinitesimally small. When

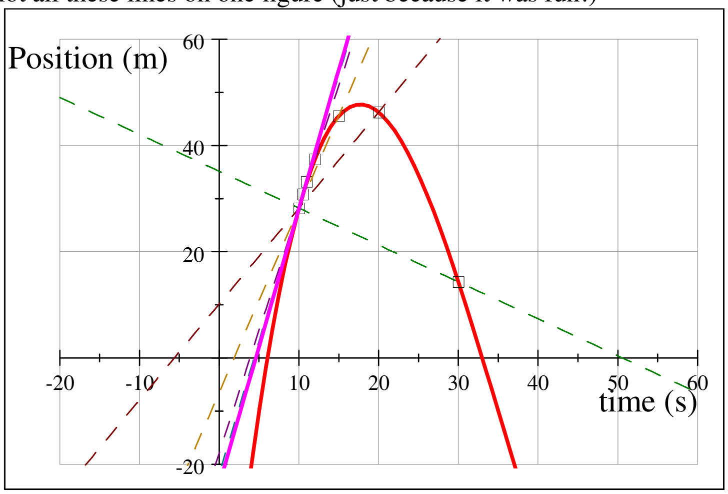
$$v = \frac{\Delta x}{\Delta t}$$

has a Δt that is infinitesimally small we write it as

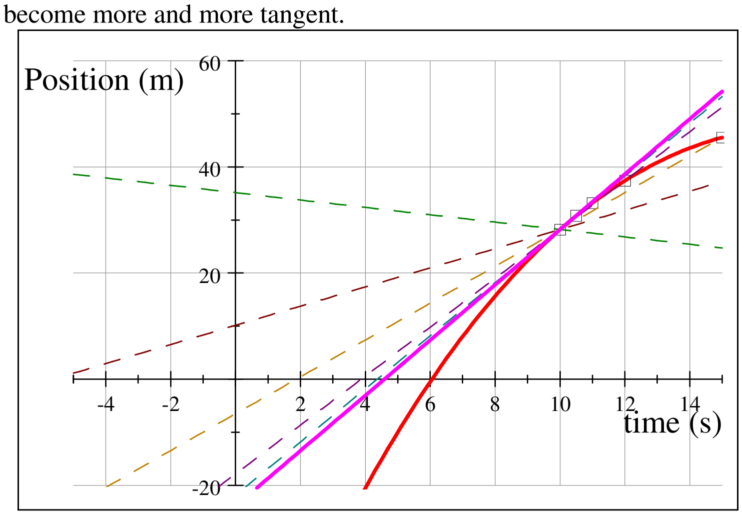
$$v = \frac{dx}{dt}$$

to show that Δt is really small. The "d" tells us this delta is really really small. But notice that this is just the notation of our new derivative math! And this similarity in the notation is on purpose. This is great! The instantaneous velocity can be found using our new derivative math.

We can plot all these lines on one figure (just because it was fun!)



and even zoom in on the part right around our starting point. We see that the dashed lines do become more and more tangent.



This situation is general for any function (well, any function for which we can find the derivative). We can state that

> The instantaneous velocity is defined as the slope of the line tangent to the position vs. time curve at a given time.

Definition 4.2 *The instantaneous speed is the magnitude of the instantaneous velocity.*

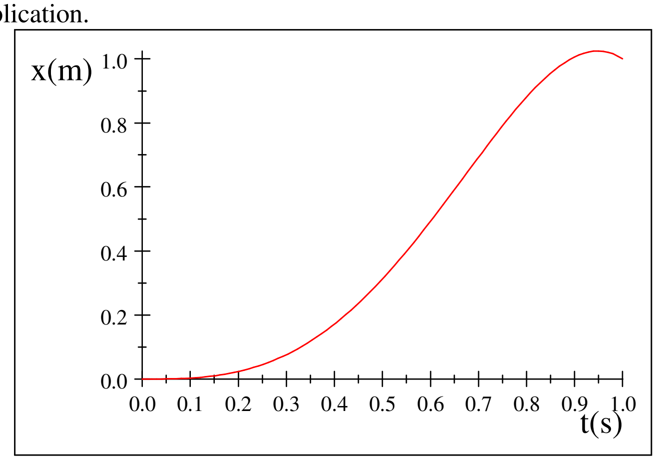
Remember that speed has no direction. This instantaneous speed is usually what we just call "speed."

Let's try an example:

Suppose the position vs. time has a functional form like

$$x\left(t\right)=-\left(2\frac{\mathrm{m}}{\mathrm{s}^5}\right)t^5+\left(3\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^3$$

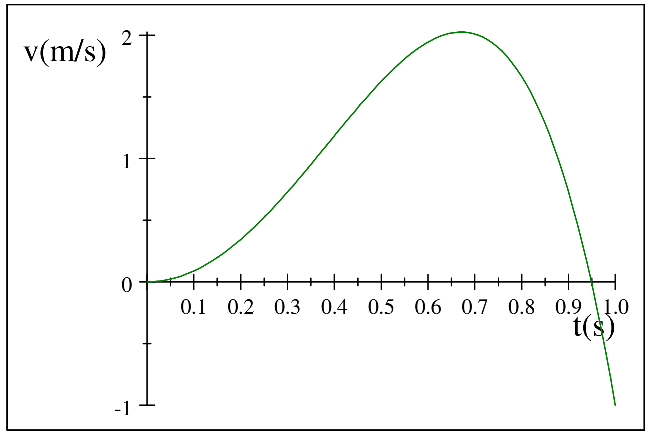
Recall that $x\left(t\right)$ means that x is a function of t, that is, x depends on t. It is not multiplication.



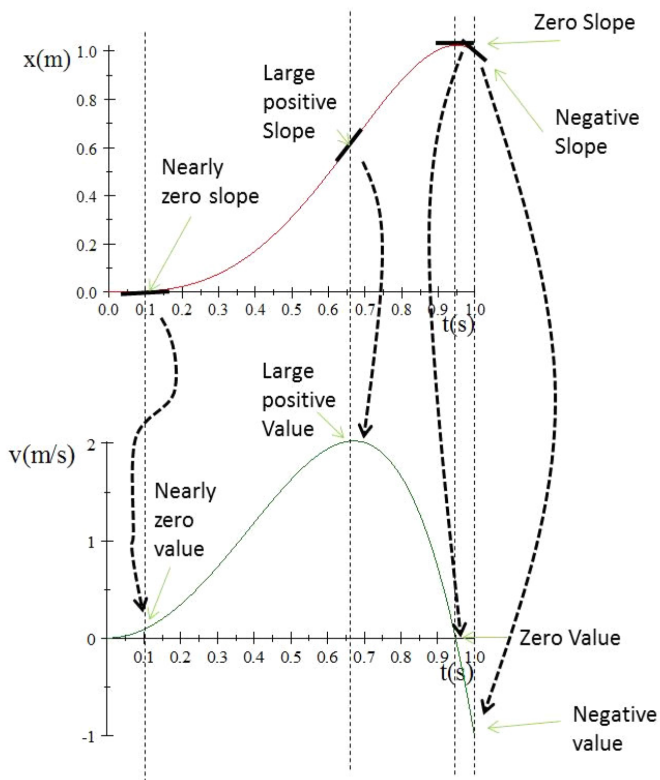
what would the velocity as a function of time be?

We can use our new math! the speed will be

$$\begin{aligned} v &= \frac{dx}{dt}=\frac{d}{dt}\left(-\left(2\frac{\mathrm{m}}{\mathrm{s}^5}\right)t^5+\left(3\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^3\right) \\ &= -\left(10\frac{\mathrm{m}}{\mathrm{s}^5}\right)t^4+\left(9\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2 \end{aligned}$$



Let's look at these two graphs. The $v(t)$ graph should be the derivative, that is the slope, graph of $x(t)$. So if we find the slope at several places along the $x(t)$ graph, the value of that slope should correspond to the velocity on the $v(t)$ graph below.



So at about $0.1\,\mathrm{s}$ we see we have a small, nearly zero positive slope on the $x(t)$ graph. And in the $v(t)$ graph at $t = 0.1\,\mathrm{s}$ we have a small positive value for v. At about $0.65\,\mathrm{s}$ we have a large positive slope on the $x\left(t\right)$ graph, and at $0.65\,\mathrm{s}$ on the velocity vs. time graph we see a large positive value. At $0.95\,\mathrm{s}$, we have zero slope, and in the $v\left(t\right)$ graph we have a speed of zero at $0.95\,\mathrm{s}$. At $1\,\mathrm{s}$ we have a negative slope on the $x\left(t\right)$ graph, and at $1\,\mathrm{s}$ we have a negative value on the $v\left(t\right)$ graph. Since our one-dimensional graph of speed can have positive or negative values, we realize that this is really a velocity graph! So we are justified in writing our derivative equation as

$$\overrightarrow{\mathbf{v}} = \frac{d\overrightarrow{\Delta x}}{dt}$$

including direction.

Note that this new math is a powerful new tool to describe motion of a moving object. We will use the idea of a derivative of position as our velocity for the rest of this course, and if you are lucky and are majoring in physics, for many courses to come!

Speed, Average Speed, Instantaneous Speed

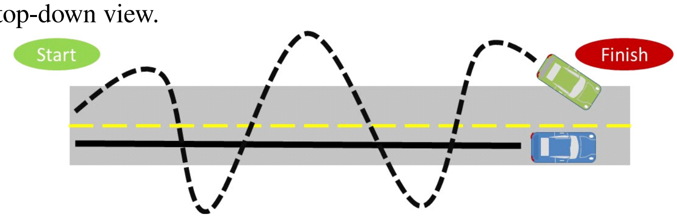
It's probably time to clarify our physics language a bit. Speed is how fast we are going. But we could define speed as how far we have gone (a distance) divided by how long it took (a duration).

$$\text{speed} = \frac{\text{distance}}{\Delta t}$$

We could also define velocity as

$$v = \frac{\Delta x}{\Delta t}$$

which tells us how far we have gotten toward some goal. Consider the car race shown below in a top-down view.



One car (the green one if you are seeing this in color) has clearly gone farther. Since the two cars are arriving at the finish at the same time, the green car must have gone faster since it traveled farther on it's zigzag path. To find out how fast the green car went we would indeed take

$$\text{speed} = \frac{\text{distance}}{\Delta t}$$

But in some since, the two cars started at the same place and ended at the same place, and they did this in the same time. Their average speeds must be the same!

$$v_{ave} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

We can see that both ideas, speed and average speed, are valuable. For this class, if a problem just asks for speed, we want distance traveled over time (duration). If the problem asks for average speed, we will report the magnitude of the displacement over duration.

Of course each of our cars has a speedometer reading the instantaneous speed of the cars. This instantaneous speed might be very different than the average speed or the speed of the car for the entire trip. For example, the blue car might have slowed down to avoid hitting a deer (that happens in Idaho!). Then he would have speed up again. During some part of the blue car's path, the blue car instantaneous speed would have been zero! But just for the time the car was stopped for the deer.

We will have to keep these three ideas in our mind. Speed, average speed, and instantaneous speed (and of course, average velocity and instantaneous velocity as well).

5 Position from velocity

So far we have found how far we go from

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

but solving for

$$\begin{aligned} \Delta x &= \bar{v}\Delta t \\ x_f &= x_i + \bar{v}\Delta t \end{aligned}$$

but this assumed constant motion. What if our motion isn't constant? We need more new math!

More New Math

We now know how to take a derivative, but you were probably wondering if there is a way to undo the process of taking a derivative. We could call that an anti-derivative. Such a thing might be useful. If we consider that

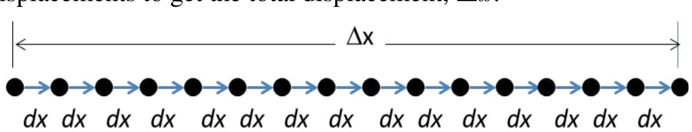
$$v(t) = \frac{dx}{dt} \tag{5.1}$$

then undoing this process might be able to give us $x(t)$ if we know $v(t)$. Say, you record your speed as you travel from your speedometer, but you want to know how far you have gone.

There is such a process. Let's think about what the process might be. From equation () we can solve for the small displacement dx

$$dx = vdt$$

This is just a small amount of displacement dx in a small amount of time, dt, so if I want to know how far my object will go in some larger amount of time, I need to add up all the small displacements to get the total displacement, Δx.



Think for a minute that a large Δt would be the sum of many, many little dt units. Then the large displacement that corresponds to the Δt would be the sum of many little displacements, dx.

So we can write our displacement like this

$$\Delta x = dx_1 + dx_2 + dx_3 + \cdots dx_N$$

where dx_N is the last of our small dx pieces. In math we can write this as

$$\Delta x = \sum_{i=1}^{N} dx_i$$

but since the dx units are so small, we give this a special, calculus notation for summation

$$\Delta x = \int_1^N dx$$

where the curly thing, $\int$, has replaced the Σ. The curly thing is a stylized "s" for sum. The Σ is a Greek "s" for sum. So they are doing the same operation, but $\int$ works on infinitesimally small displacements, dx. We still go from the first dx to the N^{th} dx and that is shown by the "lower limit," 1 and the "upper limit," N on the $\int$.

Recall that $\Delta x = x_f - x_i$ so

$$\begin{aligned} x_f &= x_i + \Delta x \\ &= x_i + \int_1^N dx \end{aligned}$$

but each dx is given by

$$dx = vdt$$

and when we change from the counting over the number if dx elements to counting over time elements, dt, we also changed the limits. The first dx was traveled at t_i, the first time. The last was traveled at the final N^{th} time so the upper limit is t_f, so

$$x_f = x_i + \int_{t_i}^{t_f} vdt$$

This is our notation for taking the anti-derivative of $v(t)$. But we don't want to actually have to measure each vdt and sum them by hand. We need a procedure for finding the result of $\int_{t_i}^{t_f} vdt$.

Once again I will let your calculus professor explain why this works, for this class I will just give a procedure. Let's take a function

$$u = at^n$$

then the anti-derivative of u between two times t_1 and t_2 is

$$\int_{t_1}^{t_2} udt = \int_{t_1}^{t_2} at^n dt = \left.\frac{at^{n+1}}{n+1}\right|_{t_1}^{t_2} = \frac{at_2^{n+1}}{n+1} - \frac{at_1^{n+1}}{n+1}$$

The strange bar $|_{t_1}^{t_2}$ just keeps track of our upper and lower limits. But notice that the way we use the upper and lower limits is to substitute *both* limits into the equation that is sitting just before the bar, and subtract the equation with the lower limit from the equation with the higher limit. This isn't so strange if you remember that we are finding something like Δx which would have a x_f and an x_i and we would subtract the two.

Like with derivatives, antiderivatives of sums of functions are just the sums of the anti-derivatives

$$\int_{t_1}^{t_2} (u + w)\, dt = \int_{t_1}^{t_2} udt + \int_{t_1}^{t_2} wdt$$

It is also important to notice that antiderivatives require us to know a function, just like derivatives do. In our case, we need to know the function $v\,(t)$, that is, the velocity as a function of time.

Let's try this on the case of constant motion. Then $v\,(t) = v$ where v is some constant speed. In this case,

$$x_f = x_i + \int_{t_1}^{t_2} vdt$$

doesn't seem to have a t in it. But remember $1 = t^0$, so we can write our x_f equation as

$$x_f = x_i + \int_1^N vt^0 dt$$

and use our formula for antiderivatives

$$\begin{aligned} x_f &= x_i + \int_{t_1}^{t_2} vt^0 dt \\ &= x_i + \left.\frac{vt^{0+1}}{0+1}\right|_{t_1}^{t_2} \\ &= x_i + vt|_{t_1}^{t_2} \\ &= x_i + vt_2 - vt_1 \\ &= x_i + v\Delta t \end{aligned}$$

which is not much of a surprise. This is just

$$v = \frac{\Delta x}{\Delta t}$$

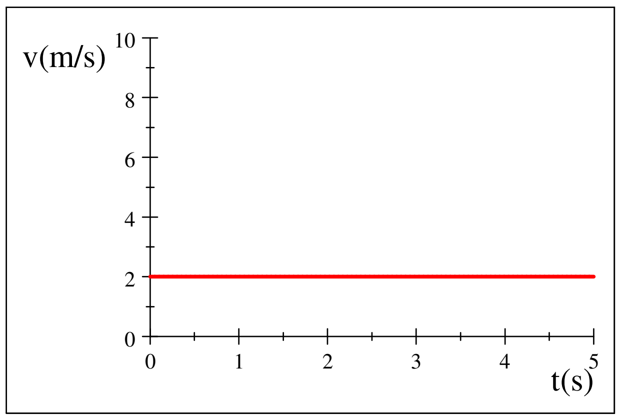
rearranged a bit!

This seems like a lot of trouble to go through for constant motion. We got just what we would expect to get, and we already knew the equation. But not all motion is constant motion! And in many cases we have a $v(t)$ that really depends on t. In such a case, our antiderivative method is the only way to get an answer for $x(t)$ knowing $v(t)$.

Before we take on such a case, there are a few more things about antiderivatives we should know.

The first is that the official modern name for an antiderivative is *"integral"* and to find an antiderivative is called "taking an integral" or simply *"integrating."*

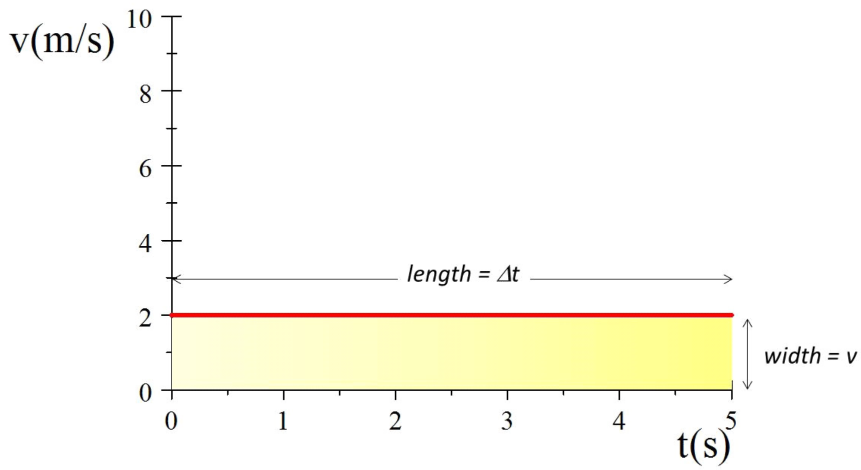
The second thing is a graphic interpretation for integrating. Let's graph the velocity vs. time for a constant motion case. Let's say we have an object leaving the origin at $t_i = 0$ and traveling at $2\,\mathrm{m/s}$. The graph would look like this



Let's calculate the final position when $t_f = 5\,\mathrm{s}$. Our equation gives us

$$
\begin{aligned}
x_f &= x_i + v\Delta t \\
&= 0 + v\Delta t \\
&= v\Delta t \\
&= \left(2\frac{\mathrm{m}}{\mathrm{s}}\right)(5\,\mathrm{s} - 0\,\mathrm{s}) \\
&= 10\,\mathrm{m}
\end{aligned}
$$

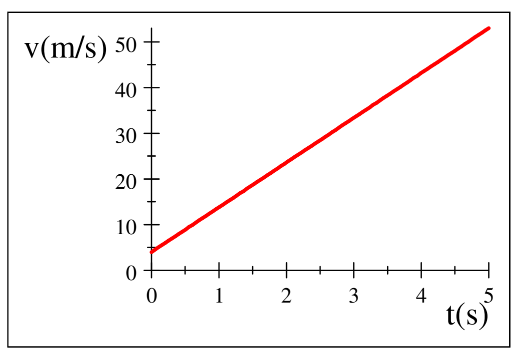
Looking at our graph we see that one axes is the time axes and the other is the velocity axis. We could define a kind of area for the surface of the graph, the part where we do the plotting, that would be length times width. But the length would be Δt and the width would be v. So the total "area" would be just $v\Delta t$.



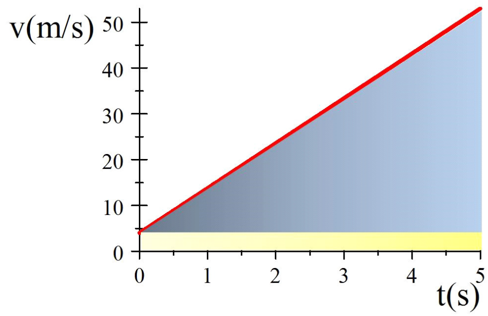
Notice that this "area" would be

$$\text{“area”} = \Delta t \times v = (5\,\mathrm{s} - 0\,\mathrm{s})\left(2\frac{\mathrm{m}}{\mathrm{s}}\right) = 10\,\mathrm{m}$$

It seems that the area under the red $v(t)$ line is equal to the final position, x_f! Of course real areas are constructed of lengths and widths that are both in the same units. So we don't mean that actual rectangle area as measured on the page with a ruler. What we mean is that if we take the value from the graph for the length, Δt, and multiply by the width, v, we get x_f. And this is what anti-derivatives or integrals do. they find the area under a curve. And this works for a $v(t)$ that is not constant as well. Consider the next graph.



We could find the "area under the curve" by splitting up the "area" into two pieces. A rectangle and a triangle.



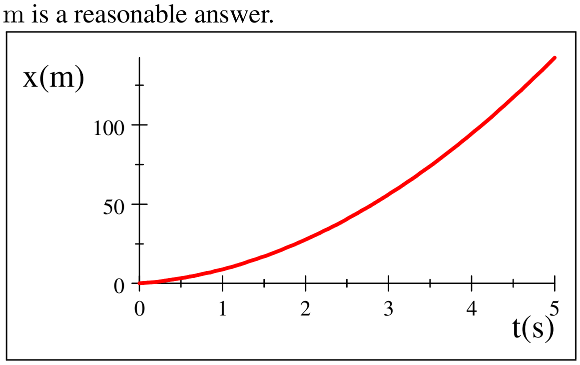
The yellow rectangle has a width of $4\,\mathrm{m/\,s}$ and a length of $5\,\mathrm{s}$. So the rectangle has an "area" of $20\,\mathrm{m}$. The triangle has a height of $53.0\frac{\mathrm{m}}{\mathrm{s}} - 4\frac{\mathrm{m}}{\mathrm{s}} = 49.0\frac{\mathrm{m}}{\mathrm{s}}$ and a base of $5\,\mathrm{s}$. So the area is

$$
\begin{aligned}
A_{triangle} &= \frac{1}{2}bh \\
&= \frac{1}{2}(5\,\mathrm{s})\left(49.0\frac{\mathrm{m}}{\mathrm{s}}\right) \\
&= 122.5\,\mathrm{m}
\end{aligned}
$$

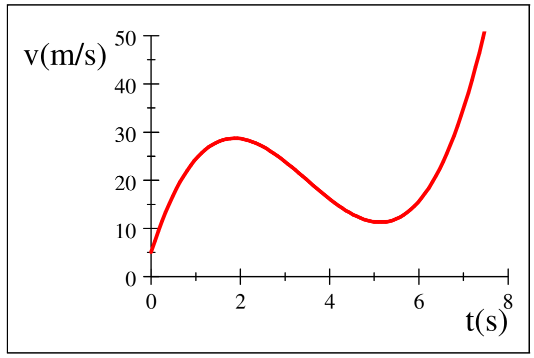
so the entire "area" is

$$
\begin{aligned}
x_f &= 20\,\mathrm{m} + 122.5\,\mathrm{m} \\
&= 142.5\,\mathrm{m}
\end{aligned}
$$

And, indeed, if we were to plot the position vs. time graph for this motion we would see that $x_f = 152.5\,\mathrm{m}$ is a reasonable answer.



Of course we could do much more complicated motions.



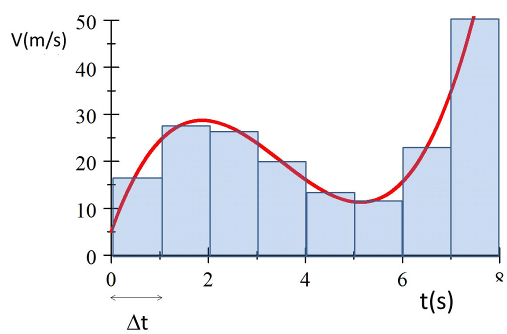
and with such a complicated motion we can see why our limiting process can turn a sum

$$\Delta x = \sum_{i=1}^{N} v\Delta t$$

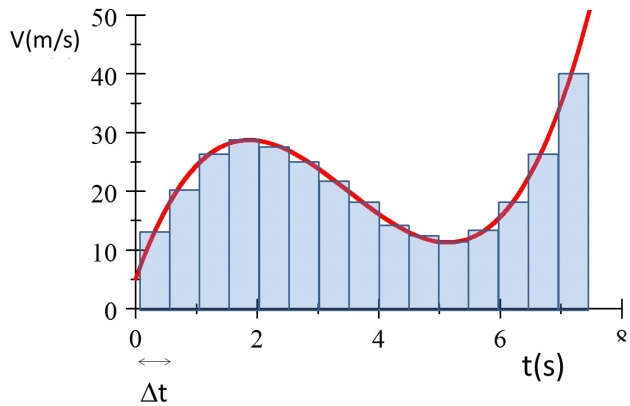
into an integral

$$\Delta x = \int_{t_i}^{t_f} v dt$$

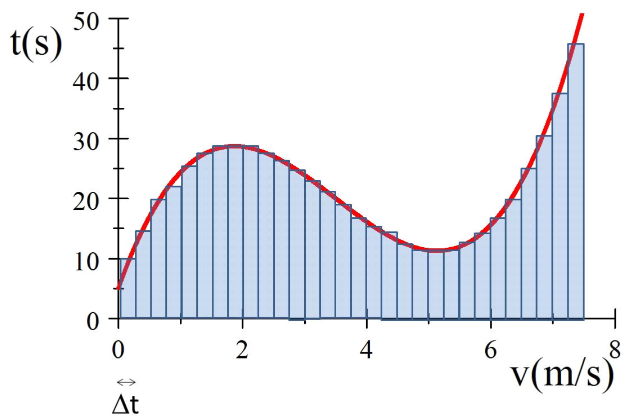
If Δt is large, our "areal under the curve" is only approximate because $v\Delta t$ is a box shape and although we could take many boxes and add them up (that is what the sum says to do) we miss pieces or get extra bits that are not under the curve



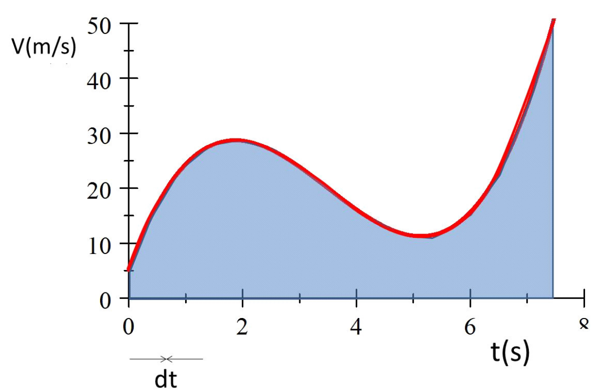
But if we let Δt get smaller our sum becomes a better approximation to the actual "area under the curve."



and each time we shrink Δt it gets better



until we finally get to an infinitesimal length for Δt, which we called dt.



and now the "area under the curve" is exact.

If I have a complicated velocity vs. time graph, then my object *must* be changing its speed or direction (or both). And we have a name for changing speed or direction. This is acceleration. Now that we have more powerful ways to think about changing velocity, it's time to reconsider acceleration. Let's take on a special case. Let's let the velocity change, but let's consider the case of constant acceleration. This is a reasonable first step in dealing with changing velocity.

Examples of integration

Now it's time to us our integral procedure on some example functions. Suppose we have the function

$$v(t) = \left(5\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2$$

What is the integral from $t_i = 0\,\mathrm{s}$ to $t_f = 3\,\mathrm{s}$?

Our procedure tells us to do the following:

$$\begin{aligned}
\int_{t_i}^{t_f} u dt &= \int_0^{3s} \left(5\frac{\mathrm{m}}{\mathrm{s}^3}\right) t^2 dt \\
&= \left.\frac{\left(5\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^3}{3}\right|_0^{3s} \\
&= \frac{5\frac{\mathrm{m}}{\mathrm{s}^3}\left(3\,\mathrm{s}\right)^3}{3} - \frac{5\frac{\mathrm{m}}{\mathrm{s}^3}\left(0\right)^3}{3} \\
&= 45.0\,\mathrm{m}
\end{aligned}$$

Let's try another. Suppose we have the function

$$v(t) = 5\frac{\mathrm{m}}{\mathrm{s}}$$

What is the integral from $t_i = 0\,\mathrm{s}$ to $t_f = 3\,\mathrm{s}$?

This function doesn't seem much like a function, but we remember that $1 = t^0$ so we could write our function as

$$v(t) = 5\frac{\mathrm{m}}{\mathrm{s}}t^0$$

Then we just use our integration procedure:

$$\begin{aligned}
\int_{t_i}^{t_f} u dt &= \int_0^{3s} \left(5\frac{\mathrm{m}}{\mathrm{s}}\right) t^0 dt \\
&= \left.\frac{\left(5\frac{\mathrm{m}}{\mathrm{s}}\right) t^1}{1}\right|_0^{3s} \\
&= \frac{5\frac{\mathrm{m}}{\mathrm{s}}\,(3\,\mathrm{s})^1}{1} - \frac{5\frac{\mathrm{m}}{\mathrm{s}}\,(0)^1}{1} \\
&= 15\,\mathrm{m}
\end{aligned}$$

We will have many more opportunities to practice this new math skill!

Constant Acceleration

Thinking about driving again, we must get the car going. When come to our destination, we must slow down. We need to change our velocity. We called a change in velocity *acceleration*. Recall that the average acceleration $\bar{a}$ during time interval Δt is the change in velocity Δv divided by Δt

Algebraically we may write

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t} = \frac{\overrightarrow{\mathbf{v}}_f - \overrightarrow{\mathbf{v}}_i}{t_f - t_i}$$

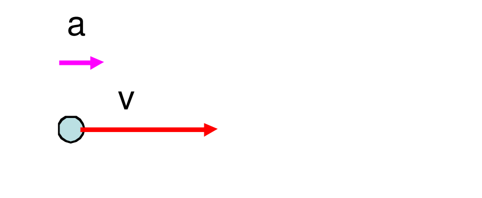
Since we are dealing with velocity, we would expect acceleration to be a vector quantity and we know that is true. In one dimension, we can indicate direction with a plus or minus sign. So we could write the equation above without the vector signs for one dimensional motion,

$$\mathbf{a}_{ave} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i}$$

but we should remember that acceleration is a vector. Just for review, we should recall what the direction of acceleration means.

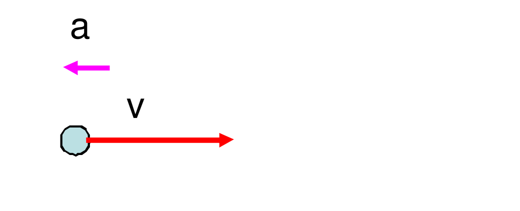
We have to consider the direction of *both* the acceleration and the velocity before we can determine the effect of acceleration on the motion of the object.

Let's start with a positive velocity and a positive acceleration.



In this case, we have a positive initial velocity, and we would expect the velocity to get larger, so $v_f - v_i$ is positive.

Suppose we have a positive velocity and a negative acceleration.



This means that $v_f - v_i$ is negative ($t_f - t_i$ is never negative). For this to be true v_f must be smaller than v_i. The object is slowing down!

If we use average velocity as an example, we can guess that if the acceleration is constant, then the acceleration is the slope of the line in a *velocity* vs. time graph.

There are many physical phenomena that can be represented as a system with constant acceleration. Neglecting air resistance, all bodies attracted by gravitation act under uniform acceleration (we will find out why later!).

We can build our list of basic problem types by adding the special case of constant acceleration. Let's do that now. We will add four new equations to our list, but I suggest you label these equations "constant acceleration" or something like that, since we are going to assume constant acceleration when we form the equations, they will *only* work for constant acceleration cases. We are also adding a new problem type: *constant acceleration*!

Lets start with our definition of average acceleration

$$a_{ave} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} \tag{5.2}$$

since we are limiting ourselves to one dimension, we will use a $+$ or a $-$ sign to indicate direction. Also, since acceleration is constant, we have

$$a_{ave} = a \tag{5.3}$$

so

$$a = \frac{v_f - v_i}{t_f - t_i} \tag{5.4}$$

Remember, this is for our special case! It is **not true** in general! This equation only works if the acceleration does not change. We defined before $\Delta t = t_f - t_i$ so let's use it here

$$a = \frac{v_f - v_i}{\Delta t} \tag{5.5}$$

Rearranging gives

$$v_f = v_i + a\Delta t \qquad \text{Constant } a \tag{5.6}$$

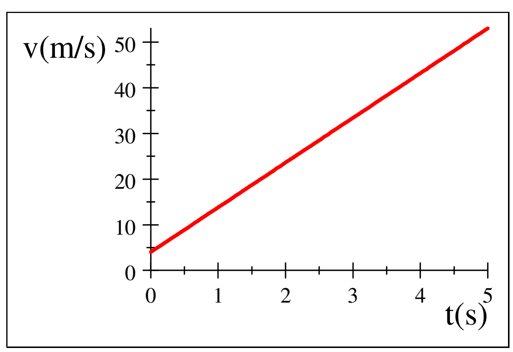
this is the first of our new set of basic equations.

Notice that it is in the form of

$$y = mx + b \tag{5.7}$$

$$v_f = at + v_i \tag{5.8}$$

For constant velocity problems, the velocity is a straight line with the acceleration as the slope (just as we would expect!) on a velocity vs. time graph. We saw this situation in an earlier example:



If we have constant acceleration we can write the average velocity as

$$\bar{v} = \frac{v_f + v_i}{2} \qquad \text{Constant } a \tag{5.9}$$

This is the second of our new set of equations.

Now, how far will a falling object go in time t? This is harder to find. Let's write our displacement as

$$\Delta x = x_f - x_i \tag{5.10}$$

Let's take the equation for average velocity

$$v_{ave} = \frac{x_f - x_i}{t_f - t_i} \tag{5.11}$$

and write it as

$$v_{ave} = \frac{x_f - x_i}{\Delta t} \tag{5.12}$$

then

$$x_f - x_i = v_{ave}\Delta t \tag{5.13}$$

Using our equation for average velocity under constant a,

$$x_f - x_i = \left(\frac{v_f + v_i}{2}\right)\Delta t \tag{5.14}$$

or, by rearranging,

$$x = x_i + \frac{1}{2}v_i\Delta t + \frac{1}{2}v_f\Delta t \qquad \text{Constant } a \tag{5.15}$$

We can add this equation to our list of basic equations for constant acceleration problems. We could write this more compactly as

$$\Delta x = \frac{1}{2}\left(v_f + v_i\right)\Delta t \tag{5.16}$$

But we usually see this written another way. Let's take equation

$$v_f = v_i + a\Delta t$$

and substitute it into equation .

$$\begin{aligned} x &= x_i + \frac{1}{2}v_i\Delta t + \frac{1}{2}\left(v_i + a\Delta t\right)\Delta t && (5.17) \\ &= x_i + \frac{1}{2}v_i\Delta t + \frac{1}{2}v_i\Delta t + \frac{1}{2}a\Delta t^2 && (5.18) \end{aligned}$$

$$x_f = x_i + v_i\Delta t + \frac{1}{2}at^2 \qquad \text{Constant } a \tag{5.19}$$

This one we definitely want in our new list of basic equations for constant acceleration problems. It tells us how far we get from our starting point as we move with constant acceleration.

There is one more, let's go back to equation

$$v_f = v_i + a\Delta t \tag{5.20}$$

and solve for Δt

$$\begin{aligned} v_f - v_i &= a\Delta t && (5.21) \\ \frac{v_f - v_i}{a} &= \Delta t && (5.22) \end{aligned}$$

now let's use this equation for Δt in equation

$$\begin{aligned} x_f - x_i &= \left(\frac{v_i + v_f}{2}\right)\Delta t && (5.23) \\ &= \left(\frac{v_i + v_f}{2}\right)\left(\frac{v_f - v_i}{a}\right) && (5.24) \\ &= \frac{1}{2}\left(v_f + v_i\right)\frac{1}{a}\left(v_f - v_i\right) && (5.25) \\ &= \frac{1}{2a}\left(v_f^2 - v_i^2\right) && (5.26) \end{aligned}$$

so

$$\Delta x = \frac{1}{2a}\left(v_f^2 - v_i^2\right) \tag{5.27}$$

or

$$2a\Delta x = -\left(v_f^2 - v_i^2\right) \tag{5.28}$$

Finally we can write this as

$$v_f^2 = v_i^2 + 2a\Delta x \qquad \text{Constant } a \tag{5.29}$$

This is the final equation of our new set. It is a good equation for problems where you are not given the time.

The Kinematic equations for constant acceleration

We have derived four main equations

$$v_f = v_i + a\Delta t \qquad \text{Constant } a \tag{5.30}$$

$$x_f = x_i + v_i t + \frac{1}{2}a\Delta t^2 \qquad \text{Constant } a \tag{5.31}$$

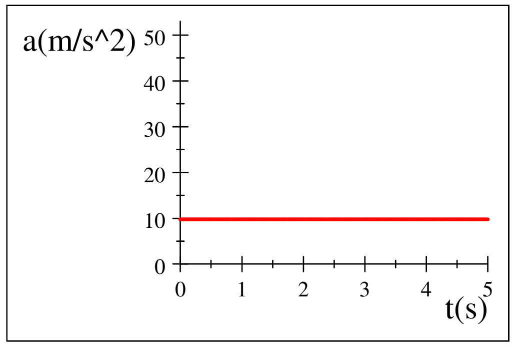
$$v_f^2 = v_i^2 + 2a\Delta x \qquad \text{Constant } a \tag{5.32}$$

$$\Delta x = \frac{1}{2}\left(v_f + v_i\right)\Delta t \qquad \text{Constant } a \tag{5.33}$$

The last one was used in deriving the second and third equation, so the four equations are not independent. The first three are the most useful. The first two are the most important for this chapter. The third is often convenient. The strategy for solving kinematic problems in this chapter should include (after restating the problem, drawing a diagram, and stating variable) selecting an equation from these three or four equations. This set of equations is so useful that it has a name. They are called the *kinematic equations*.

Graphs of kinematic equations

The graph of constant acceleration on and acceleration vs. time graph is not very exciting.



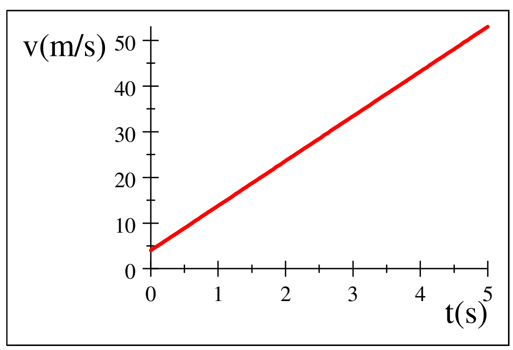
and knowing that

$$a = \frac{\Delta v}{\Delta t}$$

we can see that the acceleration will be the slope of the velocity vs. time graph. We can also see this from equation ()

$$v_f = v_i + a\Delta t$$

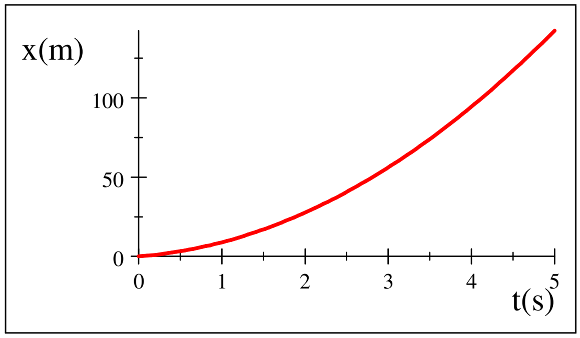
which is linear in t. The slope of a $v(t)$ vs. time graph will be the value from our a vs. time graph.



We know that the velocity is the slope of a position vs. time graph, and now our velocity is changing. But we can use one of our kinematic equations ()

$$x_f = x_i + v_i t + \frac{1}{2} a\Delta t^2$$

to realize that the position vs. time curve must be quadratic. It is a parabola!



Now we know how to draw figures for constant acceleration problems, and we have constant acceleration equations. Let's try an example!

You build a new electric car. It has an acceleration of $0.5\,\mathrm{m/\,s^2}$. You start on your way to class with an initial velocity of $v_i = 0\,\mathrm{m/\,s}$. After $\Delta t = 3\,\mathrm{s}$, how fast are you going?

We can use equation ()

$$v_f = v_i + a\Delta t$$

$$\begin{aligned} v_f &= 0 + \left(0.5\,\mathrm{m/\,s^2}\right)(3\,\mathrm{s}) \\ &= 1.5\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

If we define your house to be the $x_i = 0$ position, how far from home are you after $\Delta t = 3\,\mathrm{s}$?

Now we use equation ()

$$x_f = x_i + v_i t + \frac{1}{2}a\Delta t^2$$

$$\begin{aligned} x_f &= 0 + 0 + \frac{1}{2}\left(0.5\,\mathrm{m/\,s^2}\right)(3\,\mathrm{s})^2 \\ &= 2.25\,\mathrm{m} \end{aligned}$$

You probably know that the Earth pulls on us to keep us from flying away as it turns through the Solar system. This pull is called gravity. The pull causes a constant acceleration for things falling near the Earth's surface. And that is just the sort of situation our new equations can handle. We will take up this case in the next lecture.

6 Problem Solving in Physics

So far we have learned to draw motion diagrams, and we have learned that we can know a lot about the motion of an object by drawing a motion diagram. But we want to apply the power of mathematics to finding the motion of an object. It is time to get ready to do that.

We already know a few basic equations, like our equations for displacement and velocity, and we will learn many more. In what follows for this lecture, I will outline a proven problem solving process, then explain the parts of the process, then give an example of following that process.

Parts of a problem:

Here are the steps of our problem solving process. You received a copy of this process as part of the Syllabus package on the first day and it is posted on I-Learn under *Course Description*.

Restate the Problem

The first thing to do when working a homework problem (or a problem given to you in your job, or a problem to test with an experiment, or whatever problem you are assigned to solve), is to restate the problem. You don't get as many points if you solve a different problem than was asked! It is common, especially on tests, to misread the problem. So take some time and make sure you are answering what was asked. In industry, I used to email my boss a restatement of each assignment to make sure I understood what I had been assigned!

Identify the type of Problem

If you can look at the problem and see it as part of a class of problems, then you know

Problem Solving Process

Process Step	Purpose	Value if Present	Value if Absent
1. Label the problem with chapter and problem number	This is essential if I am to figure out what problem to grade	0	0 to -20
2. **Restate the problem in your own words.** One line may do! List any assumptions you are making.	Most major mistakes come from misinterpreting the problem. This step asks you to slow down and determine what the problem really is asking	1	-1
3. **Draw a picture**, label items, define coordinate systems, etc.	Many mistakes happen because we do not have a clear picture of the problem. This step may save hours of grief. Also, many physics problems will have different symbolic answers because of the freedom to choose coordinate systems, etc. Drawing a diagram gives the reader the ability to understand your vision of the problem.	2	-2
4. **Define variables** used, Identify known and unknown quantities	Choose reasonable names for physical quantities, and let me know what they are. Don't forget to include units.	2	-2
5. **List basic equations** that apply to the problem	This step gives you a firm starting place.	2	-2
6. **Solve the problem algebraically** starting from the basic equations.	This is the heart of the solution. The symbolic answer tells you the relationships between physical quantities.	10	-10
7. **Determine numerical answer**	The specific numerical answer is not the point of doing the problem in this class, but is a great indicator that you have succeeded in understanding the physics.	1	-1
8. **Check units.** If you have not done the algebra on the units earlier, do it here.	Many mistakes are evident in a units analysis. It is a good habit to always check units.	1	-1
9. **Determine if the numerical answer is reasonable.** Indicate if you are comfortable with the result, if you have little experience with the result and can't tell if it is reasonable, or if it is not reasonable, but you don't know why (or else you would fix it).	From your understanding of the physics, state whether the answer is reasonable. For example, if you are calculating the mass of a ping pong ball, and get an answer that is many times the mass of the earth, you should note that there may be a problem even if you do not know where you went wrong.	1	-1 to -25
Total Possible		**20**	**-25**

2.

which equations to try, and what techniques to attempt. So far all we have are displacement, velocity, and acceleration problems. But soon we will have many other types.

Suppose you are asked to find the displacement of an object that starts at position $x_i = 5\,\mathrm{m}$ and ends at position $x_f = 12\,\mathrm{m}$. The equation

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\Delta \mathbf{v}}}{\Delta t}$$

is a great resource if you are looking for acceleration, but not so great if you trying to find the displacement. Identifying that the problem asks for displacement helps you realize that the equation

$$\Delta x = x_f - x_i$$

might help.

Drawing the question

We have spent three lectures learning how to draw diagrams describing motion. We will spend more lectures learning how to draw diagrams for force problems, and equilibrium problems, and rotational problems. You would think it was an art class!

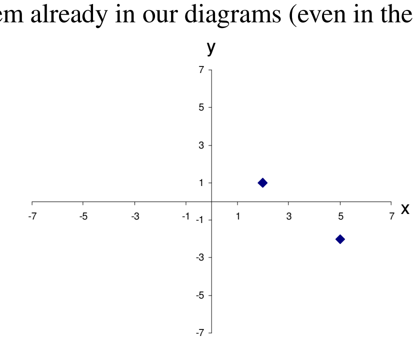
But seriously, learning to express the problem as a visualization is a large part of "doing physics," and the diagram is often the key to seeing how to solve the problem. It is tempting to skip this step. But you must convince yourself to learn how to make the diagrams and to use the diagrams in solving problems.

Coordinate Systems

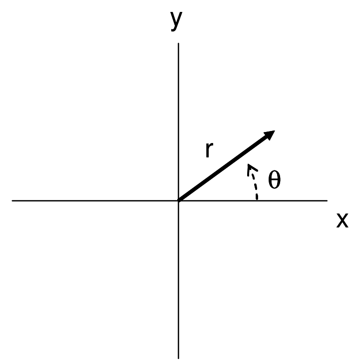
We need a context in which to describe motion graphically. Like a city map, there needs to be a way to tell where we are going and where we have come from. In a western city, we often find a city grid with addresses marked with a number of streets from a central location. Here in Rexburg we have such a system. We have addresses like 300N and 200E. We need such a system for our use.

Often we will use a Cartesian coordinate system (much like the Rexburg system, which is patterned after a Cartesian system).

We have used this system already in our diagrams (even in the context of city blocks).



We could also make a coordinate system by using the distance from the origin and an angle from the x-axis.



I'm sure you will recognize this as the polar coordinate system. And of course you recognize our position vectors as just position expressed in polar coordinates!

We could extend these to three dimensions (and we will later). Einstein used a very complicated curved three dimensional coordinate system to describe General Relativity. So what seems like a simple idea can become very complicated. Fortunately, we can usually use Cartesian coordinates for most of what we will do.

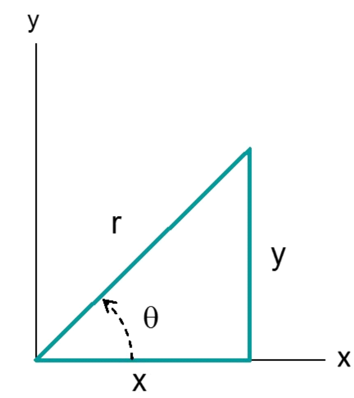
We will need to think about coordinates a bit. Is there really a zero point in the universe? If not, then are we always free to choose one?

For distances, we do not believe there is a ultimate zero point. When we get to other quantities (like temperature) we will see that sometimes there is a physical zero point.

Question 1.4
Question 1.5
Question 1.6
Question 1.7
Question 1.8
Question 1.9

We have already noticed that our position vector is really just a use of polar coordinates. And in remembering polar coordinates, you probably recall that there is trigonometry involved.

Given the triangle below, we recall that



$$\sin(\theta) = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{y}{r} \tag{6.1}$$

$$\cos(\theta) = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{x}{r} \tag{6.2}$$

$$\tan(\theta) = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{y}{x} \tag{6.3}$$

Suppose I know r and θ but not y, I can find y using

$$y = r\sin(\theta) \tag{6.4}$$

We can use cosine and tangent in a similar way.

Suppose we know r and y, but not θ. We can find this using

$$\theta = \arcsin\left(\frac{y}{r}\right) \tag{6.5}$$

which is often written as

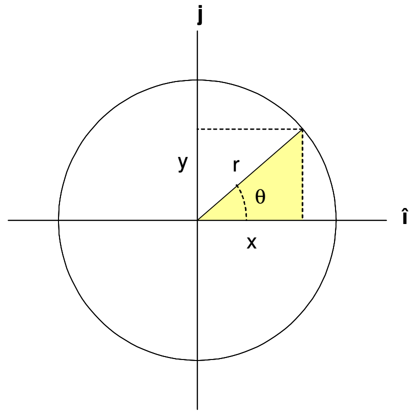
$$\theta = \sin^{-1}\left(\frac{y}{r}\right) \tag{6.6}$$

Also recall the Pythagorean theorem

$$r^2 = x^2 + y^2 \tag{6.7}$$

The combination of these ideas will be used over and over in our study of vectors. If you are a little rusty with trig, it is a good idea to look at the review in the back of our text book.

A nice way to remember the trig functions is to think of our triangle inscribed in a unit circle. The cosine of the angle gives the projection of r onto the x axis. Likewise, the sine functions the projection of r onto the y axis.



As an example of the use of trig functions, we can use them to convert from Cartesian coordinates (x, y) to polar coordinates (r, θ). We start with

$$r = \sqrt{x^2 + y^2} \tag{6.8}$$

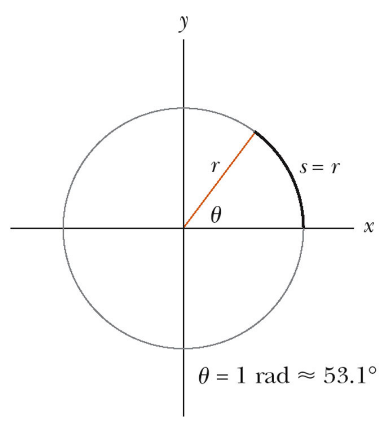
from the Pythagorean theorem,

Then we note that

$$\tan(\theta) = \frac{x}{y} \tag{6.9}$$

to yield

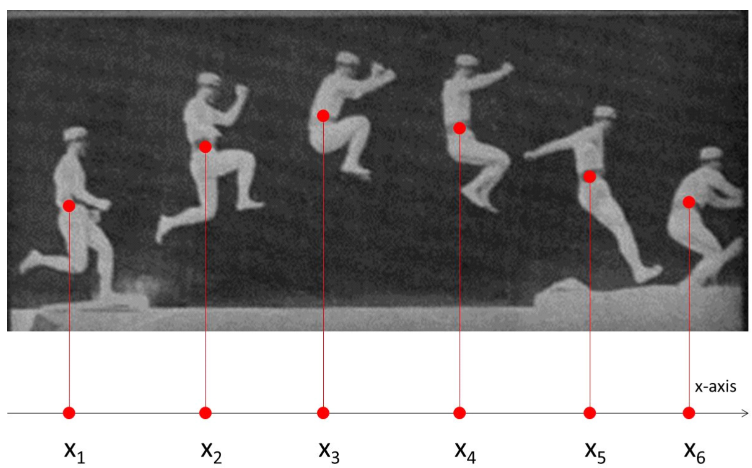
$$\theta = \tan^{-1}\left(\frac{x}{y}\right) \tag{6.10}$$



You probably remember that we divide circles into angles as shown in the figure above.

We often divide the circle into $360\,^{\circ}$, like 360 pieces of pizza. By adding up 360^{th}'s if a circle we can describe larger angles. This is one way to describe angles. If you took trigonometry, you remember that there are other ways to divide the circle. One that will be very important to us is the *radian*. It is just a slightly bigger pizza piece as a base unit (think of pizza pieces for large foot ball players, you want to start with larger basic units for them!). Your calculator will do trigonometry in degrees or radians, but you have to change the settings to tell the calculator which you want. We will deal with radians a great deal later, but for now you should find out how to set your calculator in degrees mode.

Defining Variables



Let's look at our jumping man again. It is a great help to realize what you know. Suppose that we know $x_i = 5\,\mathrm{m}$ and $x_f = 12\,\mathrm{m}$ for our man. And suppose we want to find the man's total displacement. It is much easier to see the known values if we call them out

$$\begin{aligned} x_i &= 5\,\mathrm{m} \\ x_f &= 12\,\mathrm{m} \end{aligned}$$

By placing these on your paper so you can see them, it is easier to find an answer. Seeing the positions like this it is obvious that all you need to do is subtract to get the answer. As the problems become more complicated, this will be an even more important step. It also lets the grader (or in your job, the reader of your report) know what the variable symbols mean. Not every field uses the same letters for the same quantities. And we reuse some letters! The letter T could be "tension" or "period of oscillation." By writing down what the letters mean, you avoid confusing yourself and others.

But what kinds of things are variables? Let's take some time to see what quantities we might define.

Objects

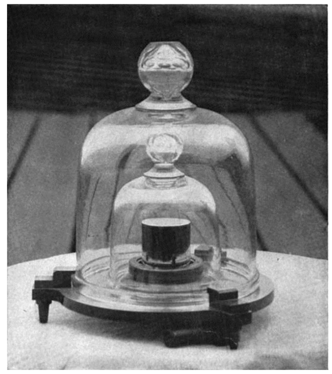
We have been talking about objects, like our jumping man, the bird, or a ball. But what is an object? What is the universe made of?

The startling answer is, we're not entirely sure! Oh, we know that the universe is made of stars and planets and galaxies and dust and many other things, but what are the things made of?

Answering that question is the job of particle physics, and the answer is still in the making. For our study of motion, we will assume there are fundamental things in the universe, and more complicated things are made of these fundamental things. Lets look at three quantities as our initial building blocks. *Mass, Length*, and *Time*.

Mass

When we think of an object, we usually think of something that has mass. Mass is an amount of matter. Exactly what matter is, still somewhat of a question (Job security for Physicists). Einstein equated mass and energy. The great experiments at the Conseil Européen pour la Recherche Nucléaire (CERN) are trying to understand exactly what mass is. You may have heard of the *Higgs boson*, a particle discovered at CERN that gives us a hint that our theory of what mass is might be right. But that is current on-going research. So for this class we shall take mass as just the amount of matter and trust our intuition on what that means. The standard unit of mass is the kilogram, abbreviated "kg." It is the mass of a standard piece of platinum alloy, again kept in France. Question 1.1.3



US National Institute of Standards standard kilogram.

Note that mass and weight are very different quantities. you can see this if we use a bathroom scale. On earth the scale gives a reading that is proportional to the amount of matter in our bodies, but if we took it on a space craft in orbit, it would not measure any weight at all. Yet the amount of matter in our bodies has not changed!

Question 1.1.4

Length

Perhaps we should really say "space" here. We need to have some idea of how far away things are or how long or tall things are. In this class our view of space will be that it is a container in which things happen. When you study Einstein's Relativity we will change that a little, but for now space is a container, and length is a measure of how far away in this container something is.

In ancient Egypt, the standard of measuring length was when Pharaoh took his ceremonial reed and measured the length of the foundation for the temple (a little like our standard kilogram for mass). This might sound strange, but in essence this is what we all did until 1960. Prior to this, a meter, our unit of length, was defined as one ten millionth of the distance from the North Pole to the equator. Since this was not a very practical day-to-day measuring device, a standard "reed" (this time made of platinum) was kept in France, and meter sticks were made to match this standard. There are terrible problems with this! Each stick of a different materials changes length with temperature! So in 1983 the meter was defined as the distance light traves in vacuum during a time interval of

$$\frac{1}{299792458}\,\mathrm{s}$$

The abbreviation for meter is "m."

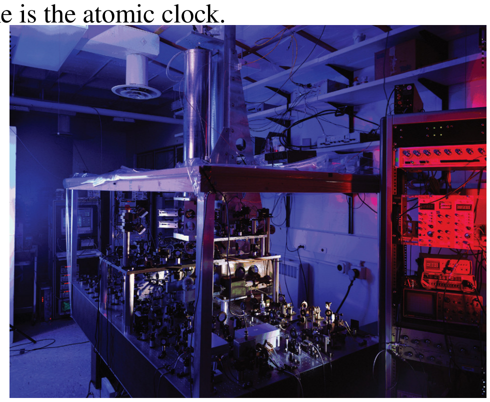
Time

You might object that time is not a thing. But we have already used time in our study of motion. It must be something! We should define it before we get to familiar with using time. But what is time? It turns out that time is hard to define. We usually use the idea that time is how long we wait.[2] That can be tricky to measure. Let's start with something simple. How much time will you spend in this class today? That is a time we can wait, about an hour. But it is harder to answer questions like "how long does it take for light to travel a foot." The answer is about a nanosecond. We cannot perceive of times this small. Likewise, we cannot wait for a million years (well, we could, but our vantage point might change after the first 70 years or so).

To measure time we use events that are *periodic*, that is, they occur at regular intervals. An early example is the pendulum of a clock. From a fundamental periodic phenomena, we can build up larger or smaller units of time.

The current unit is the *second*, abbreviated " s," which is given as 91926317000 times the period of oscillation of radiation from the cesium atom. Fortunately we can just use a clock or watch to measure seconds.

The standard for time is the atomic clock.



Atomic Clock

Derived quantities

So we have objects made of mass and space (length) and time to use in describing their motion.

Question 1.1.1

When we combine quantities we derive new quantities that are useful from the basic

[2] Feynman, Richard, Robert Leighton, Matthew Sands, *The Feynman Lectures on Physics*, Vol. I, Addison-Sesley, Reading Massachusetts, 1963

length, time, mass set. For example, speed is a combination of length and time.

$$v = \frac{\Delta x}{\Delta t} \tag{6.11}$$

quantities like acceleration, force, momentum, etc. are derived quantities.

Dimensional Analysis

Analysis of the units in a measurement can be very useful. For example, if we take our first equation

$$v = \frac{\Delta x}{\Delta t} \tag{6.12}$$

and look at the units, we find that x is a length in, say, meters. We find that t is a time in, say, seconds. Then when we calculate v we should have units of $\mathrm{m/s}$. If, instead, we have m^3 at the end of our calculation, something must have gone wrong! Sometimes it is useful to use generic units for our analysis. That is, any length is given a unit L and any time is given a unit T. So our equation gives

$$\frac{\Delta x}{\Delta t} \Rightarrow \frac{L}{T}$$

To see this lets take and acceleration . It is given by

$$a = \frac{\Delta v}{\Delta t} \Rightarrow \frac{\frac{L}{T}}{T} = \frac{L}{T^2}$$

so from dimensional analysis, we expect that acceleration would be something like

$$a = c\frac{x}{t^2}$$

and we could deduce that

$$x = \frac{1}{c}at^2$$

note that I included a constant, c. Dimensional analysis cannot tell you the constants in an equation. We will see later that in this case $c = 2$ so

$$x = \frac{1}{2}at^2$$

We will find out that our dimensional analysis did not hit too far off the mark. This is part of the equation for position for an accelerating object.

Units

No value in physics is useful without a unit. For example, if I tell you to jump from a height of 100, it makes a difference whether it is $100\,\mathrm{cm}$ or $100\,\mathrm{m}$! Units tell us what standard was used to make the measurement so all who see the result can correctly interpret what it means.

System of Units

You will notice that we have only given metric units. We will use the *Système International* or *SI* units. There are, of course, other systems of units. We will try to ignore them in this class. Occasionally we may use feet for length and slugs (yes, slugs) for mass. we will usually use the following SI units for our basic quantities.

Quantity	Unit	Symbol
Mass	Kilogram	kg
Length	meter	m
Time	second	s

The SI system makes use of prefixes to modify the basic unit, like *centi*meter to mean $1/100$ of a meter. You should be familiar with the following prefixes.

Prefix	Symbol	Power		Prefix	Symbol	Power
nano-	n	10^{-9}		giga-	G	10^{9}
micro-	μ	10^{-6}		mega-	m	10^{6}
mili-	m	10^{-3}		kilo-	k	10^{3}
centi-	c	10^{-2}		deka-	da	10^{1}
deci-	d	10^{-1}				

We can already see that our unit for mass, the kilogram, must be 1000 grams. A centimeter must be $1/100^{th}$ of a meter. We obviously will need to be able to convert from centimeters to meters from time to time. We should be able to convert from any prefixed unit to any other prefixed unit. We nee a strategy to do this.

Unit Conversions

Let's do a unit conversion that most of you do in your head. Let's convert hours to seconds. We know that

$$1\,\mathrm{h} = 60\,\mathrm{min}$$

and we know that

$$1\,\mathrm{min} = 60\,\mathrm{s}$$

Suppose we have 5 hours. How many seconds is this?

Most of us would say multiply by 3600, and that is right, but let's do it one step at a time so you can see the process. I want to multiply 5 by something, but I can't change the duration. At the end of our calculation, it still has to be a wait of $5\,\mathrm{h}$ even though we now give the value in seconds. For a person waiting 5 hours and a second one waiting $18\,000\,\mathrm{s}$ they must feel the same amount of time. So we need to adjust our 5 hours by something that does not change the wait.

I think you will agree that if I multiply by 1 nothing changes

$$5 \times 1 = 5$$

We can do this with units

$$5\,\mathrm{h} \times 1 = 5\,\mathrm{h}$$

now let's take our equation relating hours to minutes.

$$1\,\mathrm{h} = 60\,\mathrm{min}$$

and let me divide by $1\,\mathrm{h}$

$$\begin{aligned} \frac{1\,\mathrm{h}}{1\,\mathrm{h}} &= \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \\ 1 &= \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \end{aligned}$$

I can multiply by the right hand side of this equation and all I am doing is multiplying by 1!

$$5\,\mathrm{h} \times \frac{60\,\mathrm{min}}{1\,\mathrm{h}} = 5\,\mathrm{h}$$

This still must be true, but let's do the math

$$5\,\mathrm{h} \times \frac{60\,\mathrm{min}}{1\,\mathrm{h}} = 300\,\mathrm{min}$$

this is how many minutes are in 5 hours. We can play the same trick with minutes to seconds

$$\begin{aligned} \frac{1\,\mathrm{min}}{1\,\mathrm{min}} &= \frac{60}{1\,\mathrm{min}}\,\mathrm{s} \\ 1 &= \frac{60\,\mathrm{s}}{1\,\mathrm{min}} \end{aligned}$$

so we can take our $300\,\mathrm{min}$ and find out how many seconds we have!

$$300\,\mathrm{min} \times \frac{60\,\mathrm{s}}{1\,\mathrm{min}} = 18000.0\,\mathrm{s}$$

Now we could do this all in one equation, using our strange way of writing 1 that converts from hours to minutes and our strange way of writing 1 that converts from minutes to seconds.

$$\begin{aligned} 5\,\mathrm{h} \times 1 \times 1 &= 5\,\mathrm{h} \\ 5\,\mathrm{h} \times \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \frac{60\,\mathrm{s}}{1\,\mathrm{min}} &= 18000.0\,\mathrm{s} \end{aligned}$$

Notice that the units cancel like variables in algebra!

We will treat units like algebraic quantities that can be canceled. Let's do another example. We want to convert $10\,\mathrm{mi}$ (ten miles) to kilometers. We know that

$$1\,\mathrm{mi} = 1609\,\mathrm{m} \tag{6.13}$$

and we know that

$$1\,\mathrm{km} = 1000\,\mathrm{m} \tag{6.14}$$

Start with conversion from miles to meters. We recognize that with a small use of algebra

$$1 = \frac{1609\,\mathrm{m}}{1\,\mathrm{mi}} \tag{6.15}$$

then we can write

$$10\,\mathrm{mi} = 10\,\mathrm{mi}\frac{1609\,\mathrm{m}}{1\,\mathrm{mi}} = 16\,090\,\mathrm{m} \tag{6.16}$$

Then we also recognize that

$$1 = \frac{1000\,\mathrm{m}}{1\,\mathrm{km}} \tag{6.17}$$

or

$$1 = \frac{1\,\mathrm{km}}{1000\,\mathrm{m}} \tag{6.18}$$

then

$$16\,090\,\mathrm{m} = 16\,090\,\mathrm{m}\frac{1\,\mathrm{km}}{1000\,\mathrm{m}} = 16090.0\,\mathrm{m} \tag{6.19}$$

so

$$10\,\mathrm{mi} = 16\,\mathrm{km} \tag{6.20}$$

We could do this all in one large, chained, conversion

$$10\,\mathrm{mi} = 10\,\mathrm{mi}\frac{1609\,\mathrm{m}}{1\,\mathrm{mi}}\frac{1\,\mathrm{km}}{1000\,\mathrm{m}} = 16\,\mathrm{km} \tag{6.21}$$

If you think about it, to convert units, we have multiplied by 1 several times. So as you multiply to convert units, make sure your factors multiply are equal to 1.

Uncertainty in measurements

In science, we must face the fact that no measurement is completely accurate. The reasons for uncertainty are limitations in our human sensory system or sensing apparatus. For example, if I measure a square of metal with a ruler. I am likely not able to tell the length to better than a tenth of a centimeter ($1\,\mathrm{mm}$). This is because of inaccuracies in the ruler and in my own ability to see the ruler clearly and consistently. So suppose I have a measurement of $16.3\,\mathrm{cm}$. I can really only tell you that the measurement is between $16.4\,\mathrm{cm}$ and $16.2\,\mathrm{cm}$. we could write this as $16.3 \pm 0.1\,\mathrm{cm}$. We will study uncertainty in measurements in some detail in PH150. But for PH121 we will need some provisional rules that let us make a guess on now good our answers are.

Significant figures

Scientists have devised a clever way to include the level of uncertainty in the statement of the measurement result. This is referred to as *significant figures* and it basically means to keep only the digits in a number that contain well known information. In the above example, we would say that in $16.3\,\mathrm{cm}$ that the 3 is the least significant digit. Now suppose we use the same ruler to measure the same object, but I tell you that the measurement is $16.3259357\,\mathrm{cm}$. If we know the measurement is only good to $\pm 0.1\,\mathrm{cm}$, what can we say about the digits 259357? We can say they are worthless! They are nonsense, so we cleverly leave them off! There are a series of rules to tell us which digits are significant. It is important to realize that zeros that just mark where the decimal place goes are not significant (e.g. in $0.00163\,\mathrm{cm}$ the three 0's are not significant, but in $1.400\,\mathrm{cm}$ the digits mean that the measurement is known to $\pm 0.001\,\mathrm{cm}$).

We usually express numbers in scientific notation.

Propagation of uncertainty

Suppose we take two measurements, like measuring the sides of a rectangle.

$$\begin{aligned} l &= (2.3 \pm 0.1)\ \mathrm{cm} \\ w &= (4.5 \pm 0.1)\ \mathrm{cm} \end{aligned}$$

and we wish to find the area

$$A = l \times w$$

$$\begin{aligned} A &= 2.3\,\mathrm{cm} \times 4.5\,\mathrm{cm} \\ &= 10.35\,\mathrm{cm}^2 \end{aligned}$$

But we were a little uncertain about the length and width, wouldn't we also be uncertain about the area that we made from the uncertain length and the width? Of course there is some uncertainty in the area. Let's see how we could deal with this.

The length could have been as much as

$$l = 2.3\,\mathrm{cm} + 0.1\,\mathrm{cm} = 2.4\,\mathrm{cm}$$

and the width could be as much as

$$w = (4.5 + 0.1)\ \mathrm{cm} = 4.6\,\mathrm{cm}$$

So the area could be as much as

$$\begin{aligned} A_{+} &= 2.4\,\mathrm{cm} \times 4.6\,\mathrm{cm} \\ &= 11.04\,\mathrm{cm}^2 \end{aligned}$$

But the length could be as little as

$$l = 2.3\,\mathrm{cm} - 0.1\,\mathrm{cm} = 2.2\,\mathrm{cm}$$

and the width could be as little as

$$w = (4.5 - 0.1)\ \mathrm{cm} = 4.4\,\mathrm{cm}$$

So the area could be as little as

$$\begin{aligned} A_{-} &= 2.2\,\mathrm{cm} \times 4.4\,\mathrm{cm} \\ &= 9.\,68\,\mathrm{cm}^2 \end{aligned}$$

We can see that these differ by about $\pm 1\,\mathrm{cm}^2$ total.

$$A_{+} - A_{-} = 11.\,04\,\mathrm{cm}^2 - 9.\,68\,\mathrm{cm}^2 = 1.\,36\,\mathrm{cm}^2$$

Thus the tenths and hundredths places in our calculated area cannot be very certain. We drop these and write

$$A = 10 \pm 1\,\mathrm{cm}$$

Notice that our length and width had two digits,

$$\begin{aligned} l &= (2.3 \pm 0.1)\ \mathrm{cm} \\ w &= (4.5 \pm 0.1)\ \mathrm{cm} \end{aligned}$$

with the uncertainty in the second digit, and notice that our answer for the area has two digits, and the uncertainty is in the second digit!

In general:

> *In multiplying or dividing two quantities, the number of significant figures in the product or quotient is the same as the number of significant figures in the least accurate of the factors being combined.*

In our example, l and w both have two significant figures, so the result should be limited to two significant figures

For addition and subtraction the rules is:

> *The number of decimal places in the result should equal the smallest number of decimal places in any term in the sum or difference*.
> These two rules will help us determine how many digits to keep for most problems. You may know that there are several other rules, and thought we wont derive them like we did for multiplication, we will use them in our

problems. Here they are in a table

Significant Figure Rules
1. Non-zero digits are always significant
2. Embedded (i.e. captive) zero-digits are always significant
3. For the $number$ zero only the zero-digits after the decimal are significant
4. Leading zero-digits are not significant
5. Trailing zero digits:
If the number has a decimal, the trailing zero digits are significant
If the number does not have a decimal, the trailing zero digits are not significant
6. The final result of multiplication or division operation should have the same number of significant digits as the measured quantity with the least number of significant figures used in the calculation.
7. The final result of an addition or subtraction operation should have the same number digits to the right of the decimal as the measured quantity with the least number of decimal used in the calculation.
8. For a mixture of operations, work from left to right, do mathematical hierarchy of operations ($\times$ or $\div$, then $+$ or $-$).

Basic Equations

Equations are relationships in physics. The equation

$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta \mathbf{x}}}{\Delta t}$$

tells us how the displacement and duration combine to form velocity. These equations are our way of expressing motion. They are the tools in our toolbox for solving problems. Once you have identified the type of problem you have, you can quickly write down a list of equations (tools) that you could use to solve that problem. You might write down more equations than you end up using for a particular problem. That's OK. You don't empty your tool box of all tools but the ones you think you might use when you start a fix-up job in your house! You should not do so when starting a problem. List your equations, then choose the ones that seem to work given your known values in your list of variables.

Solving with symbols

For years now, you have worked with numbers and answers that are numbers. And that is what the teacher was looking for. But in physics the equation is the important thing. It tells you how things relate to each other. Let's try an example:

`The initial speed of an object is` $3\,\mathrm{m/s}$ `and it's acceleration is` $2\,\mathrm{m/s^2}$ `in the same direction as the velocity. Find the speed half a second after the experiment start.`

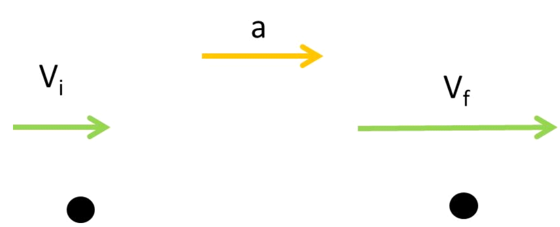
We want to start by restating the problem:

`Find the final speed knowing` a `and` v_i

Next identify the type of problem. I think it is an acceleration problem:

`PT acceleration`

Next we want to draw the picture



Our variables list is next:

`VAR`

$$\begin{aligned} v_i &= 3\frac{\mathrm{m}}{\mathrm{s}} \\ a &= 2\frac{\mathrm{m}}{\mathrm{s}^2} \\ \Delta t &= 0.5\,\mathrm{s} \end{aligned}$$

and now basic equations:

`BE:`

$$\underline{\vec{\mathbf{a}}} = \frac{\overrightarrow{\Delta \mathbf{v}}}{\underline{\Delta t}}$$

$$\overrightarrow{\Delta\mathbf{v}} = \overrightarrow{\mathbf{v}}_f - \underline{\overrightarrow{\mathbf{v}}_i}$$
$$\overrightarrow{\mathbf{v}} = \frac{\overrightarrow{\Delta\mathbf{x}}}{\Delta t}$$
$$\overrightarrow{\Delta\mathbf{x}} = \overrightarrow{\mathbf{x}}_f - \overrightarrow{\mathbf{x}}_i$$

Note that I underlined the known values from my list variables. The first two equations have v_f in them and they contain my known values, so it looks like they are the ones to use in my solution.

Solve Algebraically

Now we try to solve for the speed, but we do so symbolically. We already believe that the first two equations in our basic equation list will be helpful, so let's start with

$$\underline{\overrightarrow{\mathbf{a}}} = \frac{\overrightarrow{\Delta\mathbf{v}}}{\underline{\Delta t}}$$

and put in

$$\overrightarrow{\Delta\mathbf{v}} = \overrightarrow{\mathbf{v}}_f - \underline{\overrightarrow{\mathbf{v}}_i}$$

$$\underline{\overrightarrow{\mathbf{a}}} = \frac{\overrightarrow{\mathbf{v}}_f - \underline{\overrightarrow{\mathbf{v}}_i}}{\underline{\Delta t}}$$

At this point I recognize that I can solve for v_f and everything else is known. I could plug things in my calculator and have it solve for v_f using numbers, but we won't do that! We will continue with algebra

$$\overrightarrow{\mathbf{a}}\Delta t = \overrightarrow{\mathbf{v}}_f - \overrightarrow{\mathbf{v}}_i$$

and

$$\overrightarrow{\mathbf{a}}\Delta t + \overrightarrow{\mathbf{v}}_i = \overrightarrow{\mathbf{v}}_f$$

or

$$\overrightarrow{\mathbf{v}}_f = \overrightarrow{\mathbf{v}}_i + \overrightarrow{\mathbf{a}}\Delta t$$

Since $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{v}}_i$ are in the same direction, their magnitudes just add so

$$v_f = v_i + a\Delta t$$

This is the symbolic answer. It has the thing I want, v_f, an equals sign, and then symbols for what v_f is equal to.

Numeric answer

The numeric answer is easy. Just plug in numbers to your symbolic answer

$$v_f = v_i + a\Delta t$$

$$\begin{aligned} v_f &= 3\frac{\mathrm{m}}{\mathrm{s}} + \left(2\frac{\mathrm{m}}{\mathrm{s}^2}\right)(0.5\,\mathrm{s}) \\ &= 4.0\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Reasonableness check

If we had gotten $400000000000\,\mathrm{m}/\,\mathrm{s}$ in our example, we would know something went wrong. Nothing can go this fast! It is good to check your answer and see if it seems to make sense. But how do we do that? One way is to do a quick estimate.

Estimates

Question 1.3

There are times when we simply do not have all the information we need, but we need a number for something anyway. There are also times when we wish to make a quick calculation (like checking on the reasonableness of a calculation). In such cases, we estimate. Many people in science are somewhat uncomfortable with estimates, because they are not "correct" (In business and politics, people may be a little too comfortable with estimates).

Let's do a few examples together.

Example 1: How many sheets of paper fit between the Earth and Moon?

To do this calculation, we need to know how far away the Moon is and how thick a piece of paper is.

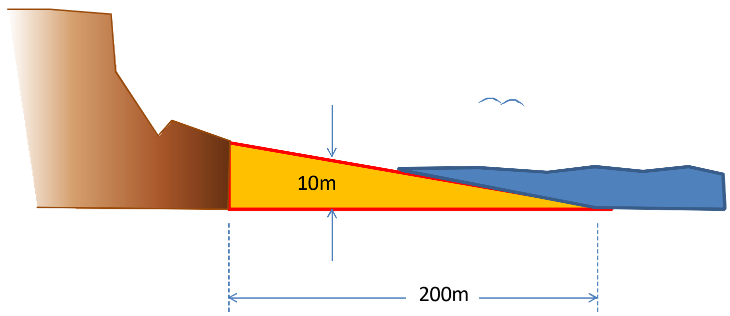
$$D_{EM} = 4000000\,\mathrm{km} \tag{6.22}$$

$$\begin{aligned} t &= \frac{3\,\mathrm{mm}}{28} = 1.071\,4 \times 10^{-4}\,\mathrm{m} \quad (6.23) \\ &\approx 1 \times 10^{-4}\,\mathrm{m} \end{aligned}$$

so the number of pieces of paper would be

$$\begin{aligned} N &= \frac{D_{EM}}{t} = \frac{4000000\,\mathrm{km}}{1 \times 10^{-4}\,\mathrm{m}}\frac{1000\,\mathrm{m}}{\mathrm{km}} \quad (6.24) \\ &= 40\,000\,000\,000\,000 \\ &= 4.0 \times 10^{13} \\ &\approx 1 \times 10^{13} \quad (6.25) \end{aligned}$$

Is this reasonable? Well look at Example 1.7 in the book and see what you think (and



3.

explain why our answers are different).

Example 2 How much sand is in the world's beaches?

We start by looking for a fundamental element of a beach, say, a grain of sand. We can calculate the total volume of the beaches, and divide by the volume of a grain of sand. This will tell us how many grains of sand there are in the world's beaches. If we can get an estimate of the mass of a grain of sand, then we can answer how much sand is in the world's beaches.

Lets estimate the grain of sand to have a mass of

$$m = 0.005\,\mathrm{kg} \tag{6.26}$$

Let's guess a volume of

$$V_s = 1.0 \times 10^{-15}\,\mathrm{m}^3 \tag{6.27}$$

We can estimate the length of the world's coastline to be

$$L = 40000000000\,\mathrm{m} \tag{6.28}$$

we need the volume of the coast line, lets say the beach is

$$w = 200\,\mathrm{m} \tag{6.29}$$

wide and

$$d = 10\,\mathrm{m} \tag{6.30}$$

deep.

Then the volume of the world's beaches would be

$$V = Lwd = (40000000000\,\mathrm{m})\,(200\,\mathrm{m})\,(10\,\mathrm{m}) \tag{6.31}$$

$$= 8.0 \times 10^{13}\,\mathrm{m}^3 \tag{6.32}$$

and the number of grains of sand would be

$$N = \frac{V}{V_s} = \frac{Lwd}{V_s} = 8.0 \times 10^{28} \tag{6.33}$$

which gives us

$$M_{beach} = Nm = \left(8.0 \times 10^{28}\right)(0.005\,\mathrm{kg}) \tag{6.34}$$

$$= 4.0 \times 10^{26}\,\mathrm{kg} \tag{6.35}$$

Mass of the Earth

$$M_E = 5.98 \times 10^{24}\,\mathrm{kg} \tag{6.36}$$

where did we go wrong?

Consider silicon oxide. It has a density of

$$\rho = 2200\frac{\mathrm{kg}}{\mathrm{m}^3} \tag{6.37}$$

If our estimate of

$$m = 0.005\,\mathrm{kg} \tag{6.38}$$

is good, then we should have used a volume of

$$V = \frac{m}{\rho} = \frac{0.005\,\mathrm{kg}}{2200\frac{\mathrm{kg}}{\mathrm{m}^3}} \tag{6.39}$$

$$= 2.2727 \times 10^{-6}\,\mathrm{m}^3 \tag{6.40}$$

So one problem is that our estimate of the volume of a grain of sand is very bad.

In general, you can be creative in making estimates, but you do have to be careful.

Units Check

We already have discussed units. But it is important to check your units in your final answer. In our case, the final units must be a length unit divided by a time unit. For speed this must be the case. If we had gotten a length unit divided by a time squared, then we would know something went wrong in our algebra. If the units don't work the answer is wrong. So especially on a test (or in your real job) checking units is important!

Problem Solving Process

I have assembled all of the problem solving pieces that we have studied into a process that leads us to our solution. Here is the process we will use in a table:

Problem Solving Process

Process Step	Purpose	Value if Present	Value if Absent
1. Label the problem with chapter and problem number	This is essential if I am to figure out what problem to grade	0	0 to -20
2. **Restate the problem in your own words.** One line may do! List any assumptions you are making.	Most major mistakes come from misinterpreting the problem. This step asks you to slow down and determine what the problem really is asking	1	-1
3. **Draw a picture**, label items, define coordinate systems, etc.	Many mistakes happen because we do not have a clear picture of the problem. This step may save hours of grief. Also, many physics problems will have different symbolic answers because of the freedom to choose coordinate systems, etc. Drawing a diagram gives the reader the ability to understand your vision of the problem.	2	-2
4. **Define variables** used, Identify known and unknown quantities	Choose reasonable names for physical quantities, and let me know what they are. Don't forget to include units.	2	-2
5. **List basic equations** that apply to the problem	This step gives you a firm starting place.	2	-2
6. **Solve the problem algebraically** starting from the basic equations.	This is the heart of the solution. The symbolic answer tells you the relationships between physical quantities.	10	-10
7. **Determine numerical answer**	The specific numerical answer is not the point of doing the problem in this class, but is a great indicator that you have succeeded in understanding the physics.	1	-1
8. **Check units.** If you have not done the algebra on the units earlier, do it here.	Many mistakes are evident in a units analysis. It is a good habit to always check units.	1	-1
9. **Determine if the numerical answer is reasonable.** Indicate if you are comfortable with the result, if you have little experience with the result and can't tell if it is reasonable, or if it is not reasonable, but you don't know why (or else you would fix it).	From your understanding of the physics, state whether the answer is reasonable. For example, if you are calculating the mass of a ping pong ball, and get an answer that is many times the mass of the earth, you should note that there may be a problem even if you do not know where you went wrong.	1	-1 to -25
Total Possible		**20**	**-25**

Notice that I have assigned point values to each part. So if you leave out a part you will know how many points you will miss. Also notice that there are a lot of points for the symbolic answer and only a one for the numeric answer. Physics is about relationships that describe motion, the numbers just aren't that important. So it is not a winning strategy to only give me the numeric answer for a problem. you only get one out of twenty five points!

Also notice that if I ask for the mass of a ping pong ball, and you give me an answer that is three times the mass of Jupiter, I can take off more than one point for the very wrong numeric answer! Since you will be doing a reasonableness check, you won't have this problem. But what if you don't know if an answer is reasonable? Then say you don't know! You will act differently as a physicist, engineer, doctor, etc. if you admit in your calculations that you are not certain of the result. And this is very valuable! It can save your job! So if you are not sure, say so.

We will use this process for the rest of the semester, and this or a similar process for PH123, and PH220 (or PH223) and if you are a physicist for the rest of your career. This is also the process I used in engineering in industry. So it is worth practicing in our problems. It is also how I will grade the tests!

7 Motion near the Earth's Surface

You may have noticed that falling things speed up as they fall. You may even have noticed that heavy things like rocks tend to fall with about the same motion. It would be convenient if falling objects experienced constant acceleration. Then we could use our equation set for falling object problems.

At the end of the 16^{th} century Galileo Galilei tried an experiment to see how falling things fell. The legend says that Galileo dropped two balls with different masses off of the leaning tower of Pisa. Galileo predicted that the motion of the two balls would be the same. Galileo's experiment was successful in that he was able to show his model for motion was more correct than Aristotle's. But it was only an approximation to motion with constant acceleration. The reason for this is that on the Earth there is an atmosphere, and the air get's in the way. You may have heard of "air resistance," and this resistance due to the air getting in the way slows the fall of the balls, but not by much. By using smooth round balls, the air resistance was limited, so Galileo did not notice the problem.

But the experiment was repeated a few centuries later–on the Moon–where there is no air. In the figure below, you can see the Apollo 15 astronauts dropping two objects, a hammer and a feather. The feather would have been strongly effected by air as it fell, but there is no air on the moon, so both the hammer and the feather experienced constant acceleration.

Applo 15 test of Galeleo's experiment on the moon. For a video, go to
http://nssdc.gsfc.nasa.gov/planetary/lunar/apollo_15_feather_drop.html

We will give a name to such a situation, where objects fall with nothing, not even air, to get in their way. We will also use the same name for situations where the motion is so close to having no impediments that we would not know the difference, like Galileo's experiment. We will call such a motion *free fall.*

Free Fall

If we don't go too far above the surface, and if we pick an object for which we can neglect air resistance (like a smooth ball, and not like a feather), then we can use our constant acceleration equations for the falling object! Under these conditions, falling objects act like they are under the influence of constant acceleration.

It is a little counter intuitive, but the mass of the object does not matter in this type of problem! Think of dropping a large rock and a small rock off a bridge. They seem to fall at the same rate even though one rock is much less massive than the other. All objects that are truly in free fall have the same acceleration. We will call this acceleration *free fall acceleration.*

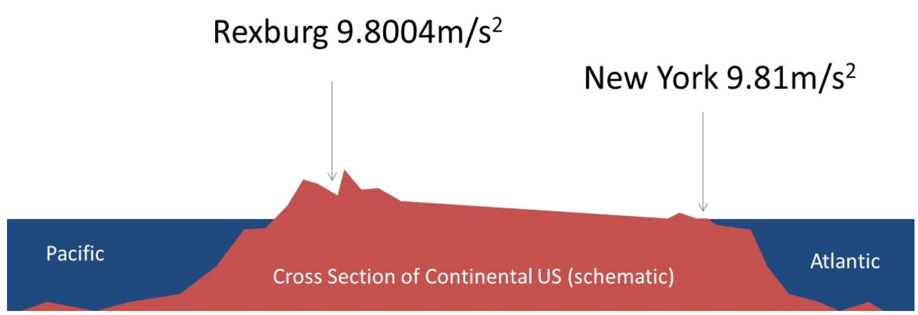
For the case of free fall near the surface of the Earth, we even give free fall acceleration a specific symbol, the letter "g." The value for this free fall acceleration near the Earths surface is about

$$g = 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \tag{7.1}$$

but for rough estimations

$$g \approx 10\frac{\mathrm{m}}{\mathrm{s}^2} \tag{7.2}$$

Because g really varies with height, it is closer to $9.81\,\mathrm{m/\,s^2}$ in New York and is about $9.8004\,\mathrm{m/\,s^2}$ in Rexburg. For our class $9.8\,\mathrm{m/\,s^2}$ is close enough most of the time.



Tradition guides us to choose an x-axis parallel to the ground. This leaves a chose between y and z for the axis perpendicular to the ground. Tradition again tells us to choose y. So heights are measured on the y axis[3].

Now you can define the axis any way you want, but if you do something different, you should warn people who might read your work (like our grader). With this choice of axis, "down" is in a negative y-direction. So our free fall acceleration near the Earth's surface is

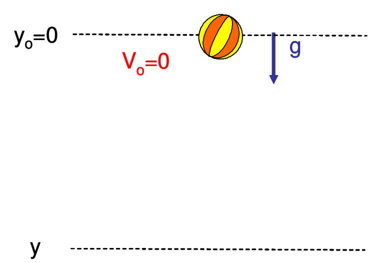
$$\overrightarrow{\mathbf{a}}_{ff} = -g = -\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right) \tag{7.3}$$

[3] This will change later in our course and in higher level physics classes.

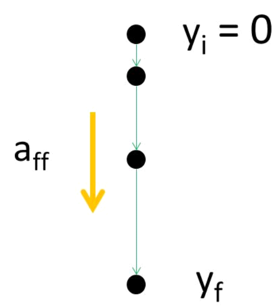
where g is the magnitude of the vector, and the minus sign is telling us that a_{ff} points down

Let's try a problem

An object is dropped from rest near the Earth's surface and we can say that free fall conditions apply. Determine the position of the object after $4.00\,\mathrm{s}$.



We can choose $y_i = 0.00$ as the point where the ball starts.



We could choose anywhere for $y = 0.00$, our origin, and the math would work out just fine. But for this problem I choose $y_i = 0.00$. We recognize that $v_i = 0.00$ (that is what is meant by the words"from rest").

So here is what we know

$$
\begin{aligned}
y_i &= 0.00\,\mathrm{m} \\
v_i &= 0.00\frac{\mathrm{m}}{\mathrm{s}} \\
\Delta t &= 4.00\,\mathrm{s} \\
g &= 9.80\frac{\mathrm{m}}{\mathrm{s}^2} \\
a_{ff} &= -g = -\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)
\end{aligned}
$$

Since free-fall motion is constant acceleration motion, we can use our constant acceleration equation set:

$$\begin{aligned} v_f &= v_i + a\Delta t \qquad \text{Constant } a \\ x_f &= x_i + v_i\Delta t + \frac{1}{2}a\Delta t^2 \qquad \text{Constant } a \\ v_f^2 &= v_i^2 + 2a\Delta x \qquad \text{Constant } a \end{aligned}$$

but we can write them in terms of y because our motion is in the y-direction

$$\begin{aligned} v_f &= v_i + a\Delta t \qquad \text{Constant } a \\ y_f &= y_i + v_i\Delta t + \frac{1}{2}a\Delta t \qquad \text{Constant } a \\ v_f^2 &= v_i^2 + 2a\Delta y \qquad \text{Constant } a \end{aligned}$$

We know v_i, Δt, and we know y_i and we know a, and we want y_f.

$$\begin{aligned} v_f &= \underline{v_i} + \underline{a\Delta t} \qquad \text{Constant } a \\ y_f &= \underline{y_i} + \underline{v_i\Delta t} + \frac{1}{2}\underline{a\Delta t^2} \qquad \text{Constant } a \\ v_f^2 &= \underline{v_i^2} + 2\underline{a}\Delta y \qquad \text{Constant } a \end{aligned}$$

It looks like the second equation in our constant acceleration set will do for the final position

$$y_f = \underline{y_i} + \underline{v_i\Delta t} + \frac{1}{2}\underline{a\Delta t^2} \tag{7.4}$$

only we need to change a to $-g$

$$y_f = \underline{y_i} + \underline{v_i\Delta t} - \frac{1}{2}\underline{g\Delta t^2} \tag{7.5}$$

If a known value is zero, input it now. If it is not zero, it is better to wait to use it. We will use $y_i = 0$ and $v_i = 0$ now.

$$y_f = -\frac{1}{2}\underline{g\Delta t^2} \tag{7.6}$$

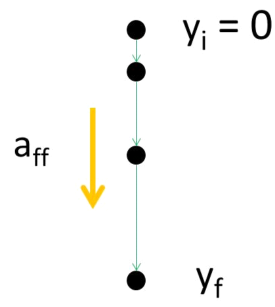
This is our symbolic solution! We can now put in the rest of our numbers

$$\begin{aligned} y &= -\frac{1}{2}\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4.00\,\mathrm{s})^2 && (7.7) \\ &= -78.4\,\mathrm{m} && (7.8) \end{aligned}$$

Let's do another example:

What is the velocity of the object (same as in the last example) at $t = 4.0\,\mathrm{s}$?

Since everything is the same as in the previous example, we know that



$$\begin{aligned}
y_i &= 0.00\,\mathrm{m}\\
v_i &= 0.00\frac{\mathrm{m}}{\mathrm{s}}\\
\Delta t &= 4.00\,\mathrm{s}\\
g &= 9.80\frac{\mathrm{m}}{\mathrm{s}^2}\\
a_{ff} &= -g = -\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)
\end{aligned}$$

and we can use the same equation set:

$$v_f = \underline{v_i} + \underline{a\Delta t} \qquad \text{Constant } a$$
$$y_f = \underline{y_i} + \underline{v_i\Delta t} + \underline{\frac{1}{2}a\Delta t^2} \qquad \text{Constant } a$$
$$v_f^2 = \underline{v_i^2} + 2\underline{a}\Delta y \qquad \text{Constant } a$$

We know v_i, a, and Δt, so we can use the first equation

$$v_f = \underline{v_i} + \underline{a\Delta t}$$

but we know $a = -g$

$$v_f = \underline{v_i} - \underline{g\Delta t}$$

We now use any zero terms

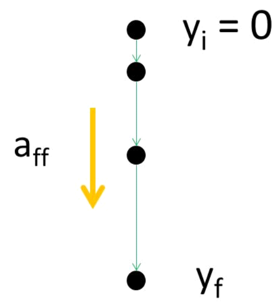
$$v_f = -g\Delta t$$

and we have a symbolic solution. We input numbers

$$\begin{aligned}
v_f &= -\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4.00\,\mathrm{s})\\
&= -39.2\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

Let's take a third example: An object is dropped from rest near the Earth's surface and we can say that free fall conditions apply. A short time later the object has dropped $80.0\,\mathrm{m}$. What is the velocity of the object at this point?

This is still free-fall



But this time we don't have a time

$$\begin{aligned} y_i &= 0.00\,\mathrm{m} \\ y_f &= -80.0\,\mathrm{m} \\ v_i &= 0.00\frac{\mathrm{m}}{\mathrm{s}} \\ g &= 9.80\frac{\mathrm{m}}{\mathrm{s}^2} \\ a_{ff} &= -g = -\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right) \end{aligned}$$

We can use the same set of equations

$$v_f = \underline{v_i} + \underline{a}\Delta t \qquad \text{Constant } a$$

$$y_f = \underline{y_i} + \underline{v_i}\Delta t + \frac{1}{2}\underline{a}\Delta t^2 \qquad \text{Constant } a$$

$$v_f^2 = \underline{v_i^2} + 2\underline{a}\,\underbrace{\Delta y} \qquad \text{Constant } a$$

$$\underbrace{\Delta y} = \underline{y_f} - \underline{y_i}$$

but since we know v_i, Δy, and a, and we want v_f, we should choose the third equation

$$v_f^2 = \underline{v_i^2} + 2\underline{a}\,\underbrace{\Delta y}$$

and solve for v_f

$$v_f = \sqrt{\underline{v_i^2} + 2\underline{a}\,\underbrace{\Delta y}}$$

and use our zeros

$$v_f = \sqrt{2\underline{a}\,\underbrace{\Delta y}}$$

and recall that $a = a_{ff} = -g$

$$v_f = \sqrt{-2g\,\underbrace{\Delta y}}$$

$$v_f = \sqrt{-2g\,(y_f - y_i)}$$

this is the algebraic answer. It might worry you that we have a negative sign inside a square root. Is this a problem? Won't we get an imaginary number?

But let's put in our values and we will see that this is not the case.

$$\begin{aligned} v_f &= \sqrt{-2\left(9.80\frac{\mathrm{m}}{\mathrm{s}^2}\right)(-80.0\,\mathrm{m}-0.00\,\mathrm{m})} \\ &= \pm 39.598\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

We have to remember that if we set $y=0$ at y_i then our y_f is negative, so we have another negative sign that cancels the first! Note that we got two answers, because from a square root we don't know if the quantity that was squared was positive or negative, think

$$\begin{aligned} 2^2 &= 4 \\ (-2)^2 &= 4 \end{aligned}$$

but we can have only one velocity at a time for our object! We need to choose the answer that fits our situation. We know (from our picture) that the object is falling, so we will choose the negative value

$$v_f = -39.6\frac{\mathrm{m}}{\mathrm{s}}$$

Constant acceleration and Free-fall

Let's say that you have joined the BYU-I rocket team. You launch the rocket (and this time it does not blow up). The rocket moves upward with a constant acceleration $a=30.0\,\mathrm{m}/\,\mathrm{s}^2$ as long as the rocket fuel lasts, which is $\Delta t_1 = 4.00\,\mathrm{s}$. Once the fuel is spent, the rocket continues up for a while, then it begins to fall back to the ground (the parachute did not open. . . again). find

a) the velocity of the rocket and at the time when the fuel is spent

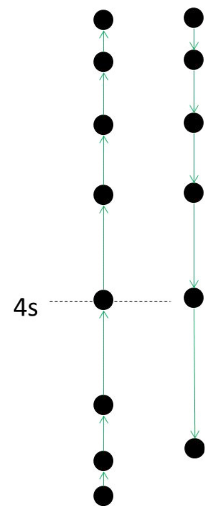
b) the position at the time when the fuel is spent

c) the maximum height of the rocket

We can only do problems for constant acceleration. But in this problem the acceleration changes! Fortunately, the change in acceleration is instantaneous, and the rest of the time we have constant acceleration, but we have *two* constant accelerations. To deal with this we divide this problem into two phases of it's motion, one for each constant acceleration. In effect we make our one problem into two related problems. The first problem is the powered ight up. The second problem is after the fuel is spent and the rocket is in free fall.

Solution:

PT: constant acceleration problem *and* a free-fall problem. While the rocket engine is working it is constant acceleration, afterward, it is free-fall. We will divide the problem into two parts, one constant motion, the other free-fall.



Here is what we know from the problem statement

$$a_1 = 29.4\,\frac{\mathrm{m}}{\mathrm{s}^2}$$
$$t_1 = 4.00\,\mathrm{s}$$
$$a_2 = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$$
$$y_{1i} = 0.00 \text{ (ground)}$$

Let's use y as position. We will choose $y = 0$ to be the ground. We will use v and a as velocity and acceleration, and g as the acceleration due to gravity. Our basic equations are

$$v_f = v_i + a\Delta t \qquad \text{Constant } a \tag{7.9}$$

$$y_f = y_i + v_i\Delta t + \frac{1}{2}a\Delta t^2 \qquad \text{Constant } a \tag{7.10}$$

$$v_f^2 = v_i^2 + 2a\Delta y \qquad \text{Constant } a \tag{7.11}$$

Parts a) and b) Find the rocket's velocity and position after $4.00\,\mathrm{s}$ which is when the rocket fuel has just run out. These values are both from the first part of the problem where we have an acceleration a_1.

$$a_1 = 29.4\,\frac{\mathrm{m}}{\mathrm{s}^2}$$
$$t_1 = 4.00\,\mathrm{s}$$
$$y_{1i} = 0.00 \text{ (ground)}$$

We choose from our kinematic equations. For position, we might use

$$y_{1f} = y_{1i} + v_{1i}\Delta t_1 + \frac{1}{2}a_1\Delta t_1^2 \tag{7.12}$$

The subscript 1 means this is the first position value we will find for part 1 of the motion. We can guess that $v_{1i} = 0$ (when the rocket starts, it is not moving) and $y_{1i} = 0$, and we are given $a_1 = 29.4\frac{\mathrm{m}}{\mathrm{s}^2}$ so

$$y_{1f} = \frac{1}{2}a_1\Delta t_1^2 \tag{7.13}$$

Now we can use numbers

$$\begin{aligned} y_{1f} &= \frac{1}{2}\left(29.4\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4.00\,\mathrm{s})^2 && (7.14)\\ &= 235.2\,\mathrm{m} && (7.15)\\ &= 235.\ \mathrm{m} && (7.16)\end{aligned}$$

Now let's find the velocity at this altitude (when the rocket engine just runs out of fuel). We choose the equation

$$v_{1f} = v_{1i} + a_1\Delta t_1 \tag{7.17}$$

and recognize that v_{1i} is still zero, so

$$v_{1f} = a_1\Delta t_1 \tag{7.18}$$

Now we use numbers

$$\begin{aligned} v_{1f} &= \left(29.4\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4.00\,\mathrm{s}) && (7.19)\\ &= 117.6\frac{\mathrm{m}}{\mathrm{s}} && (7.20)\\ &= 118\frac{\mathrm{m}}{\mathrm{s}} && (7.21)\end{aligned}$$

Part c) Find the maximum Height. We will use the ending values from part a) to be our starting values for part b). We will go back to all the digits we had before we rounded for sig-figs. That is because we keep all the digits from a calculation and only round at the end. At the end of part a) we did round because we reported v_{1f} and y_{1f}, but now we are continuing to calculate, so let's keep all the digits we have until the end once more.

$$v_{1f} = v_{2i} = 117.6\,\frac{\mathrm{m}}{\mathrm{s}}$$
$$y_{1f} = y_{2i} = 235.2\,\mathrm{m}$$
$$a_2 = -9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

We know the initial speed, position, and acceleration. Notice that we no longer have the engine operating, so the acceleration is now $-g$. We are in free fall (but we are going up!). We know that at the highest point in the flight the velocity must be zero, so we can write another known value

$$v_{2f} = 0$$

We again need our kinematic equations, marked with what we know, and a subscript "2" to distinguish this set from the new part 2 set from part 1. We want y_{2f}.

$$\underline{v_{2f}} = \underline{v_{2i}} + \underline{a_2}\Delta t_2 \qquad \text{Constant } a \tag{7.22}$$

$$y_{2f} = \underline{y_{2i}} + \underline{v_{2i}}\Delta t_2 + \frac{1}{2}\underline{a_2}\Delta t_2^2 \qquad \text{Constant } a \tag{7.23}$$

$$\underline{v_{2f}}^2 = \underline{v_{2i}}^2 + 2\underline{a_2}\Delta y_2 \qquad \text{Constant } a \tag{7.24}$$

$$\Delta y_2 = y_{2f} - \underline{y_{2i}}$$

It looks like we could use the second equation

$$y_{2f} = y_{2i} + v_{2i}\Delta t_2 + \frac{1}{2}a_2\Delta t_2^2$$

but we don't know what Δt_2 is. But we can solve for Δt_2 from the first equation. Notice how we did this. After marking our equations with what we know, we looked for a way to solve for what we want, y_{2f}. But we could not do it with any one equation on it's own. So we looked at the other equations to find the missing parts in our equation for y_{2f}. Let's get the time Δt_2 first,

$$\underline{v_{2f}} = \underline{v_{2i}} + \underline{a_2}\Delta t_2$$

using our zeros

$$0 = \underline{v_{2i}} + \underline{a_2}\Delta t_2$$

and $-g$

$$\underline{v_{2i}} = g\Delta t_2$$

so

$$\Delta t_2 = \frac{\underline{v_{2i}}}{\underline{g}}$$

$$\Delta t_2 = \frac{\underline{v_{1f}}}{\underline{g}}$$

We don't know need to know the value of Δt_2, but let's calculate it anyway.

$$\Delta t_2 = \frac{120.0\,\frac{\text{m}}{\text{s}}}{9.80\,\frac{\text{m}}{\text{s}^2}} \tag{7.25}$$

$$= 12.245\,\text{s} \tag{7.26}$$

so the rocket goes up for $12.2\,\text{s}$ after the fuel runs out!

Now that we know Δt_2 we can use the second of our equations to find y_{2f}

$$
\begin{aligned}
y_{\max} &= y_{2f} = y_{2i} + v_{2i}\Delta t_2 + \frac{1}{2}a_2\Delta t_2^2 && (7.27)\\
&= y_{1f} + v_{1f}\Delta t_2 - \frac{1}{2}g\Delta t_2^2 && (7.28)\\
&= y_{1f} + v_{1f}\left(\frac{v_{1f}}{g}\right) - \frac{1}{2}g\left(\frac{v_{1f}}{g}\right)^2 && (7.29)\\
&= y_{1f} + v_{1f}\left(\frac{v_{1f}}{g}\right) - \frac{1}{2g}\left(v_{1f}\right)^2 && (7.30)
\end{aligned}
$$

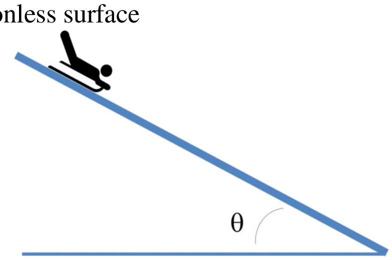
Notice that even though I calculated the value of Δt_2 I still put in the symbolic Δt_2 into my equation. Now we are ready for numbers

$$
\begin{aligned}
y_{\max} &= 240.0\,\mathrm{m} + \frac{\left(120.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{9.8\frac{\mathrm{m}}{\mathrm{s}^2}} - \frac{1}{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)}\left(120.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2 && (7.31)\\
&= 974.69\,\mathrm{m} && (7.32)
\end{aligned}
$$

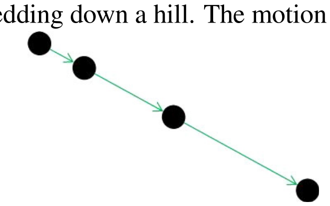
$$
\begin{aligned}
y_1 &= 240.0\,\mathrm{m}\\
v_1 &= 120.0\frac{\mathrm{m}}{\mathrm{s}}\\
t &= 12.245\,\mathrm{s}
\end{aligned}
$$

Motion on an inclined plane

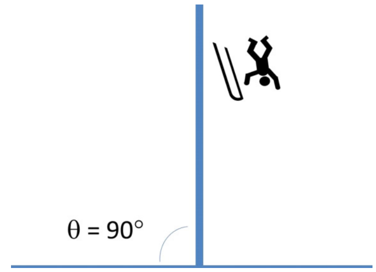
Let's take on an example that is almost free-fall. Suppose we have a person sledding down a (perfectly) frictionless surface



would this be free-fall? First, we should test for constant acceleration. So suppose we record video of a person sledding down a hill. The motion diagram looks like this



which we recognize as acceleration. But only if the slope angle θ were 90° would we have free-fall



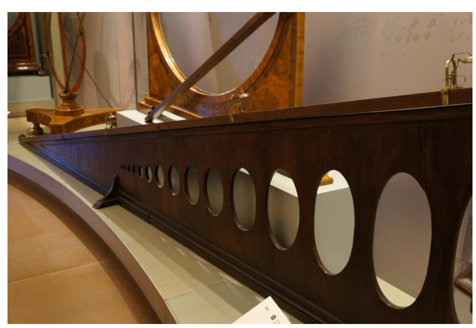
And our intuition tells us that the guy who tried to sled off a vertical cliff will reach the bottom faster than the guy that slid down the slope. So we can guess that our slope has an acceleration that is less than the $a_{ff} = g = 9.8\,\mathrm{m/\,s^2}$. It is as though only part of the free-fall acceleration is able to pull the sled down the hill. The hill seems to be getting in the way!

Being physics students The obvious thing to do is to record video of sledders going down many different hills with different slopes to see if the acceleration is dependent on the steepness of the slope. You might guess that the steeper the slope the larger the acceleration.

Such an experiment has been done, and the result is

$$a_{slope} = g\sin\theta$$

From our trigonometry experience, we recall that $\sin\theta$ goes between 0 and 1 as the slope angle goes from $0\,^\circ$ to $90\,^\circ$. Indeed, we seem to have only part of the free-fall acceleration. We have a fraction of the total possible falling acceleration with the fraction given by $\sin\theta$.



Even Galileo actually used ramps to do his motion studies. By using a ramp (shown above) he could have constant acceleration, but a smaller constant acceleration than g. That made it easier to see the motion. Galileo did not have digital cameras to record the motion, instead he used bells along the ramp to tell where the balls were so he could make his motion diagrams.

Instantaneous Acceleration

But what if the acceleration is not constant? Recall that we started with constant motion,

$$v = \frac{\Delta x}{\Delta t}$$

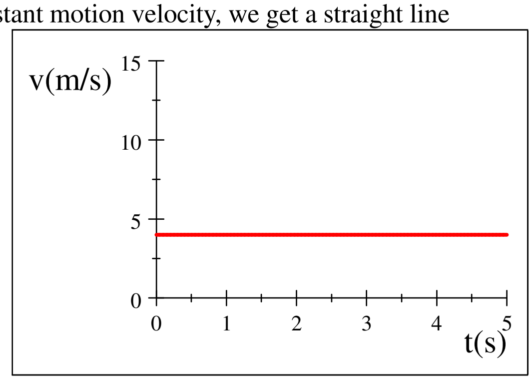
and we found that for constant motion

$$v = \frac{dx}{dt}$$

and we rewrote this to be

$$x_f = x_i + v\Delta t$$

If we plot the constant motion velocity, we get a straight line



and then we interpreted $v\Delta t$ as the area under a v vs. t graph

$$x_f = x_i + \text{area under a } v \text{ vs. } t \text{ graph}$$

and we found that if we added up many $v\Delta t$ segments with Δt very small so we called it dt, we could always find the area under a v vs. t graph using an integral

$$x_f = x_i + \int_{t_i}^{t_f} v dt$$

We have been working with constant acceleration. And we recall that acceleration is

$$a = \frac{\Delta v}{\Delta t}$$

We could rewrite this as

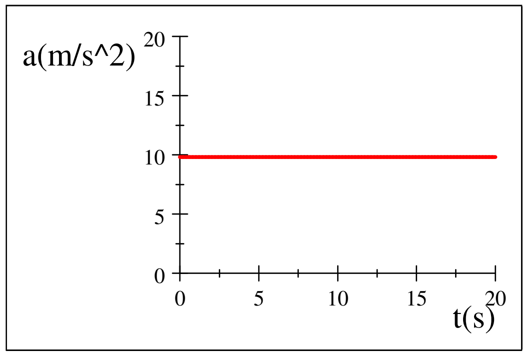
$$v_f = v_i + a\Delta t$$

and we recognize this as one of our constant acceleration equation set of equations. If we make Δt so small that we could call it dt, we would have

$$a(t) = \frac{dv}{dt}$$

and we have something grand! We have found that the derivative of the velocity as a function of time is the acceleration! We get to use our new math again!

Further, if we plot a constant acceleration, we get a straight line



It sure looks like we could interpret $a\Delta t$ as the area under a a vs. t graph

$$v_f = v_i + \text{area under a } v \text{ vs. } t \text{ graph}$$

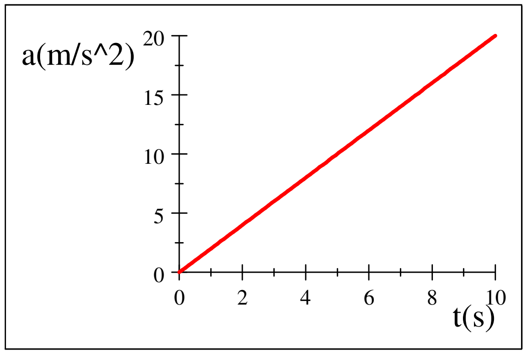
If we add up many $a\Delta t$ segments with Δt very small so we called it dt, we can find the area under a a vs. t graph using an integral

$$v_f = v_i + \int_{t_i}^{t_f} a dt$$

We have a way to find the velocity as a function of time knowing the acceleration! We get to use our integral process again!

Let's try some problems using these new ideas.

Look at the graph below. If the initial velocity is zero, what would be the final velocity?



The "area" under the curve would be

$$\begin{aligned} A_{curve} &= \frac{1}{2}bh \\ &= \frac{1}{2}(10\,\mathrm{s})\left(20\frac{\mathrm{m}}{\mathrm{s}^2}\right) \\ &= 100\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Let's try the same problem, but using the integral. The acceleration function is a straight line

$$a(t) = \left(2\frac{\mathrm{m}}{\mathrm{s}^3}\right)t + 0$$

so

$$v_f = v_i + \int_{t_i}^{t_f} a dt$$

would be

$$\begin{aligned} v_f &= 0 + \int_0^{10\,\mathrm{s}} \left(2\frac{\mathrm{m}}{\mathrm{s}^3}\right) t dt \\ &= \left.\frac{\left(2\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2}{2}\right|_0^{10\,\mathrm{s}} \\ &= \left.\left(\frac{\mathrm{m}}{\mathrm{s}^3}\right)t^2\right|_0^{10\,\mathrm{s}} \\ &= \left(\frac{\mathrm{m}}{\mathrm{s}^3}\right)(10\,\mathrm{s})^2 - 0 \\ &= 100\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

What we have done is quite profound. We developed a process that will allow us to find the acceleration as a function of time given the velocity as a function of time, and the velocity as a function of time given the acceleration as a function of time. And we have a process to find the velocity as a function of time if we know the position and position as a function of time if we know the velocity.

position and velocity		velocity and acceleration	
$v = \frac{dx(t)}{dt}$	$x(t) = x_i + \int_{t_i}^{t_f} v(t)\,dt$	$a = \frac{dv(t)}{dt}$	$v(t) = v_i + \int_{t_i}^{t_f} a(t)\,dt$

These equations work for constant motion, constant acceleration, and all motion in general!

You may think, though, that not all motion is in straight lines. Some motion is in two dimensions. How do we deal with this type of motion? We will start to tackle this problem next.

8 Vectors

We have learned about vectors already. But there is more to know! Let's review what we know so far, then we will launch into new vector understanding.

We already know that some things in the universe have direction. And that direction is important. If you want to buy groceries in IF but end up in Ashton instead, things are not so good!

We will use the mathematical idea of a vector to describe these things that have direction. We draw a vector with an arrow. The arrow seems natural because it has both a direction that it points and a length. We can use the length of the arrow to show the *magnitude* of the vector. For example, a velocity is a vector quantity. We might travel at $26\,\mathrm{m/s}$ north. The $26\,\mathrm{m/s}$ is the magnitude of the vector. The "north" is the direction. We could draw an arrow to represent this.

Notice that we draw the arrow starting at the point (labeled P, for "position point"). This is always true for instantaneous vector values. We draw the vector starting at the point. The length of the arrow represents the $26\,\mathrm{m/s}$ The direction is given by the direction the arrow points. In this system for representing vectors, the magnitude *cannot be negative*. The magnitude is the amount of something, like speed. Negative speed does not make sense. We will only use positive values for magnitudes. But, you might say, I can write a velocity for a one-dimensional problem like $-26\,\mathrm{m/s}$. But remember the minus sign is the direction. In a one-dimensional problem, the minus sign means "to the left." So it is a direction. The $26\,\mathrm{m/s}$ is still a magnitude and it is still a positive value. When you put the $26\,\mathrm{m/s}$ together with the minus sign, then it is a vector that tells you that the object is traveling to the left at with a speed of $26\,\mathrm{m/s}$.

Our notion for the magnitude of a vector is a set of absolute value signs

$$\text{magnitude of } \overrightarrow{\mathbf{v}} = \left|\overrightarrow{\mathbf{v}}\right|$$

but this is a lot to write, four individual characters required to make the symbol for a magnitude. So it is customary to write the magnitude of the vector with the same symbol as we use for the vector, but without the arrow.

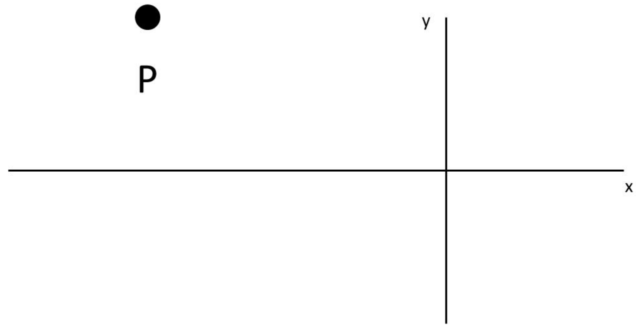
$$\text{magnitude of } \overrightarrow{\mathbf{v}} = \left|\overrightarrow{\mathbf{v}}\right| = v$$

so $\overrightarrow{\mathbf{v}}$ is a vector and v is a magnitude. Now you see why v is the symbol for instantaneous speed! Speed is the magnitude of a velocity. So naturally the symbol for speed is v.Note that in one dimensional problems being this careful with notion was not so important because we used the symbol for the vector as the magnitude and a plus or minus sign for the direction. But as we go to two and three dimensions, just using a plus or minus for direction won't work. So it pays to be careful and write $\overrightarrow{\mathbf{v}}$ for a vector and v for a magnitude.

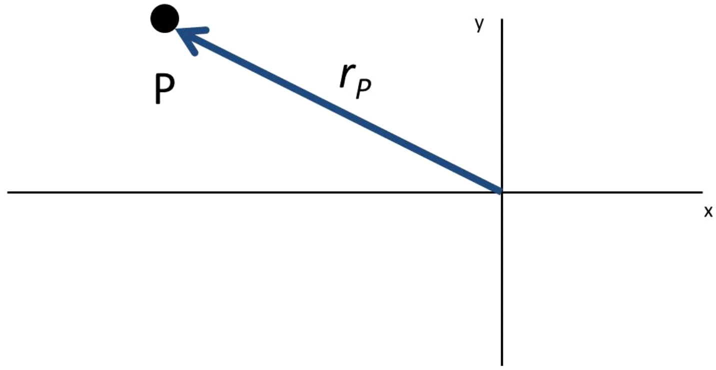
It is *very important* to realize that both the magnitude and direction of an instantaneous vector quantity like velocity or acceleration represent values *at a single point, P.* The vector stretches to the right beyond the point, P, but the magnitude is the speed when the object was *at point* P and it does not tell us about any other point. For example, here is a drawing with the velocity show for a nearby point, Q drawn in blue so we can see it.

The green velocity vector from point P would completely cover the point Q and it's velocity vector! Notice that the speed at Q is much less than it is at point P. The green vector $\overrightarrow{\mathbf{v}}_P$ does not represent the speed at Q. A vector will extend beyond the point, but it only gives us information about what happens at its single point.

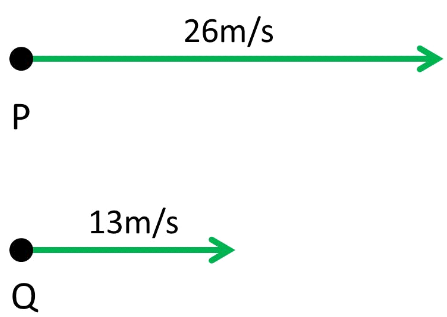
You might wonder how long to make the arrows for our drawings. For position vectors it is easy to know. A position vector must be as long as the displacement between a point and the origin. For example, here our point P again, but this time with an origin drawn so we can see where it is.



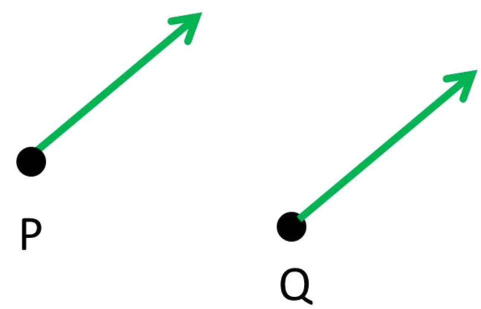
The position vector, $\overrightarrow{\mathbf{r}}_p$, must reach from the origin to the point P



This is different than the rule we just learned about velocities where the velocity vector would start at the point, P. But how long should the velocity vector be? Really we can choose any length we want for a velocity vector. But once we have chosen a length for a particular problem, we have to draw all other velocity vectors for that problem to the same scale. So, given our velocity at point P of $26\,\mathrm{m/s}$, a velocity of $13\,\mathrm{m}/s$ would have to be half as long.



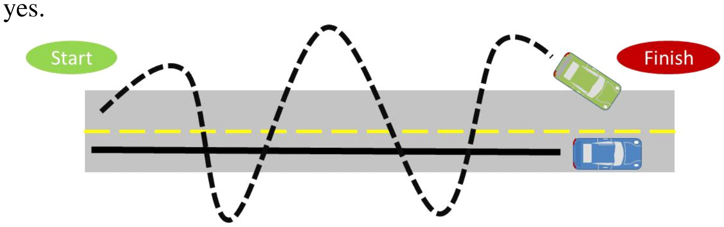
I is also important to realize that by "direction" we are talking about a direction like a compass direction. Both of the vectors in the next figure are pointing the same direction.

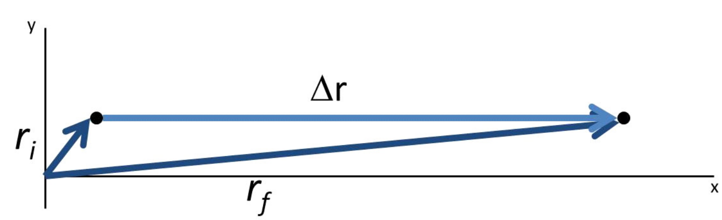


Notice that the object passing through P and the object passing through Q are going the same direction but will never meet. The will not even get closer together! Going the same compass direction is not "going to the same place."

Of course there are quantities in physics that don't have direction. Examples are mass and temperature. Mass does not have a direction at all. We don't say "your mass is $75\,\mathrm{kg}$ due north." That would make no sense. Temperature might seem to have a direction. We say it temperature is going up or going down. But this is not a compass direction. By this we only mean the temperature is rising or lowering, not that it is going north or south. Such quantities are called *scalar* quantities.

Let's consider an example. Suppose we are again watching a race from above. One car travels the straight-line path from *Start* to *Finish.* Another travels the curved line. Both arrive after the same time interval Δt. Do both have the same displacement Δx? The answer is, yes.





We can see that vector displacement is equivalent to a straight line path, what we might colloquially call "as the crow flies." But vector displacement might not be the actual path taken by the object.

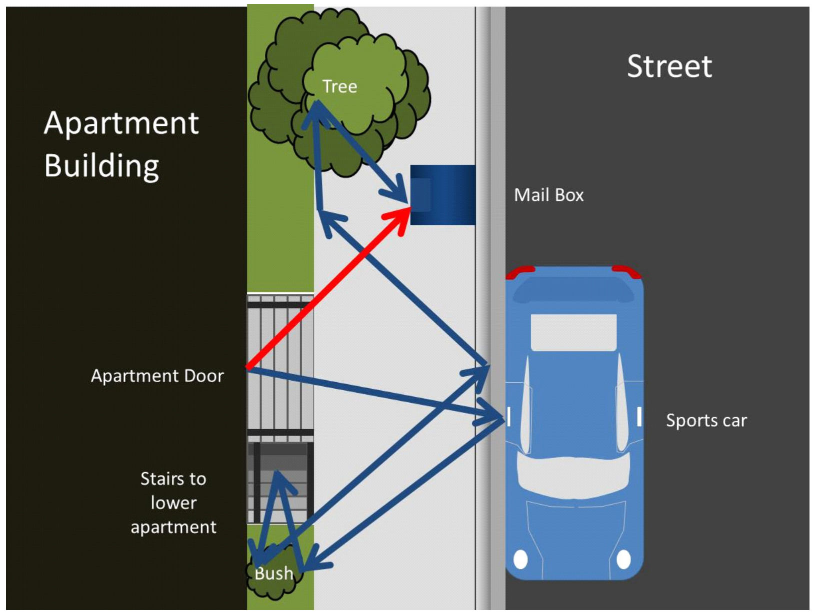
$$\Delta \overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r}}_f - \overrightarrow{\mathbf{r}_i}$$

only deals with the beginning and ending points.

Vector Addition revisited

You have landed a nice job on campus grading papers for PH121 (after you get your "A" in this class). Suppose you agreed to work for \$8.50 an hour. And campus jobs allow you to work for 20 hours per week. Then in a month you would earn about \$680. But when you get your pay check, will it be for \$680 dollars? Of course you know the answer is "no." The \$680 is called your "gross" pay. It is how much you earned. But you have to pay taxes, and the taxes, and insurance costs, and so forth are taken out of your paycheck before you ever see the money. You might get \$340 dollars that actually comes to you. The \$340 is called your "net" pay. I'm not sure where these words come from. I don't consider the larger amount to be "gross." I would like to have all that money to pay for food and rent, etc. But these are the words use by the financial people to describe our pay. And one of these words is useful in physics.

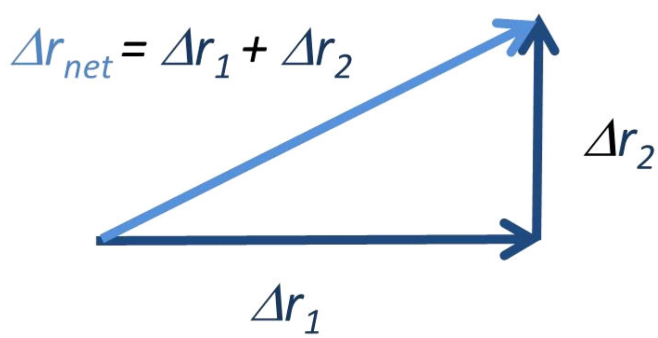
The word "net" is the part of the pay check you can do things with. The word "net" in physics is the part that actually mattered. Let's take our race car example again. All that mattered was how far the cars went toward the finish line. Not the actual distance traveled. Our displacement is a "net" value, because it is the part that actually mattered. Let's take another example. Suppose we send a child to a mail box to post a letter. Here is the path of the child in blue.



The path you wanted the child to travel is in red. The red arrow is the path that would get the job done. The part that actually mattered to you, the parent. It is the net displacement you wanted (get the letter to the box).

Notice that if we use vector addition to add up all the blue, child, displacements, we would end up with the red displacement! We will use the word "net" to mean "add vectors" in this class.

Our child going to the mailbox is fun to think about, but to learn the math of vector addition let's take on a simpler example.



and let's say that

$$\begin{aligned}\Delta r_1 &= 40\,\mathrm{m}\\ \Delta r_2 &= 30\,\mathrm{m}\end{aligned}$$

what would $\Delta\overrightarrow{r}_{net} = \Delta\overrightarrow{r}_1 + \Delta\overrightarrow{r}_2$ be? To answer this we need to find both the magnitude and the direction of $\Delta\overrightarrow{r}_{net}$. Let's start with the magnitude.

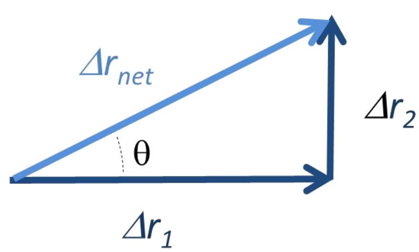
From our trigonometry experience we can see that finding the magnitude is an easy problem. Our position vectors form a right triangle. We use the Pythagorean theorem

$$\Delta r_{net} = \sqrt{\Delta r_1^2 + \Delta r_2^2}$$

to get

$$\begin{aligned}\Delta r_{net} &= \sqrt{(40\,\mathrm{m})^2 + (30\,\mathrm{m})^2}\\ &= 50\,\mathrm{m}\end{aligned}$$

Now we need to find the direction of $\Delta\overrightarrow{r}_{net}$. For our class, we will define the direction like a compass direction measured from the x-axis. We use Greek letters for angles, so our direction is the angle θ shown in the next figure.



Recalling our trig knowledge, we can see that

$$\tan\theta = \frac{\Delta r_2}{\Delta r_1}$$

so the direction θ would be

$$\theta = \tan^{-1}\left(\frac{\Delta r_2}{\Delta r_1}\right)$$

We can use this to solve for the direction in our example

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{30\,\mathrm{m}}{40\,\mathrm{m}}\right)\\ &= 0.643\,5\,\mathrm{rad}\\ &= 36.870\,^{\circ}\\ &= 37\,^{\circ}\end{aligned}$$

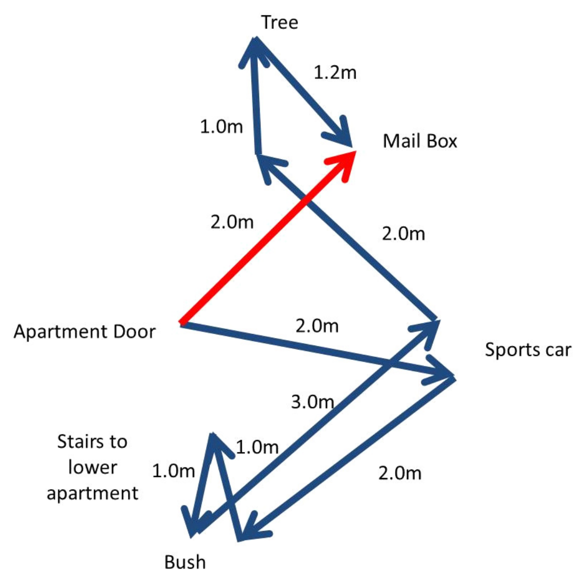
And we have our solution,

$$\Delta\overrightarrow{r}_{net} = 50\,\mathrm{m} \text{ at } 37\,^{\circ}$$

The answer **needs both parts**, magnitude and direction.

Of course, this problem would have been harder if our displacement $\Delta\overrightarrow{r}_1$ and $\Delta\overrightarrow{r}_2$ did not form a right angle. For such a problem, we could use the law of sines or law of cosines from trigonometry (*but we won't*, we are working our way toward a better way to do this!).

Let's go back to our child mailing a letter. Suppose the displacements involved are as shown in the next figure



What is the magnitude of the velocity of the child if the trip took $10\,\mathrm{min}$? Velocity uses net displacement. It only cares about what got the job done. So only our starting and ending positions matter. Then

$$\Delta\overrightarrow{r}_{net} = \overrightarrow{r}_{mailbox} - \overrightarrow{r}_{door}$$

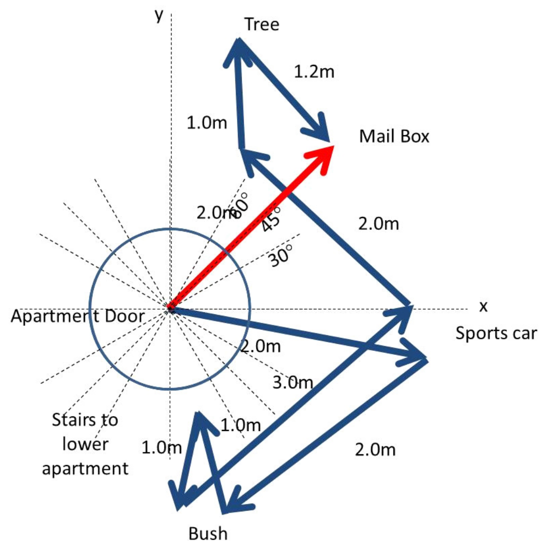
and according to the figure, $\Delta r_{net} = 2.0\,\mathrm{m}$. The duration Δt is $10\,\mathrm{min} = 600.0\,\mathrm{s}$. So the magnitude of the velocity is

$$v = \frac{2.0\,\mathrm{m}}{600.0\,\mathrm{s}} = 3.3 \times 10^{-3}\,\frac{\mathrm{m}}{\mathrm{s}}$$

which is small, but considering all the stops the child actually made, it is reasonable.

Direction would be harder, needing us to actually add up all the vector displacements.

Or we could use a compass or protractor to measure the direction. Let's do the latter.



It looks like $\Delta \overrightarrow{r}_{net}$ has a direction of about $45\,^{\circ}$.

It would be nice if there were a better way to find the magnitude and direction of the net displacement without resorting to quite so much trigonometry or giving up and doing a measurement.

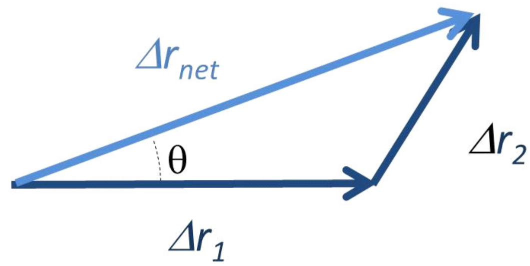
Components of vectors

We have been using vectors for some time now. And we know how to add and subtract vectors graphically. In a previous problem we were able to use math to find the magnitude and direction of the net displacement, but that was a special case because the displacements $\Delta \overrightarrow{r}_1$ and $\Delta \overrightarrow{r}_2$ formed a right angle. But in our mailbox problem, this was not the case. It would have been nice if our child had only made turns at right angles. But it is not realistic to require that of all moving objects (especially a child).

It would be nice if there were a way to turn a hard problem like the child's displacement into an easier problem like the right angle displacements.

And there is....

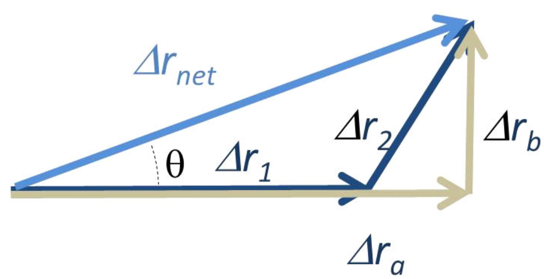
We will work our way up to making our displacement problems all easier using a clever trick, one we learned about with our sledder a few lectures ago. It will be easier to see how this works if we have a problem to work on, so here is one that is simpler than our child's displacement.



We wish to find

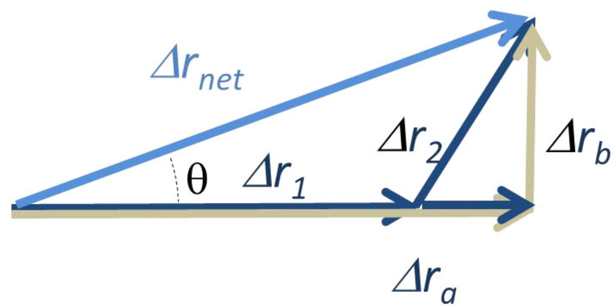
$$\Delta\overrightarrow{r}_{net} = \Delta\overrightarrow{r}_1 + \Delta\overrightarrow{r}_2$$

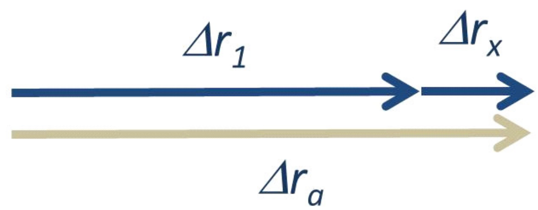
where $\Delta\overrightarrow{r}_1 = 40\,\mathrm{m}$ and now $\Delta\overrightarrow{r}_2 = 35\,\mathrm{m}$. The angle between $\Delta\overrightarrow{r}_1$ and $\Delta\overrightarrow{r}_2$ must now be larger than $90\,°$. And our intuition tells us that $\Delta\overrightarrow{r}_{net}$ must be larger in this case than it was in our previous problem. But we know that $\Delta\overrightarrow{r}_{net}$ could come from many different paths, and $\Delta\overrightarrow{r}_{net}$ only depends on the stopping and starting points. So suppose that we had two new vectors $\Delta\overrightarrow{r}_a$ and $\Delta\overrightarrow{r}_b$ that also add up to $\Delta\overrightarrow{r}_{net}$. And suppose $\Delta\overrightarrow{r}_a$ and $\Delta\overrightarrow{r}_b$ form a right angle. Then if we knew $\Delta\overrightarrow{r}_a$ and $\Delta\overrightarrow{r}_b$ we could use the Pythagorean theorem and tangent function just like in our previous problem.



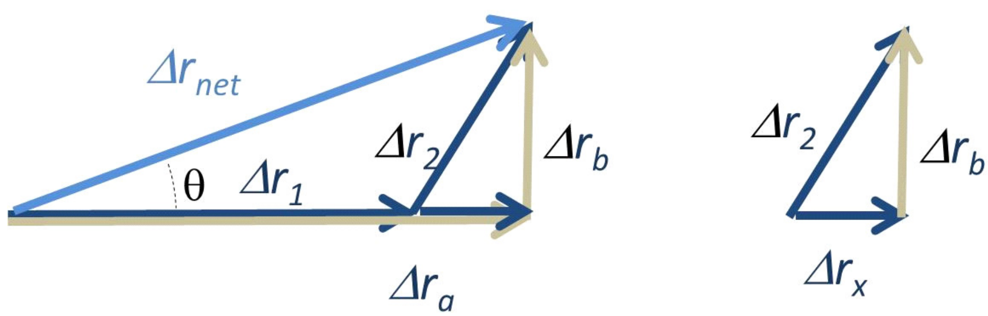
The only problem is that we don't know the length of $\Delta\overrightarrow{r}_a$ and $\Delta\overrightarrow{r}_b$. But notice that $\Delta\overrightarrow{r}_a = \Delta\overrightarrow{r}_1 +$ a little bit more. Let's call the little bit more $\Delta\overrightarrow{r}_x$ so then

$$\Delta\overrightarrow{r}_a = \Delta\overrightarrow{r}_1 + \Delta\overrightarrow{r}_x$$





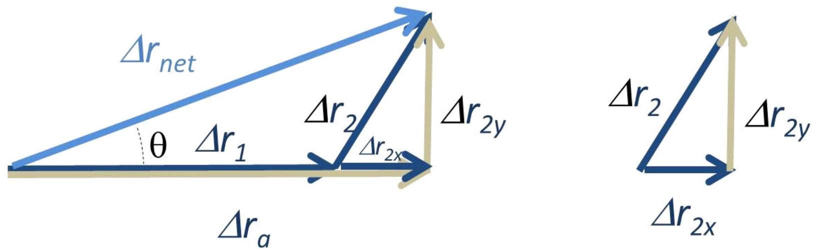
And further notice that there is a relationship between $\Delta\overrightarrow{r}_2$, $\Delta\overrightarrow{r}_b$, and $\Delta\overrightarrow{r}_x$



they form a right triangle. Then we can say that

$$\Delta\overrightarrow{r}_2 = \Delta\overrightarrow{r}_b + \Delta\overrightarrow{r}_x$$

We have broken $\Delta\overrightarrow{r}_2$ into two parts $\Delta\overrightarrow{r}_b$, and $\Delta\overrightarrow{r}_x$ that form a right angle. Since $\Delta\overrightarrow{r}_b$, and $\Delta\overrightarrow{r}_x$ are really parts of $\Delta\overrightarrow{r}_2$, let's relabel them



We will call $\Delta\overrightarrow{r}_x = \Delta\overrightarrow{r}_{2x}$ so we know it is part of $\Delta\overrightarrow{r}_2$. The x is appropriate because $\Delta\overrightarrow{r}_{2x}$ is along the x-axis. We will also call $\Delta\overrightarrow{r}_b = \Delta\overrightarrow{r}_{2y}$ because it is part of $\Delta\overrightarrow{r}_2$ but it is in the y-direction. Now we can rewrite our solution for $\Delta\overrightarrow{r}_{net}$

$$\Delta\overrightarrow{r}_{net} = \Delta\overrightarrow{r}_1 + \Delta\overrightarrow{r}_{2x} + \Delta\overrightarrow{r}_{2y}$$

We could even separate this into two parts, a part along the x-axis and a part along the y-axis.

$$\Delta\overrightarrow{r}_{net} = (\Delta\overrightarrow{r}_1 + \Delta\overrightarrow{r}_{2x}) + (\Delta\overrightarrow{r}_{2y})$$

and we could name the parts

$$\Delta\overrightarrow{r}_{net_x} = \Delta\overrightarrow{r}_1 + \Delta\overrightarrow{r}_{2x}$$
$$\Delta\overrightarrow{r}_{net_y} = \Delta\overrightarrow{r}_{2y}$$

and if you have been followng carefully you will recognize that

$$\begin{aligned}\Delta\overrightarrow{r}_{net_x} &= \Delta\overrightarrow{r}_a \\ \Delta\overrightarrow{r}_{net_y} &= \Delta\overrightarrow{r}_b\end{aligned}$$

so we have found a new way to write our two new vectors that make the problem $\Delta\overrightarrow{r}_{net} = \Delta\overrightarrow{r}_1 + \Delta\overrightarrow{r}_2$ easier. Now all we have to do is use the Pythagorean theorem to find Δr_{net}

$$\begin{aligned}\Delta r_{net} &= \sqrt{\Delta r_{net_x}^2 + \Delta r_{net_y}^2} \\ &= \sqrt{(\Delta r_1 + \Delta r_{2x})^2 + \Delta r_{net_y}^2}\end{aligned}$$

and the direction is given by

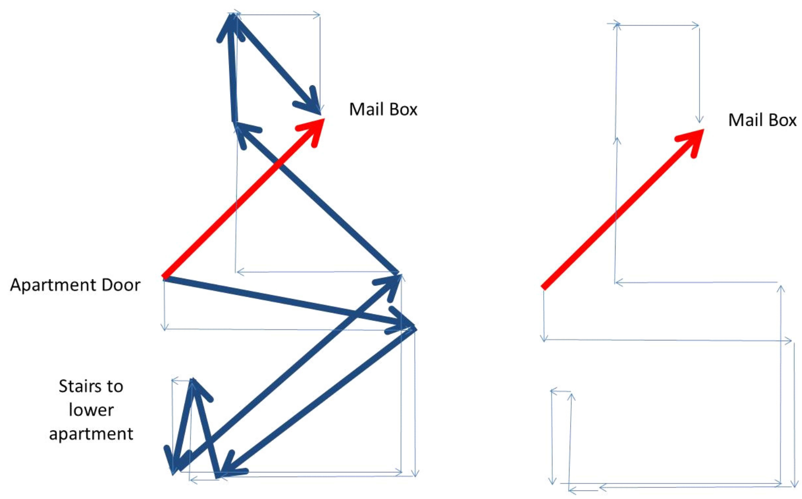
$$\theta = \tan^{-1}\left(\frac{\Delta r_{net_y}}{\Delta\overrightarrow{r}_{net_x}}\right) = \tan^{-1}\left(\frac{\Delta r_{net_y}}{\Delta r_1 + \Delta r_{2x}}\right)$$

Now, no matter how complicated the situation, we can separate every vector that is not along an axis into parts that are along an axis, and then all we have to do is to add up all the parts that are along each axis to find the net displacement.

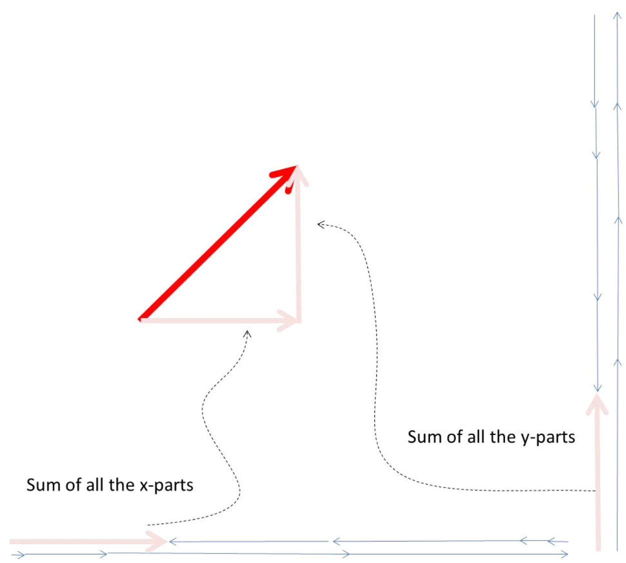
Let's take on the child displacement case!

None of the child's displacements are along an axis, so we need to find x-parts and y-

parts for every vector.



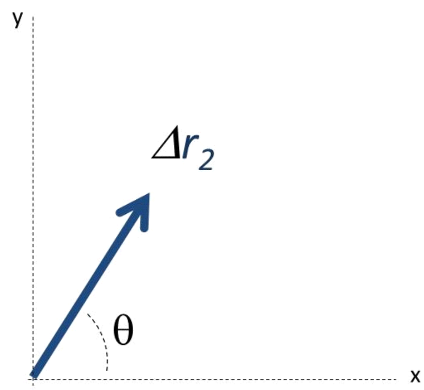
Once we have the parts, we add up all the x-parts separately, treating them like vectors so they go head to tail.



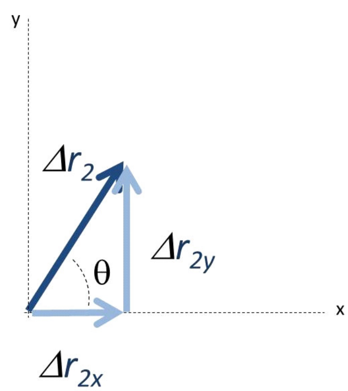
We also add up all the y-parts separately, placing them head to tail. The two resulting vectors are the x-part and the y-part of the net displacement (honest, I didn't cheat, they really worked!).

But you might have noticed that there is a part missing. We know how to find the x-part and the y-part of vectors graphically, just make right triangles. But we need to know how to mathematically find actual magnitudes for these x and y-parts. Fortunately, we know a little trigonometry! Since all our new vector-part triangles are right triangles, we know how to find the side lengths!

Suppose again that I have the following vector.



where I know from measurement, if nothing else, that $\theta = 60\,^\circ = 1.047\,2\,\mathrm{rad}$. Then if I make a right triangle with x and y-parts of the vector



then the from trig we know that

$$\cos\theta = \frac{\Delta r_{2x}}{\Delta r_2}$$

so

$$\Delta r_{2x} = \Delta r_2 \cos\theta$$

To find the x-part, all we have to do is multiply the magnitude of the vector, $\overrightarrow{\Delta r_2}$ by the cosine of the angle it makes with the x-axis! So in our case, the x-part is just

$$\begin{aligned} \Delta r_{2x} &= 35\,\mathrm{m}\cos 60\,^{\circ} \\ &= 17.5\,\mathrm{m} \end{aligned}$$

Similarly, to find the y-part notice that

$$\sin\theta = \frac{\Delta r_{2y}}{\Delta r_2}$$

so that

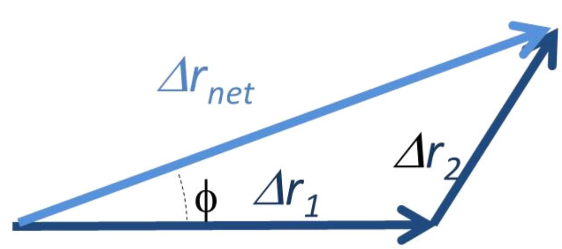
$$\Delta r_{2y} = \Delta r_2 \sin\theta$$

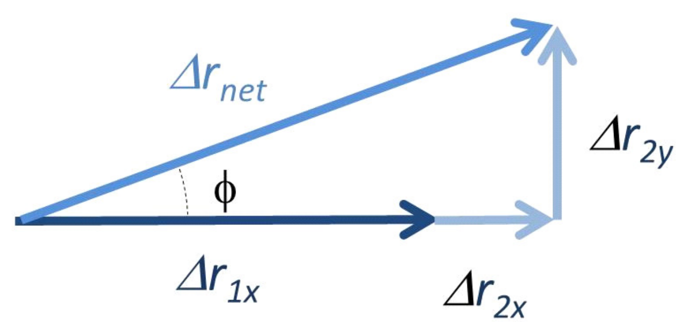
and so all we have to do to find the y-part is to multiply the magnitude of the vector, $\overrightarrow{\Delta r}_2$ by the sine of the angle it makes with the x-axis. In our case this gives

$$\begin{aligned} \Delta r_{2y} &= 35\,\mathrm{m}\sin 60\,^{\circ} \\ &= 30.311\,\mathrm{m} \end{aligned}$$

Armed with this we can do the problem we suggested so long ago.

Suppose we have two displacements $\Delta r_1 = 40\,\mathrm{m}$ along the x-axis and $\Delta r_2 = 35\,\mathrm{m}$ at an angle $\theta = 60\,^{\circ}$. What is the net displacement.





Our procedure is to divide up $\Delta\overrightarrow{r}_1$ and $\Delta\overrightarrow{r}_2$ into x and y-parts, add up the x and

y-parts, and then use these to find the magnitude of $\Delta\overrightarrow{r}_{net}$.

We first find the x-parts of both vectors. Let's start with $\overrightarrow{\Delta r}_1$

$$\begin{aligned}\Delta r_{1x} &= \Delta r_1 \cos(0)\\ &= \Delta r_1\\ &= 40\,\mathrm{m}\end{aligned}$$

this is because $\overrightarrow{\Delta r}_1$ lies right on the x-axis, so the angle from the x-axis is zero. which makes sense since $\overrightarrow{\Delta r}_1$ is along the x-axis. Now let's do the x-part of $\overrightarrow{\Delta r}_2$

$$\begin{aligned}\Delta r_{2x} &= \Delta r_2 \cos(\theta)\\ &= 35\,\mathrm{m}\cos(60\,^\circ)\\ &= 17.5\,\mathrm{m}\end{aligned}$$

Not let's find the y-parts. For $\overrightarrow{\Delta r}_1$

$$\begin{aligned}\Delta r_{1y} &= \Delta r_1 \sin(0)\\ &= 0\end{aligned}$$

as we found before. And the y-part of $\overrightarrow{\Delta r}_2$ is

$$\begin{aligned}\Delta r_{2y} &= \Delta r_2 \sin(\theta)\\ &= 35\,\mathrm{m}\sin(60\,^\circ)\\ &= 30.311\,\mathrm{m}\end{aligned}$$

To summarize:

$$\begin{aligned}\Delta r_{1x} &= 40\,\mathrm{m}\\ \Delta r_{2x} &= 17.5\,\mathrm{m}\\ \Delta r_{1y} &= 0\,\mathrm{m}\\ \Delta r_{2y} &= 30.311\,\mathrm{m}\end{aligned}$$

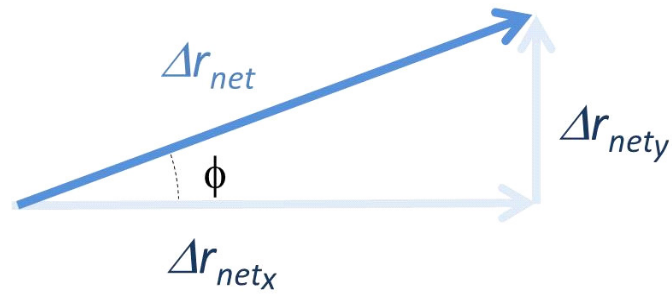
Next we add up all the x-parts

$$\begin{aligned}\Delta r_{net_x} &= \Delta r_{1x} + \Delta r_{2x}\\ &= 40\,\mathrm{m} + 17.5\,\mathrm{m}\\ &= 57.5\,\mathrm{m}\end{aligned}$$

and we add up all the y-parts

$$\begin{aligned}\Delta r_{net_y} &= \Delta r_{1y} + \Delta r_{2y}\\ &= 0 + 30.311\,\mathrm{m}\\ &= 30.311\,\mathrm{m}\end{aligned}$$

and now we have the sides for our final right triangle



and we can find Δr_{net} using the Pythagorean theorem

$$\begin{aligned}
\Delta r_{net} &= \sqrt{\left(\Delta r_{net_x}\right)^2 + \left(\Delta r_{net_y}\right)^2} \\
&= \sqrt{(57.5\,\mathrm{m})^2 + (30.311\,\mathrm{m})^2} \\
&= 65.0\,\mathrm{m}
\end{aligned}$$

But we are not done! Δr_{net} is a vector so we need a direction. Since we used the Greek letter θ already, I called the $\overrightarrow{\Delta r}_{net}$ direction ϕ. So, using our trig knowledge

$$\tan\phi = \frac{\Delta r_{net_y}}{\Delta r_{net_x}}$$

so

$$\begin{aligned}
\phi &= \tan^{-1}\left(\frac{\Delta r_{net_y}}{\Delta r_{net_x}}\right) \\
&= \tan^{-1}\left(\frac{30.311\,\mathrm{m}}{57.5\,\mathrm{m}}\right) \\
&= 0.48513\,\mathrm{rad} \\
&= 27.796\,^\circ \\
&= 27\,^\circ
\end{aligned}$$

What we have done is deeply profound. We have an easy way to find the summation of any number of vectors. The steps are as follows:

1. We convert every vector in the sum into x and y-parts using
$$\begin{aligned}
v_x &= v\cos\theta \\
v_y &= v\sin\theta
\end{aligned}$$
where θ will be different for every vector.

2. Then when we have the x and y-parts, we sum up each set (x or y) separately to find v_{net_x} and v_{net_y}.

3. We then use the Pythagorean theorem

$$v_{net} = \sqrt{v_{net_x}^2 + v_{net_y}^2}$$

and the inverse tangent

$$\phi = \tan^{-1}\left(\frac{v_{net_y}}{v_{net_x}}\right)$$

to convert the sums, v_{net_x} and v_{net_y} into a magnitude and direction for the net vector.

We will do this over and over again in this class and in PH123 and in PH 220 and forever if you are a physics major or a mechanical engineer!

The name "x-part" is not very fancy. So let's give the x and y-parts of vectors a new name. We call the parts of vectors *components* of the vectors. So the x-part is called the x-component of the vector and the y-part is called the y-component of the vector.

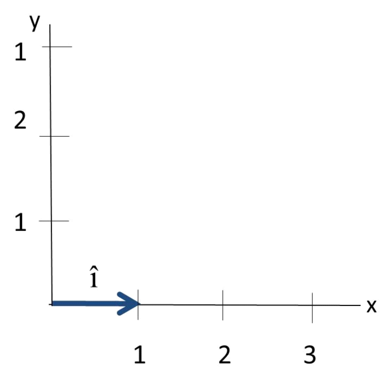
We are now all set to do motion problems in two dimensions.

9 Transition to Two Dimensions

After last lecture, you might have wondered if it wouldn't be just as well to give the components of the net vector and dispense with the magnitude and direction. And indeed, this is a perfectly fine way to express a vector. But we need a little bit of notation to help with this.

unit vectors

To help with expressing a vector as components, consider the following vector.

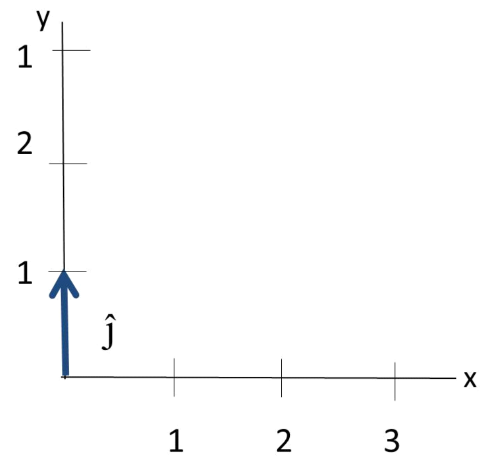


It has a magnitude of exactly 1 with no units. We give it a name, $\hat{\imath}$, pronounced "eye hat." It might sound like a restaurant name, but it is a useful little vector. Using this new vector we could write the x-component of a vector as

$$\Delta \overrightarrow{r}_{net_x} = \Delta r_{net_x} \hat{\imath}$$

the first part, Δr_{net_x} is the magnitude of the vector. The second part, our $\hat{\imath}$, has a direction, and it is along the x-axis. It also has a magnitude of 1, but multiplying anything by 1 does not change a value. So the product $\Delta r_{net_x} \hat{\imath}$ has a magnitude of Δr_{net_x} and a direction along the x-axis. By looking at $\Delta r_{net_x} \hat{\imath}$ you know the magnitude and direction of the component.

Likewise there is another useful vector, $\hat{\jmath}$



that also has a magnitude of 1 with no units, but it points in the y-direction. So we could write the y-component of a vector as

$$\Delta\overrightarrow{r}_{net_y} = \Delta r_{net_y}\hat{\jmath}$$

Although this makes sense, the beauty of $\hat{\imath}$ and $\hat{\jmath}$ may not yet be apparent. Let's write out $\Delta\overrightarrow{r}_{net}$ as the sum of $\Delta\overrightarrow{r}_{net_x}$ and $\Delta\overrightarrow{r}_{net_y}$

$$\Delta\overrightarrow{r}_{net} = \Delta\overrightarrow{r}_{net_x} + \Delta\overrightarrow{r}_{net_y}$$

but we have new expressions for $\Delta\overrightarrow{r}_{net_x}$ and $\Delta\overrightarrow{r}_{net_y}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$, so let's substitute them in

$$\Delta\overrightarrow{r}_{net} = \Delta r_{net_x}\hat{\imath} + \Delta r_{netly}\hat{\jmath}$$

This completely defines a vector. It has both the magnitude

$$\Delta r_{net} = \sqrt{\left(\Delta r_{net_x}\right)^2 + \left(\Delta r_{net_y}\right)^2}$$

and the direction

$$\phi = \tan^{-1}\left(\frac{\Delta r_{net_y}}{\Delta r_{net_x}}\right)$$

contained within it. Sometimes it is much easier to express a vector as a sum of it's components.

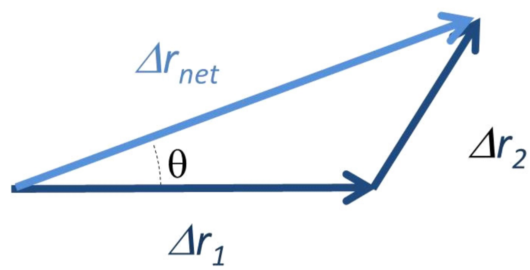
Let's take last time's example and write it in component form.

We had two vectors,

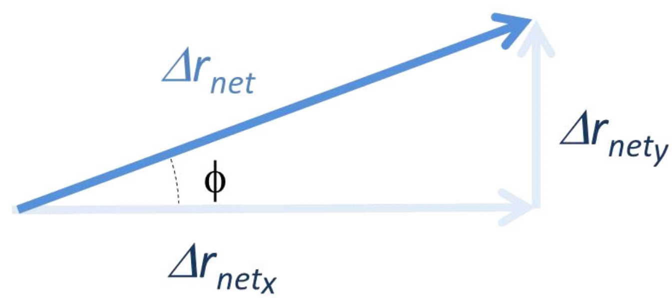
$$\Delta\overrightarrow{r}_1 = 40\,\mathrm{m}\angle 0^\circ$$

where $\angle$ is the symbol for "at the angle," and

$$\Delta\overrightarrow{r}_1 = 35\,\mathrm{m}\angle 60^\circ$$



and we turned this into two easier vectors $\Delta\overrightarrow{r}_{net_x}$ and $\Delta\overrightarrow{r}_{nety}$



and found the magnitude and direction as

$$\Delta\overrightarrow{r}_{net} = 65.0\,\mathrm{m}\angle 27^\circ$$

We got this by finding the x and y-components of $\Delta\overrightarrow{r}_1$ and $\Delta\overrightarrow{r}_2$

$$\begin{aligned}
\Delta r_{1x} &= \Delta r_1 \cos(0) \\
&= \Delta r_1 \\
&= 40\,\mathrm{m} \\
\Delta r_{1y} &= \Delta r_1 \sin(0) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\Delta r_{2x} &= \Delta r_2 \cos(\theta) \\
&= 35\,\mathrm{m}\cos(60^\circ) \\
&= 17.5\,\mathrm{m}
\end{aligned}$$

and finally

$$\begin{aligned}\Delta r_{2y} &= \Delta r_2 \sin(\theta) \\ &= 35\,\mathrm{m}\sin(60\,^\circ) \\ &= 30.311\,\mathrm{m}\end{aligned}$$

Then we added up all the x-parts

$$\begin{aligned}\Delta r_{net_x} &= \Delta r_{1x} + \Delta r_{2x} \\ &= 40\,\mathrm{m} + 17.5\,\mathrm{m} \\ &= 57.5\,\mathrm{m}\end{aligned}$$

and we added up all the y-parts

$$\begin{aligned}\Delta r_{net_y} &= \Delta r_{1y} + \Delta r_{2y} \\ &= 0 + 30.311\,\mathrm{m} \\ &= 30.311\,\mathrm{m}\end{aligned}$$

But now we see that the result could be written as

$$\begin{aligned}\Delta \overrightarrow{r}_{net} &= (\Delta r_{1x} + \Delta r_{2x})\,\hat{\imath} + (\Delta r_{1y} + \Delta r_{2y})\,\hat{\jmath} \\ &= 57.5\,\mathrm{m}\hat{\imath} + 30.311\,\mathrm{m}\hat{\jmath}\end{aligned}$$

Last lecture we found the magnitude and the direction for $\Delta \overrightarrow{r}_{net}$, we could have just left $\Delta \overrightarrow{r}_{net}$ in this new *component form* using unit vectors.

Notice that what we have done is very profound. We have written our original vectors in component form, then added up all the x-components and the y-components separately, and reported the sums as the components of the net vector. We have effectively turned a two-dimensional problem into two one-dimensional problems. This is huge! Because we know how to do one dimensional problems already! By using components, we can turn a new, complicated kind of problem into two easy problems that we know how to do. We will often do this in our class. So it might be good to do another example.

Let's add the following two displacements

$$\Delta \overrightarrow{r_1} = 25\,\mathrm{m} \text{ at } 10\,^\circ$$
$$\Delta \overrightarrow{r_2} = 30\,\mathrm{m} \text{ at } 20\,^\circ$$

To do this problem we need to take components of both vectors, but now we know how to do this:

$$\begin{aligned}
\Delta r_{1x} &= 25\,\mathrm{m}\cos\left(10\,^{\circ}\right) \\
&= 24.62\,\mathrm{m} \\
\Delta r_{1y} &= 25\,\mathrm{m}\sin\left(10\,^{\circ}\right) \\
&= 4.3412\,\mathrm{m} \\
\Delta r_{2x} &= 30\,\mathrm{m}\cos\left(20\,^{\circ}\right) \\
&= 28.191\,\mathrm{m} \\
\Delta r_{2y} &= 30\,\mathrm{m}\sin\left(20\,^{\circ}\right) \\
&= 10.261\,\mathrm{m}
\end{aligned}$$

and to find the net vector, we add up all the x-components

$$\Delta r_{net_x} = 24.62\,\mathrm{m} + 28.191\,\mathrm{m} = 52.811\,\mathrm{m}$$

and we add up all the y-components

$$\Delta r_{net_y} = 4.3412\,\mathrm{m} + 10.261\,\mathrm{m} = 14.602\,\mathrm{m}$$

Then we can write the net displacement as vector in component form

$$\Delta\overrightarrow{r}_{net} = 52.811\,\mathrm{m}\hat{\imath} + 14.602\,\mathrm{m}\hat{\jmath}$$

Notice that $\Delta\overrightarrow{r}_{net}$ has a vector sign. It *is* a vector, not just a magnitude, because we have expressed $\Delta\overrightarrow{r}_{net}$ in terms of a sum of unit vectors. So it must be a vector. Of course we could find the magnitude and direction as well

$$\begin{aligned}
\Delta r_{net} &= \sqrt{\Delta r_{net_x}^2 + \Delta r_{net_y^2}} \\
&= \sqrt{\left(52.811\,\mathrm{m}\right)^2 + \left(14.602\,\mathrm{m}\right)^2} \\
&= 54.793\,\mathrm{m} \\
&= 54.8\,\mathrm{m}
\end{aligned}$$

and

$$\begin{aligned}
\phi &= \tan^{-1}\left(\frac{\Delta r_{net_y}}{\Delta r_{net_x}}\right) \\
&= \tan^{-1}\left(\frac{14.602}{52.811}\right) \\
&= 0.26976\,\mathrm{rad} \\
&= 15.456\,^{\circ} \\
&= 15.5\,^{\circ}
\end{aligned}$$

So we could write $\Delta\overrightarrow{r}_{net}$ as

$$\Delta\overrightarrow{r}_{net} = 54.8\,\mathrm{m}\angle 15.5\,^{\circ}$$

but the component form is just as good. Often we prefer the magnitude and direction form of the vector, because it is easy for us humans to interpret. But both forms are equally valid,

Two-Dimensional Acceleration

We have learned how things move in one dimension. We called the study of motion Kinematics. So we know the Kinematics of one-dimensional motion. But if you have played futbol[4] you know that we can have motion in more than one dimension. We also have studied vectors. With the powerful mathematical notion of vectors, it is time to see how we can study motion in more than one dimension. We will start with two dimensions, but the extension to three or more dimensions is trivial[5] once we understand two dimensions.

Acceleration in Two Dimensions

We learned in one-dimensional kinematics that acceleration was very important for our understanding of motion. It is reasonable to assume that it will be just as important for our understanding of two-dimensional motion. So let's start by reviewing what we know about acceleration. Our equation for acceleration is

$$a_{ave} = \frac{\Delta v}{\Delta t}$$

but now that we know about vectors, we should ask, is acceleration a vector quantity?

Acceleration has a magnitude, we know that. it matters whether the acceleration is in the same direction as the velocity or not.

[4] Americans call this soccer, but if you have played American football, motion in more than one dimension is just as important.

[5] To physicists, "trivial" means that once you have slogged through the hard work you are doing, you should be able to see how to do a little more without it being so much work to understand the little bit more.

So acceleration *does* have a direction . Acceleration must be a vector. We can write our equation for average acceleration as

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t}$$

To get a feeling for what this means, let's practice our vector principles in one-dimension with acceleration. Suppose we have an object moving and the object's velocity is given by the motion diagram.

Let's say that the initial speed is $2\,\mathrm{m/s}$ and the final speed is $6\,\mathrm{m/s}$ and let's say that one second has transpired between our initial and final states, $\Delta t = 1\,\mathrm{s}$. What is the acceleration? We would take

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{6\,\frac{\mathrm{m}}{\mathrm{s}} - 2\,\frac{\mathrm{m}}{\mathrm{s}}}{1\,\mathrm{s}} = 4\frac{\mathrm{m}}{\mathrm{s}^2}$$

and since the answer is positive we would say that the acceleration is to the right.

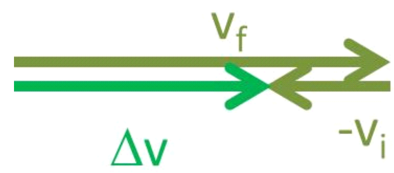
But we can do this with vectors too.

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t}$$

and we know that

$$\Delta\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}_f} - \overrightarrow{\mathbf{v}_i}$$

and we even know how to do this. We take $\overrightarrow{\mathbf{v}_i}$ and we turn it around to make it $-\overrightarrow{\mathbf{v}_i}$ and we add the result,$-\overrightarrow{\mathbf{v}_i}$, to $\overrightarrow{\mathbf{v}_f}$ by placing the tail of $-\overrightarrow{\mathbf{v}_i}$ on the tip of $\overrightarrow{\mathbf{v}_f}$. Then we draw a vector from the tail of $\overrightarrow{\mathbf{v}_f}$ to the tip of $-\overrightarrow{\mathbf{v}_i}$. It might look a little like this



We could measure out the lengths so that we have v_f has a length of 6 and v_i with a length of 2. The we could measure $\Delta\overrightarrow{\mathbf{v}}$ to have a length of 4. But that is a lot of work to do all the measuring. And since we know the magnitude of $\overrightarrow{\mathbf{v}}_f$ and $\overrightarrow{\mathbf{v}}_i$, it is easier to do the math than to draw the picture and measure. We can say that

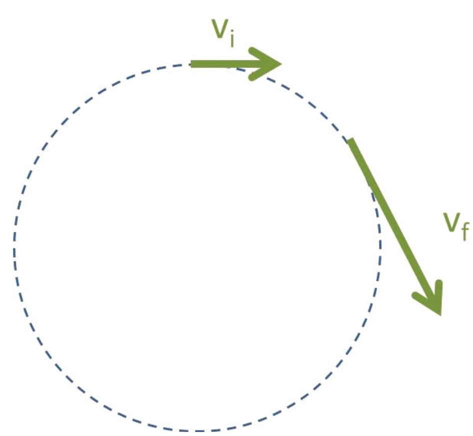
$$\overrightarrow{\mathbf{a}}_{ave} = \frac{4\,\frac{\mathrm{m}}{\mathrm{s}}}{1\,\mathrm{s}} = 4\frac{\mathrm{m}}{\mathrm{s}^2}$$

and none of this is surprising. This is really just what we have been doing all along. But we have forgotten something, we have to give a direction. And we can use our new unit vectors to provide the direction.

$$\overrightarrow{\mathbf{a}}_{ave} = 4\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\imath}$$

The drawing does remind us of the need for a direction.

You may think that all we have done is complicate things, but now let's do a problem in two dimensions. Let's consider an object that is moving in a circle. We can use the same magnitudes for v_i and v_f as before.



$v_i = 2\,\mathrm{m}/\,\mathrm{s}$ and $v_f = 6\,\mathrm{m}/\,\mathrm{s}$ with the same $\Delta t = 1\,\mathrm{s}$. But now the vectors are pointing in different directions in two dimensions.

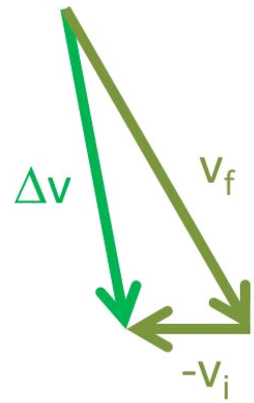
Still, we will do just the same thing we did before. Our acceleration will be

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t}$$

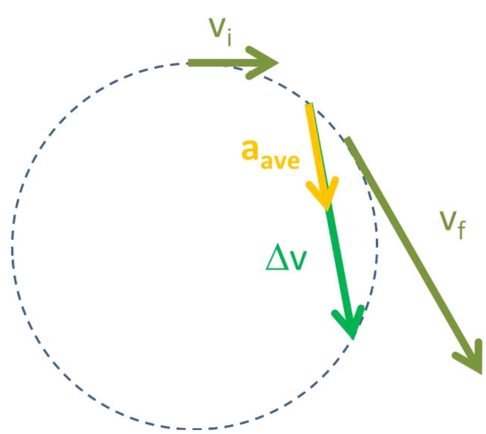
where

$$\Delta\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}_f} - \overrightarrow{\mathbf{v}_i}$$

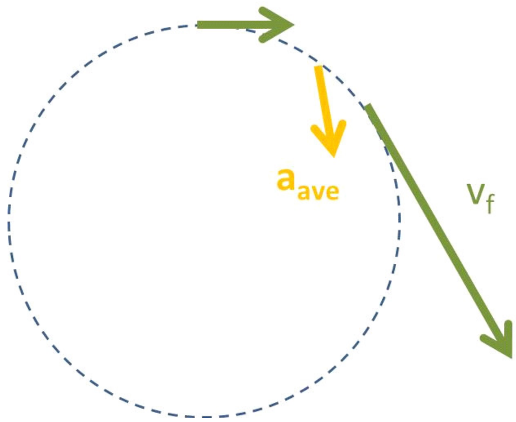
To do this we take $\overrightarrow{\mathbf{v}_i}$ and we turn it around to make it $-\overrightarrow{\mathbf{v}_i}$ and then add the result to $\overrightarrow{\mathbf{v}_f}$ by placing the tail of $-\overrightarrow{\mathbf{v}_i}$ on the tip of $\overrightarrow{\mathbf{v}_f}$. Then we draw a vector from the tail of $\overrightarrow{\mathbf{v}_f}$ to the tip of $-\overrightarrow{\mathbf{v}_i}$. It might look a little like this



Notice that I recopied my vectors. I did not do the vector addition on the original diagram. This makes it much easier to do the vector addition. Once we know the length and direction of $\Delta\overrightarrow{\mathbf{v}}$, we divide by Δt and we have our acceleration. Note that Δt does not have a direction. The direction of our acceleration must come from $\Delta\overrightarrow{\mathbf{v}}$. If we draw our $\Delta\overrightarrow{\mathbf{v}}$ and our $\overrightarrow{\mathbf{a}}_{ave}$ on our original diagram, it is easy to see they point the same direction.



Usually we just place $\overrightarrow{\mathbf{a}}_{ave}$ on our diagram in between v_i and v_f along the trajectory. That is because it is an average value.



But what do we do about the magnitude of the vector $\overrightarrow{\mathbf{a}}_{ave}$? In one dimension I could just add $v_f - v_i$ and divide by Δt. But this won't work here. Look at the length Δv compared to the length $v_{f.}$ They are nearly the same. This is really different than when they were all in the same dimension.

Working in components

The answer to our dilemma from the last section is to do our math in components of the vectors.

Think, back to our one-dimensional problem. All we had to do was just add up the numbers to get Δv (remember one was negative, so we added a negative to subtract). Wouldn't it be great if we could reduce our difficult two-dimensional problem to two one-dimensional problems? Then we would know how to do the problem, and it would be easy!

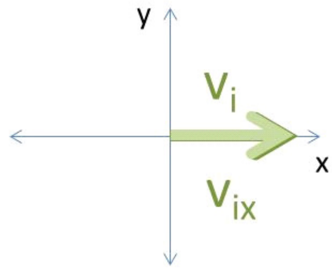
And that is just what we are going to do. suppose we have

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta \overrightarrow{\mathbf{v}}}{\Delta t}$$

and

$$\Delta \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}_f} - \overrightarrow{\mathbf{v}_i}$$

as before. But we could make vector components for $\overrightarrow{\mathbf{v}_f}$ and $\overrightarrow{\mathbf{v}_i}$. Our vector $\overrightarrow{\mathbf{v}_i}$ is shown below.



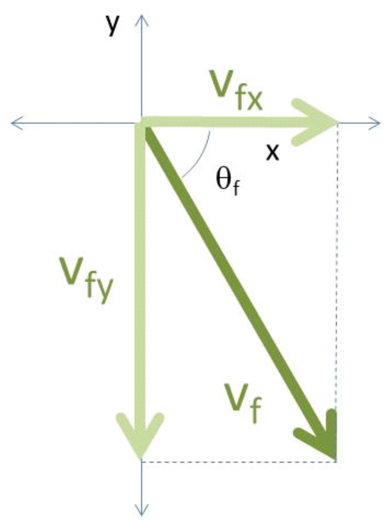
We can use our basic equations for taking components of vectors to find the components of $\overrightarrow{\mathbf{v}_i}$

$$\begin{aligned} v_{ix} &= v_i \cos\theta_i \\ v_{iy} &= v_i \sin\theta_i \end{aligned}$$

but $\theta_i = 0$ (see the diagram) so

$$\begin{aligned} v_{ix} &= v_i = 2\frac{\text{m}}{\text{s}} \\ v_{iy} &= 0 \end{aligned}$$

Now let's do the same process for $\overrightarrow{\mathbf{v}_f}$

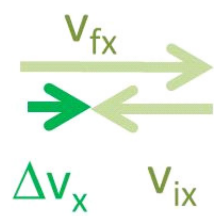


$$\begin{aligned} v_{fx} &= v_f \cos\theta_f \\ v_{fy} &= v_f \sin\theta_f \end{aligned}$$

and this one is harder because θ_f is not zero. Suppose $\theta_f = -55\,^\circ$ then

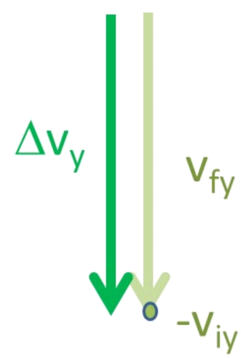
$$\begin{aligned} v_{fx} &= v_f \cos\theta_f = 6\frac{\mathrm{m}}{\mathrm{s}}\cos\left(-55\,^\circ\right) = 3.441\,5\frac{\mathrm{m}}{\mathrm{s}} \\ v_{fy} &= v_f \sin\theta_f = 6\frac{\mathrm{m}}{\mathrm{s}}\sin\left(-55\,^\circ\right) = -4.914\,9\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and now we can complete our two one-dimensional problems. First for x



$$\begin{aligned} \Delta v_x &= v_{fx} - v_i \\ &= 3.441\,5\frac{\mathrm{m}}{\mathrm{s}} - 2\frac{\mathrm{m}}{\mathrm{s}} \\ &= 1.441\,5\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and then for y



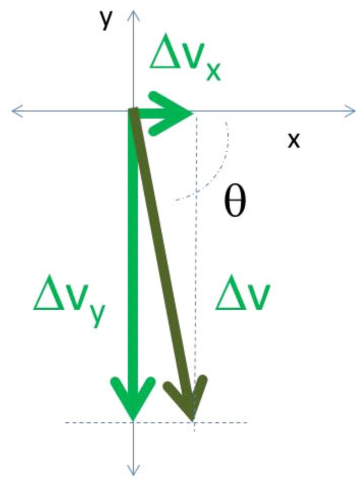
$$\begin{aligned} \Delta v_y &= v_{fy} - v_{iy} \\ &= -4.914\,9\frac{\mathrm{m}}{\mathrm{s}} - 0\frac{\mathrm{m}}{\mathrm{s}} \\ &= -4.914\,9\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and now we have Δv_x and Δv_y. But really, we wanted $\Delta\overrightarrow{\mathbf{v}}$. We need to combine our x and y-component solutions into a solution for the two-dimensional problem. And we know there are two ways we could write it. We could give components and unit vectors, or magnitude and direction. Let's do both. The first is easy

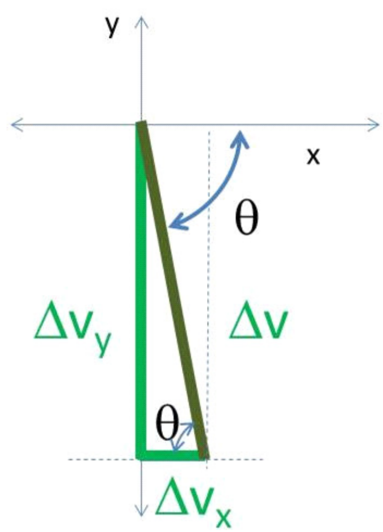
$$\Delta\overrightarrow{\mathbf{v}}=1.441\,5\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}-4.914\,9\frac{\mathrm{m}}{\mathrm{s}}\hat{\jmath}$$

And this is our answer! We have made a two-dimensional motion problem into two one-dimensional problems, and then combined the results of those two one-dimensional problems to form the answer of the two-dimensional problem.

But, of course, there is a second way to do the combination of our two one-dimensional results. We can find the magnitude and direction of $\Delta\overrightarrow{\mathbf{v}}$. To do this, we need to think a bit. The vector $\Delta\overrightarrow{\mathbf{v}}$ makes an angle, $-\theta$, with respect to the x-axis.



Note that $\Delta\mathbf{v}$, $\Delta v_{x,}$ and Δv_y form the sides of a right triangle



so to find Δv, we can use the Pythagorean theorem

$$\Delta v = \sqrt{\Delta v_x^2 + \Delta v_y^2}$$

and since it is a right triangle, we could find the direction of Δv using a tangent function

$$\tan\theta = \frac{\Delta v_y}{\Delta v_x}$$

so that

$$\theta = \tan^{-1}\left(\frac{\Delta v_y}{\Delta v_x}\right)$$

Let's try these for our case,

$$\begin{aligned}\Delta v &= \sqrt{\left(1.4415\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(4.9149\frac{\mathrm{m}}{\mathrm{s}}\right)^2} \\ &= 5.1219\frac{\mathrm{m}}{\mathrm{s}}\end{aligned}$$

and

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{-4.9149\frac{\mathrm{m}}{\mathrm{s}}}{1.4415\frac{\mathrm{m}}{\mathrm{s}}}\right) \\ &= -1.2855\,\mathrm{rad} \\ &= -73.654\,^{\circ}\end{aligned}$$

so we could report like this our vector $\Delta\overrightarrow{\mathbf{v}}$ like this

$$\Delta\overrightarrow{\mathbf{v}} = 5.1\frac{\mathrm{m}}{\mathrm{s}}\angle -73.7\,^{\circ}$$

Then our acceleration is given by

$$\begin{aligned}\overrightarrow{a} &= \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t} \\ &= \frac{5.1\frac{\mathrm{m}}{\mathrm{s}}}{1\,\mathrm{s}}\angle -73.7\,^{\circ} \\ &= 5.1\frac{\mathrm{m}}{\mathrm{s}^2}\angle -73.7\,^{\circ}\end{aligned}$$

Which is our answer. This was a little harder, but often conveys the meaning of the result better. But just the same, the magnitude and direction form of the solution is equivalent to

$$\overrightarrow{a} = 1.4\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\imath} - 4.9\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\jmath}$$

and either representation would work.

Again let me mention that we have done something profound. We have split a two-dimensional problem into two one-dimensional problems. That made it so we could solve the problem with what we already knew how to do. We did this by using components of the vectors and treating the x-components as one one-dimensional problem and the y-components as another one dimensional problem. Then at the very end we combined the results of our x-problem and our y-problem into the final vector. Doing this is the heart and soul of two-dimensional kinematics. And that is what we will take up in the next lecture.

Instantaneous Acceleration

Like instantaneous velocity, we can define the instantaneous acceleration

Definition 9.1 *The instantaneous acceleration a is the limit of the average acceleration as the time interval Δt goes to zero.*

Algebraically this is

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{9.1}$$

If we think about it a moment.

> *The instantaneous acceleration is the slope of a tangent line on a graph of v vs. t.*
> (think of how we found the instantaneous velocity was the slope of a tangent to the position vs. time graph).
> Like with instantaneous velocity, the instantaneous acceleration requires us to know the velocity as a function of time.
> In the next lecture, we will try this with our circular motion example.

10 Two-Dimensional Motion

Last lecture, we learned that we could do two-dimensional problems by splitting them into two one-dimensional problems. The mechanism for doing this was to find components of all the vectors. Then deal with all the x-components and all the y-components separately. Finally we combine the results of the x-part of the problem and the y-part of the problem together for a two-dimensional answer. What we have done so far is quite profound. For a two-dimensional problem, instead of making new mathematical techniques for solving problems, we instead just reduced our two-dimensional problem into two one-dimensional problems by taking vector components. Then we could use all we learned about one-dimensional motion to solve the two parts of our problem. To finish off the problem, we combined the individual one-dimensional solutions using simple trigonometry to form our final two-dimensional solution.

Two-Dimensional Kinematics

Armed with this technique, let's study the motion of things in two-dimensions in more detail.

Let's start with displacement in two-dimensions

$$\Delta \overrightarrow{\mathbf{r}} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$$

Notice that our displacement, itself, can be split into components.

And recall that average velocity is just

$$\begin{aligned} \overrightarrow{\mathbf{v}}_{ave} &= \frac{\Delta \overrightarrow{\mathbf{r}}}{\Delta t} = \frac{\Delta x\hat{\imath} + \Delta y\hat{\jmath}}{\Delta t} \\ &= \frac{\Delta x\hat{\imath}}{\Delta t} + \frac{\Delta y\hat{\jmath}}{\Delta t} \end{aligned}$$

but we can recognize $\Delta x/\Delta t$ as v_x and $\Delta y/\Delta t$ ad v_y so

$$\overrightarrow{\mathbf{v}}_{ave} = v_x\hat{\imath} + v_y\hat{\jmath}$$

and we also know that

$$\begin{aligned}\overrightarrow{\mathbf{a}}_{ave} &= \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t} = \frac{\Delta v_x\hat{\imath} + \Delta v_y\hat{\jmath}}{\Delta t}\\ &= \frac{\Delta v_x\hat{\imath}}{\Delta t} + \frac{\Delta v_y\hat{\jmath}}{\Delta t}\\ &= a_x\hat{\imath} + a_y\hat{\jmath}\end{aligned}$$

So our separation of our two-dimensional problem into two one-dimensional problems is totally justified. It works for displacement, velocity, and acceleration. All the motion can be described in terms of components.

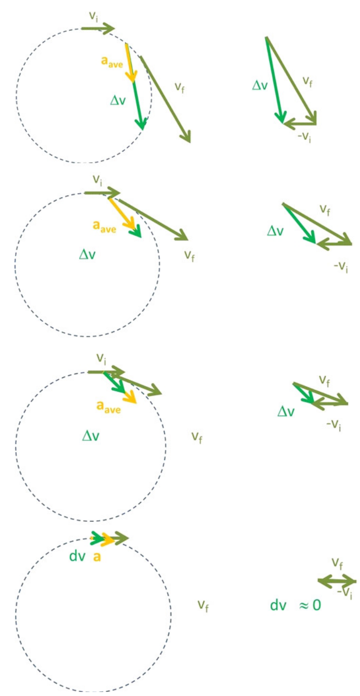
You are probably wondering if all this still works for instantaneous velocities and accelerations. And the answer is yes. We simply take the limit as $\Delta t \to 0$ and

$$\overrightarrow{\mathbf{v}} = \lim_{\Delta t\to 0}\overrightarrow{\mathbf{v}}_{ave} = \frac{d\overrightarrow{\mathbf{r}}}{dt} = \frac{dx\hat{\imath}}{dt} + \frac{dy\hat{\jmath}}{dt}$$

and

$$\begin{aligned}\overrightarrow{\mathbf{a}} &= \lim_{\Delta t\to 0}\overrightarrow{\mathbf{a}}_{ave} = \frac{d\overrightarrow{\mathbf{v}}}{dt} = \frac{dv_x\hat{\imath}}{dt} + \frac{dv_y\hat{\jmath}}{dt}\\ &= a_x\hat{\imath} + a_y\hat{\jmath}\end{aligned}$$

This brings up an interesting question, what happens to our diagram for motion as $\Delta t \to 0$? In the next figure, we shorten Δt by a factor of about 2. In the last frame, $\Delta t \to 0$. So Δt gets smaller and smaller as we go down the page in the figure. Notice that Δv also gets smaller and smaller until in the last figure $\Delta v \to 0$. But notice that the acceleration is not zero. This is what we expect from the math we did above. dt may be small but so is $d\overrightarrow{\mathbf{v}}$ so the ratio of the two is not zero.

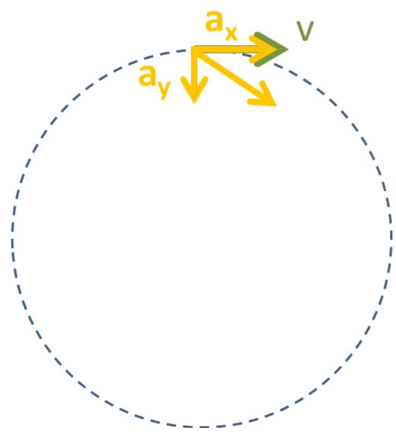


But now we have an instantaneous a vector that is right on top of the v_i vector. That is how we will draw the instantaneous value of the acceleration. Also notice that as we move along the trajectory (the path the object follows), the velocity vectors are always tangent to the trajectory path. This is also just what we expect from the math.

$$\overrightarrow{\mathbf{v}} = \frac{dx\hat{\imath}}{dt} + \frac{dy\hat{\jmath}}{dt}$$

and dx/dt is the slope of the trajectory in the x direction and dy/dt is the slope of the line in the y direction. Combining them should give the slope along the trajectory, and that slope will be tangent to the actual path.

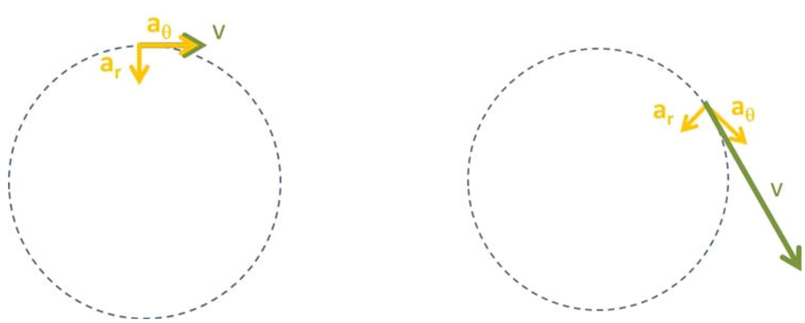
This gives us an idea. We know we can express our acceleration a in terms of a_x and a_y,



but for our circular problem, we realize that as we go around the circle, the velocity will stay tangent to the circle. Think of the component of the acceleration a_x in the picture. This component is making the object speed up because it is in the same direction as the speed. But what is the other component, a_y doing? Think that acceleration is a change in velocity

$$\overrightarrow{\mathbf{a}}_{ave} = \frac{\Delta\overrightarrow{\mathbf{v}}}{\Delta t}$$

so acceleration can change speed *or direction.* The x component is changing speed, but the y-component is making the object turn–change direction. It might be convenient to keep our acceleration in terms of the part that makes the object speed up and the part that makes the object turn. The part that is parallel to the velocity is the speed-up-part. The part perpendicular to the velocity is the turn-part. We could define a component of the acceleration that is tangent to the trajectory. That will be the speed-up-part. And we could define a component perpendicular to the tangent.



This way we always know what is doing the speeding up and what is doing the turning. Notice that the turning-part always points to the center of the turn. A line from the center to the object would be along the same line. And we call a line from the center of a circle to the edge of the circle a “radius” so it is tradition to call the turning part of the acceleration the "radial acceleration" and the speeding-up part the "tangential acceleration." These names fit our "speeding-up" and "turning" acceleration parts even when the object has moved to another part of the circle. A math savvy student would immediately recognize these terms (radial and tangential) from polar coordinates. We will often use polar coordinates for motion in a circle.

We remember that we had a set of equations for one-dimensional motion under constant acceleration.

$$\begin{aligned} \Delta x &= v_i \Delta t + \frac{1}{2} a \Delta t^2 \\ v_f &= v_i + a\Delta t \\ v_f^2 &= v_i^2 + 2a\Delta x \end{aligned}$$

But now we have acceleration in two dimensions. Still, if our acceleration is constant, then both a_x and a_y will be constant. And we can simply split our problem into two problems, one for each dimension. But we will need twice as many equations, one whole set for each one-dimensional problem part.

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x \Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y \Delta t^2 \\ v_{fx} = v_{ix} + a_x \Delta t & v_{fy} = v_{iy} + a_y \Delta t \\ v_{fx}^2 = v_{ix}^2 + 2a_x \Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y \Delta y \end{array}$$

Let's try such a problem. Suppose we have Duke Mudwalker's Q-wing fighter taking off. The Q-wing must go up and forward at the same time. Suppose the Q-wing has $a_x = 3\,\mathrm{m/s^2}$ and $a_y = 6\,\mathrm{m/s^2}$. What is the speed of the Q-wing after $\Delta t = 2\,\mathrm{s}$?

This is a two-dimensional motion problem with constant acceleration. We need to split our problem into two one-dimensional problems. In the x-direction we have

$$\begin{aligned} v_{ix} &= 0 \\ a_x &= 3\frac{\mathrm{m}}{\mathrm{s}^2} \\ \Delta t &= 2\,\mathrm{s} \end{aligned}$$

and in the y-direction

$$\begin{aligned} v_{iy} &= 0 \\ a_y &= 6\frac{\mathrm{m}}{\mathrm{s}^2} \\ \Delta t &= 2\,\mathrm{s} \end{aligned}$$

Notice that the Δt values must be the same! We need two sets of equations

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\ v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \end{array}$$

Let's do the x part first. If we underline the parts we know

$$\Delta x = \underline{v_{ix}}\underline{\Delta t} + \frac{1}{2}\underline{a_x}\underline{\Delta t}^2$$
$$v_{fx} = \underline{v_{ix}} + \underline{a_x}\underline{\Delta t}$$
$$v_{fx}^2 = \underline{v_{ix}^2} + 2\underline{a_x}\Delta x$$

We can see that the second equation in the x set will give us the final x speed

$$v_{fx} = v_{ix} + a_x\Delta t$$
$$\begin{aligned} v_{fx} &= a_x\Delta t = \left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s}) \\ &= 6.0\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Now let's do the y part.

$$\Delta y = \underline{v_{iy}}\underline{\Delta t} + \frac{1}{2}\underline{a_y}\underline{\Delta t}^2$$
$$v_{fy} = \underline{v_{iy}} + \underline{a_y}\underline{\Delta t}$$
$$v_{fy}^2 = \underline{v_{iy}^2} + 2\underline{a_y}\Delta y$$

Again the second equation in the set will work

$$v_{fy} = v_{iy} + a_y\Delta t$$
$$v_{fy} = 0 + a_y\Delta t = \left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s}) = 12\frac{\mathrm{m}}{\mathrm{s}}$$

then our the magnitude of the final velocity will be

$$\begin{aligned} v_f &= \sqrt{\left(6.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(12\frac{\mathrm{m}}{\mathrm{s}}\right)^2} \\ &= 13.416\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and the direction will be

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{12\frac{\mathrm{m}}{\mathrm{s}}}{6.0\frac{\mathrm{m}}{\mathrm{s}}}\right) \\ &= 1.1071\,\mathrm{rad} \\ &= 63.432^\circ \end{aligned}$$

We followed our pattern for solving a two-dimensional motion problem:

1. Split the two-dimensional problem into two one-dimensional problems by taking components of the vectors using the general form
$$\begin{aligned} V_x &= V\cos\theta \\ V_y &= V\sin\theta \end{aligned}$$
where θ is measured from the positive x-axis.

2. Solve the two one-dimensional problems separately

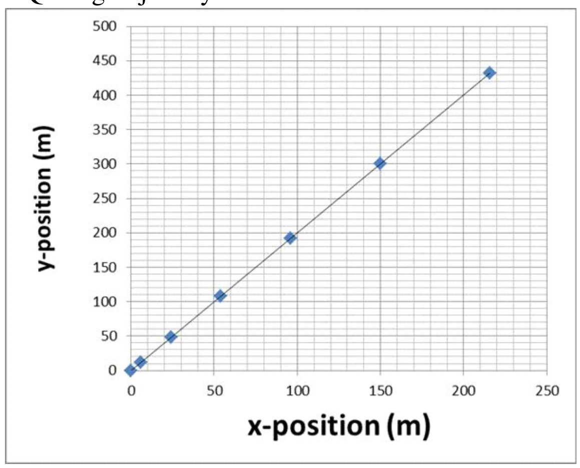
3. Combine the results of the one-dimensional problems together using

$$V = \sqrt{V_x^2 + V_y^2}$$

and

$$\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$$

for the vector V that you are solving for, whether velocity, acceleration, or even displacement. If we use the first equation in our sets for both x and y we can get the position vs. time. A plot of the x and y positions for each time gives us a trajectory plot. A plot of the Q-wing trajectory looks like this.



Is it reasonable that the Q-wing goes in a straight line?

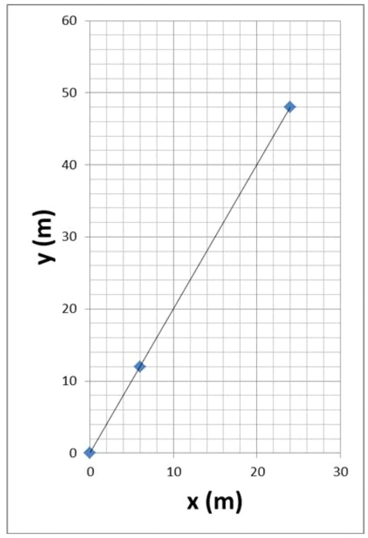
We can check this by doing a few calculations. After $\Delta t = 2\,\mathrm{s}$

$$\begin{aligned}\Delta x &= (0)\,(2\,\mathrm{s}) + \frac{1}{2}\left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s})^2 \\ &= 6.0\,\mathrm{m} \\ \Delta y &= (0)\,(2\,\mathrm{s}) + \frac{1}{2}\left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)(2\,\mathrm{s})^2 \\ &= 12.0\,\mathrm{m}\end{aligned}$$

and after $\Delta t = 4\,\mathrm{s}$

$$\begin{aligned}\Delta x &= (0)\,(4\,\mathrm{s}) + \frac{1}{2}\left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4\,\mathrm{s})^2 \\ &= 24.0\,\mathrm{m} \\ \Delta y &= (0)\,(4\,\mathrm{s}) + \frac{1}{2}\left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)(4\,\mathrm{s})^2 \\ &= 48.0\,\mathrm{m}\end{aligned}$$

and we can plot these

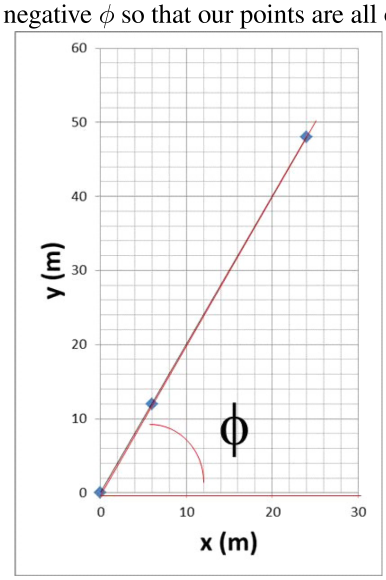


to see that we are right. It is linear.

A moment's though will tell us that this has to be right. Suppose we rotated our axes by an angle

$$\begin{aligned}\phi &= \tan^{-1}\left(\frac{6\frac{\mathrm{m}}{\mathrm{s}^2}}{3\frac{\mathrm{m}}{\mathrm{s}^2}}\right)\\ &= 1.107\,1\,\mathrm{rad}\\ &= 63.432\,^{\circ}\end{aligned}$$

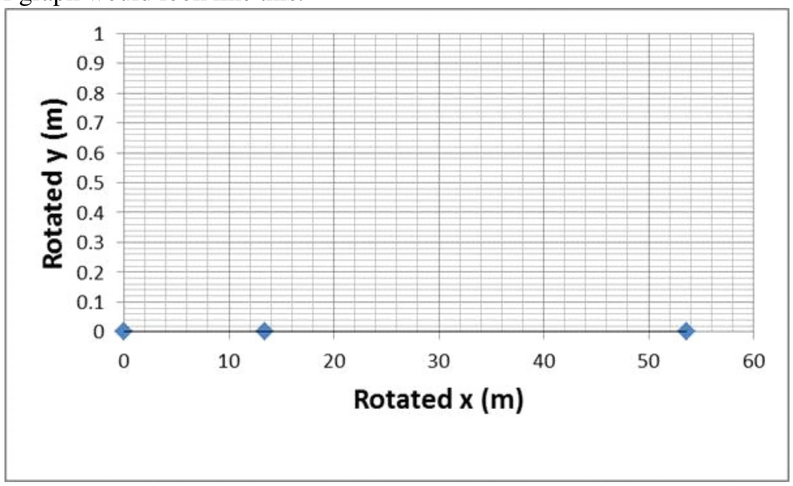
We'll, say we rotate by a negative ϕ so that our points are all on the x axes.



Then our acceleration would be a constant

$$a = \sqrt{\left(3\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2 + \left(6\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2}$$
$$= 6.7082\frac{\mathrm{m}}{\mathrm{s}^2}$$

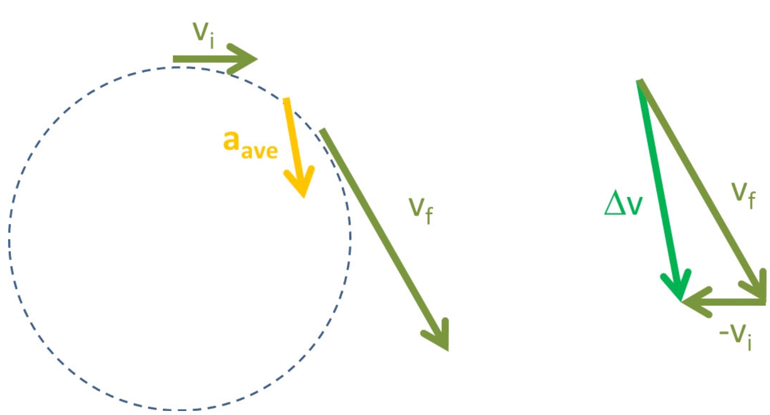
and our graph would look like this.



so indeed, we should have a straight line.

But two dimensional problems aren't always so easy. What if the Q-wing was already moving? or if it only experienced acceleration in one direction?

Let's try a more difficult problem. Suppose we have a car moving on a circular track. When we first observe the car it is going $20\,\mathrm{m/s}$ at $0\,°$. 7 seconds later, the car is going $53\,\mathrm{m/s}$ at $-50\,°$. What is the average acceleration of the car?



This is a two dimensional problem. I don't know if the acceleration is constant or not. All I have is a before and after picture. And we want an average acceleration, so I will call this an average motion problem.

Our basic equations would be

$$\overrightarrow{v}_{ave} = \frac{\Delta \overrightarrow{r}}{\Delta t}$$
$$\overrightarrow{a}_{ave} = \frac{\Delta \overrightarrow{v}}{\Delta t}$$

but we have to split these into x and y-parts. So let's write these as

$$\begin{aligned} v_{ave_x} &= \frac{\Delta x}{\Delta t} \\ v_{avey} &= \frac{\Delta y}{\Delta t} \\ a_{ave_x} &= \frac{\Delta v_x}{\Delta t} \\ a_{ave_y} &= \frac{\Delta v_y}{\Delta t} \end{aligned}$$

We have split our two-dimensional equations into two one-dimensional equations.

We have several known values from the problem statement.

$$\begin{aligned} v_i &= 20\frac{\mathrm{m}}{\mathrm{s}} \\ \theta_i &= 0^\circ \\ v_f &= 53\frac{\mathrm{m}}{\mathrm{s}} \\ \theta_f &= -50^\circ \\ \Delta t &= 7\,\mathrm{s} \end{aligned}$$

but we need to split these initial values into x and y-parts. To do this we need to include our vector components equation set.

$$\begin{aligned} v_x &= v\cos\theta \\ v_y &= v\sin\theta \end{aligned}$$
$$v = \sqrt{v_x^2 + v_y^2}$$
$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

We use these equations to turn our initial and final velocity vectors into initial and final velocity x and y-parts.

$$\begin{aligned} v_{ix} &= v_i\cos\theta_i \\ v_{iy} &= v_i\sin\theta_i \end{aligned}$$

$$\begin{aligned} v_{fx} &= v_f \cos\theta_f \\ v_{fy} &= v_f \sin\theta_f \end{aligned}$$

Now we can attempt to solve the problem. We are going to want a magnitude and a direction or at least the component form for the average acceleration. So our plan should be to find the x and the y parts of the acceleration, and then combine them for the total acceleration.

$$\overrightarrow{a}_{ave} = a_{ave_x}\hat{\imath} + a_{ave_y}\hat{\jmath}$$

We need to solve two-one dimensional problems, one for a_{ave_x} and one for a_{ave_y}. Let's start with a_{ave_x}.

$$\begin{aligned} a_{ave_x} &= \frac{\Delta v_x}{\Delta t} \\ &= \frac{v_{fx} - v_{ix}}{\Delta t} \\ &= \frac{\underline{\underline{v_f}}\cos\underline{\theta_f} - \underline{v_i}\cos\underline{\theta_i}}{\underline{\Delta t}} \end{aligned}$$

and it looks like we know all the parts, so we have solved for a_{ave_x}. Now for a_{ave_y}

$$\begin{aligned} a_{ave_y} &= \frac{\Delta v_y}{\Delta t} \\ &= \frac{v_{fy} - v_{iy}}{\Delta t} \\ &= \frac{\underline{\underline{v_f}}\sin\underline{\theta_f} - \underline{v_i}\sin\underline{\theta_i}}{\underline{\Delta t}} \end{aligned}$$

so we could report symbolically

$$\overrightarrow{a}_{ave} = \frac{\underline{\underline{v_f}}\cos\underline{\theta_f} - \underline{v_i}\cos\underline{\theta_i}}{\underline{\Delta t}}\hat{\imath} + \frac{\underline{\underline{v_f}}\sin\underline{\theta_f} - \underline{v_i}\sin\underline{\theta_i}}{\underline{\Delta t}}\hat{\jmath}$$

We have some zeros. So let's use them

$$\overrightarrow{a}_{ave} = \frac{\underline{\underline{v_f}}\cos\underline{\theta_f} - \underline{v_i}\cos(0\,^\circ)}{\underline{\Delta t}}\hat{\imath} + \frac{\underline{\underline{v_f}}\sin\underline{\theta_f} - \underline{v_i}\sin(0\,^\circ)}{\underline{\Delta t}}\hat{\jmath}$$

and we know that $\cos(0\,^\circ) = 1$ and $\sin(0\,^\circ) = 0$ so we can write our solution as

$$\overrightarrow{a}_{ave} = \frac{\underline{\underline{v_f}}\cos\underline{\theta_f} - \underline{v_i}}{\underline{\Delta t}}\hat{\imath} + \frac{\underline{\underline{v_f}}\sin\underline{\theta_f}}{\underline{\Delta t}}\hat{\jmath}$$

or

$$\overrightarrow{a}_{ave} = \frac{53\frac{\mathrm{m}}{\mathrm{s}}\cos(-50\,^\circ) - 20\frac{\mathrm{m}}{\mathrm{s}}}{7\,\mathrm{s}}\hat{\imath} + \frac{53\frac{\mathrm{m}}{\mathrm{s}}\sin(-50\,^\circ)}{7\,\mathrm{s}}\hat{\jmath}$$

$$\overrightarrow{a}_{ave} = 2.0\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\imath} - 5.8\frac{\mathrm{m}}{\mathrm{s}^2}\hat{\jmath}$$

and this seems to make sense. From looking at the $\Delta\overrightarrow{v}$ direction we can see that $\overrightarrow{a}_{ave}$ should point to the right a little and down a little more.

Of course we could report this in magnitude and direction form as well

$$
\begin{aligned}
a_{ave} &= \sqrt{\left(2.0\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2 + \left(-5.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2} \\
&= 6.1351\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}
$$

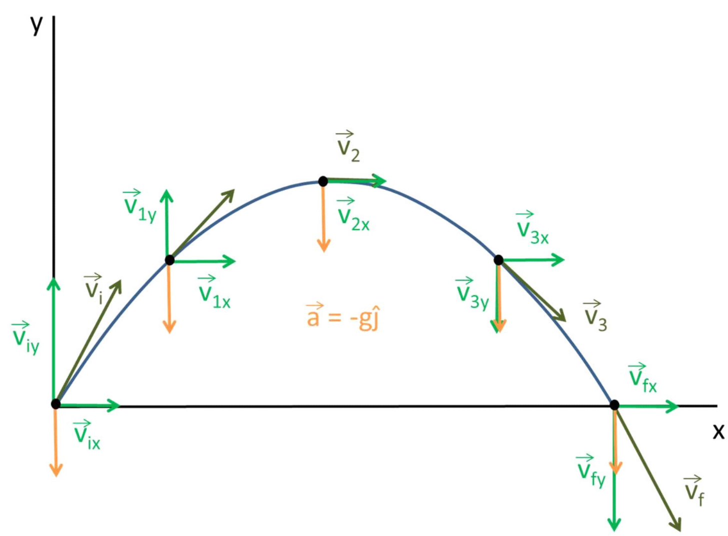
and

$$
\begin{aligned}
\phi &= \tan^{-1}\left(\frac{-5.8\frac{\mathrm{m}}{\mathrm{s}^2}}{2.0\frac{\mathrm{m}}{\mathrm{s}^2}}\right) \\
&= -1.2387\,\mathrm{rad} \\
& \quad -70.972\,^{\circ}
\end{aligned}
$$

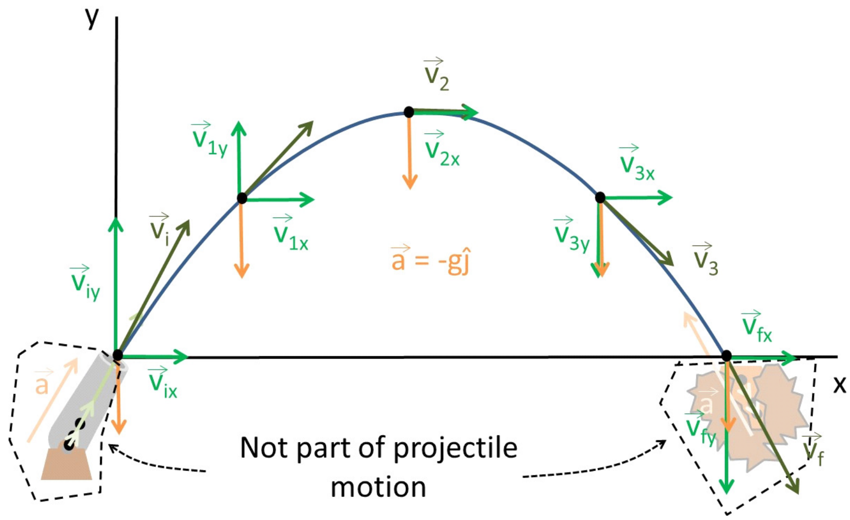
which seem reasonable.

Projectile Motion

Consider two people playing catch. One person throws the ball toward the other person. The ball rises in the air as it is thrown. As it travels, it reaches a maximum height, and then drops down into the glove of the catcher.



Let's look at this in some detail. First off, you may wonder, how did this motion start? Well of course the person throwing the ball must accelerate the ball. In the next figure, we have a cannon shooting a ball as our staring point.



While the ball is in the cannon, there is an acceleration acting on the ball. Then there is the acceleration due to gravity while the ball is in flight. And then there is a different acceleration as the ball crashes into the ground. Our kinematic equations can only deal with one constant acceleration at a time. So let's pick just one of these accelerations to do in this problem, the one where the ball is in flight. We will consider the ball's motion only after it is moving.

For our person playing catch we could say that we are studying the motion after the hand releases the ball. We will also stop our analysis of this motion before the ball hits the ground, or the other person's had if the person catches the ball.

In the case of the cannon ball, the ball would probably embed itself in the ground. Clearly there would be a different acceleration if the ball is slowing because it is plowing through the ground. We want to limit our study to just the part of the motion that is free-fall, with an acceleration of $\overrightarrow{a} = -g\hat{\jmath}$. This is a little like our rocket problem we did a few lectures ago. We split the problem into sections with, each with its own constant acceleration. We just want the middle part where the ball is flying and it experiences an acceleration of $\overrightarrow{a} = -g\hat{\jmath}$.

Notice this is a two-dimensional problem! So of course we would try to turn it into two one-dimensional problems.

First look at the horizontal motion. The figure above shows the velocity of the ball broken into components. Carefully look at the velocity component in the x direction. Notice the magnitude (size) of the vector $\overrightarrow{\mathbf{v}}_x$. It does not change. Is that a surprise?

Well, not really. To see where velocity will change we look for acceleration. We know there is an free-fall acceleration, $\overrightarrow{a} = -g\hat{\jmath}$. The $\hat{\jmath}$ tells us that this acceleration is all in the y-direction. We could say that

$$a_y = -g$$

and that is all the acceleration we have for our part of the motion that we are studying. So we know that

$$a_x = 0$$

Since $a_x = \Delta v_x/\Delta t$ so if $a_x = 0$ then $\Delta v_x = 0$. The x-component of the velocity cannot change.

Let's give this special case of free-fall with $a_y = -g$ and $a_x = 0$ its own problem type name. We will call it *projectile motion*.

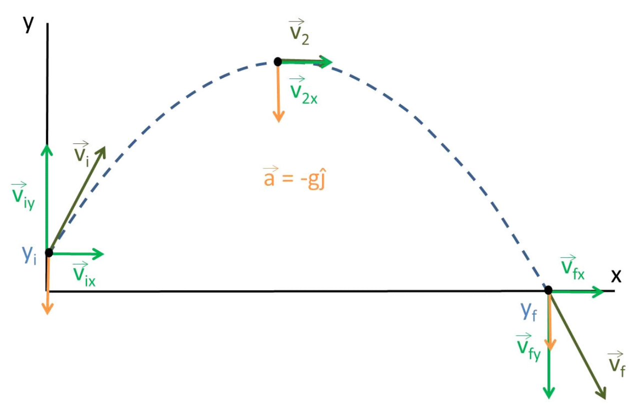
Ball Drop Demonstration

Let's try an example:

Suppose we have a small spring cannon that launches a metal ball with an initial speed of $7.00\,\mathrm{m/s}$ at an initial angle of $60.0\,^\circ$ starting from $0.270\,\mathrm{m}$ off the ground. What will the final speed be just before the ball hits the ground?

This is a PT=Projectile-motion problem. The ball is in free-fall with $a_y = -g$ and $a_x = 0$.

For our picture, let's draw the motion of the ball and mark the initial and final positions, velocities, velocity components, and accelerations.



For our variables we know

$$
\begin{aligned}
x_i &= 0 \\
y_i &= 0.270\,\mathrm{m} \\
y_f &= 0 \\
v_i &= 7.00\frac{\mathrm{m}}{\mathrm{s}} \\
\theta_i &= 60^\circ \\
a_y &= -g \\
a_x &= 0 \\
g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}
$$

For basic equations we will need two sets of kinematic equations, one for the x-part and one for the y-part.

where the angle θ_o is the angle from the horizontal, the angle at which the ball is thrown. We call θ_o the projection angle.

Then we have two sets of our motion equations

$$
\begin{array}{ll}
\Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\
v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\
v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y
\end{array}
$$

We also may need

$$
\begin{aligned}
\Delta y &= y_f - y_i \\
\Delta x &= x_f - x_i
\end{aligned}
$$

and we will need to split our problem into x and y-parts so we need the component of vectors set of equations

$$
\begin{aligned}
v_x &= v\cos\theta \\
v_y &= v\sin\theta
\end{aligned}
$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Notice that projectile motion problems may take three of our previously collected sets of equations! This is a little like an auto repair job taking three or four sets of wrenches. We collect the right set of tools for the job.

Now we are ready to try to solve for the final velocity.

The first part of our solution is to combine what we know with our equation set. So let's copy our equations and underline what we know.

$$\Delta y = \underline{y_f} - \underline{y_i}$$
$$\Delta x = \underline{x_f} - x_i$$

$$\Delta x = v_{ix}\Delta t + \frac{1}{2}\underline{a_x}\Delta t^2 \qquad \Delta y = v_{iy}\Delta t + \frac{1}{2}\underline{a_y}\Delta t^2$$
$$v_{fx} = v_{ix} + \underline{a_x}\Delta t \qquad v_{fy} = v_{iy} + \underline{a_y}\Delta t$$
$$v_{fx}^2 = v_{ix}^2 + 2\underline{a_x}\Delta x \qquad v_{fy}^2 = v_{iy}^2 + 2\underline{a_y}\Delta y$$

We don't have enough information to solve for the final velocity yet. But we realize that several of the quantities in the kinematic set can be found with the component of vectors set. We can find the components of the initial velocity, for example

$$v_{ix} = \underline{v_i}\cos\underline{\theta_i}$$
$$v_{iy} = \underline{v_i}\sin\underline{\theta_i}$$

so we can mark v_{ix} and v_{iy} as something we can know. Let's mark these and use our zeros.

$$\Delta y = 0 - \underline{y_i}$$
$$\Delta x = \underline{x_f} - 0$$

$$\Delta x = \underbrace{v_{ix}}\Delta t + 0 \qquad \underbrace{\Delta y} = \underbrace{v_{iy}}\Delta t + \frac{1}{2}\underline{a_y}\Delta t^2$$
$$v_{fx} = \underbrace{v_{ix}} + 0 \qquad v_{fy} = \underbrace{v_{iy}} + \underline{a_y}\Delta t$$
$$v_{fx}^2 = \underbrace{v_{ix}^2} + 0 \qquad v_{fy}^2 = \underbrace{v_{iy}^2} + 2\underline{a_y}\underbrace{\Delta y}$$

Now let's look at our set of equations. To find $\overrightarrow{v}_f$ we need to at least know v_{fx} and v_{fy} in order to write $\overrightarrow{v}_f$ in component form. So we need to search our set of equations and see if we know enough parts that we can solve for v_{ix} and v_{iy}.

It turns out that v_{ix} is fairy easy. From the second equation in our x-set we have

$$v_{fx} = \underbrace{v_{ix}} = \underline{v_i}\cos\underline{\theta_i}$$

It looks like from our third equation in the y-set we can find v_{fy}

$$v_{fy}^2 = \underbrace{v_{iy}^2} + 2\underline{a_y}\underbrace{\Delta y}$$

we will need to fill in the pieces from the vector components set and the displacement set

$$v_{fy}^2 = \left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(0 - \underline{y_i}\right)$$

and solve for v_{fy}

$$v_{fy} = \pm\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)}$$

where we will choose the the negative sign by looking at our picture to know that v_{fy} must be negative.

$$v_{fy} = -\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)}$$

So our $\overrightarrow{v}_f$ can be written as

$$\overrightarrow{v}_f = \left(\underline{v_i}\cos\underline{\theta_i}\right)\hat{\imath} - \left(\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)}\right)\hat{\jmath}$$

but of course we could use

$$\begin{aligned} v_f &= \sqrt{\left(\underline{v_i}\cos\underline{\theta_i}\right)^2 + \left(-\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)}\right)^2} \\ &= \sqrt{\left(\underline{v_i}\cos\underline{\theta_i}\right)^2 + \left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)} \end{aligned}$$

$$\theta_f = \tan^{-1}\left(\frac{-\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)}}{\left(\underline{v_i}\cos\underline{\theta_i}\right)}\right)$$

then

$$\begin{aligned} \overrightarrow{v}_f &= \sqrt{\left(\underline{v_i}\cos\underline{\theta_i}\right)^2 + \left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)} \\ &\quad \angle\tan^{-1}\left(\frac{-\sqrt{\left(\underline{v_i}\sin\underline{\theta_i}\right)^2 + 2\left(-g\right)\left(-\underline{y_i}\right)}}{\left(\underline{v_i}\cos\underline{\theta_i}\right)}\right) \end{aligned}$$

And we know every part of these equations except what we are solving for. Let's put in some numbers now

$$\begin{aligned} \overrightarrow{v}_f &= \left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(60\,^{\circ}\right)\right)\hat{\imath} - \left(\sqrt{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(60\,^{\circ}\right)\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(0.270\,\mathrm{m}\right)}\right)\hat{\jmath} \\ &= 3.5\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} - 6.4840\frac{\mathrm{m}}{\mathrm{s}}\jmath \\ &= 3.50\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} - 6.480\frac{\mathrm{m}}{\mathrm{s}}\jmath \end{aligned}$$

or

$$\begin{aligned} \overrightarrow{v}_f &= \sqrt{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(60\,^{\circ}\right)\right)^2 + \left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(60\,^{\circ}\right)\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(0.270\,\mathrm{m}\right)} \\ &\quad \angle\tan^{-1}\left(\frac{-\sqrt{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin\left(60\,^{\circ}\right)\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(0.270\,\mathrm{m}\right)}}{\left(\left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos\left(60\,^{\circ}\right)\right)}\right) \\ &= 7.3683\frac{\mathrm{m}}{\mathrm{s}}\angle - 1.0758\,\mathrm{rad} \\ &= 7.3683\frac{\mathrm{m}}{\mathrm{s}}\angle - 61.639\,^{\circ} \\ &= 7.37\frac{\mathrm{m}}{\mathrm{s}}\angle - 61.6\,^{\circ} \end{aligned}$$

This says that our ball is going a little faster at the end than it was at the beginning. Since it fell a little from the beginning to the end, this makes sense. It also makes sense that the angle is negative, and $-61\,^\circ$ seems reasonable looking at the picture.

You might think this was a lot of work! and it is! So why would mankind want to go through all this? From safety from falling rocks, to hunting food, to cannon balls in war, to moon launches, this process has been very useful! Perhaps even more useful is learning how to structure a solution so you can do a long complicated problem where you can't really know the answer intuitively when you start the problem. That kind of reasoning is useful in every technical field, medicine included! And it is in this king of problem that our problem solving process actually saves us time.

Let's extend our example to ask what the total displacement in the x-direction would be. We can add to our known values from the work we have done.

$$\begin{aligned}
x_i &= 0 \\
y_i &= 0.270\,\mathrm{m} \\
y_f &= 0 \\
v_i &= 7.00\frac{\mathrm{m}}{\mathrm{s}} \\
\theta_i &= 60\,^\circ \\
a_y &= -g \\
a_x &= 0 \\
g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \\
v_{xf} &= 3.50\frac{\mathrm{m}}{\mathrm{s}} \\
v_{yf} &= -6.480\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

We need an updated set of equations as well.

$$\begin{aligned}
\underbrace{v_{ix}} &= \underline{v_i}\cos\underline{\theta_i} \\
\underbrace{v_{iy}} &= \underline{v_i}\sin\underline{\theta_i}
\end{aligned}$$

so we can mark v_{ix} and v_{iy} as something we can know. Let's mark these and use our zeros.

$$\begin{aligned}
\Delta y &= 0 - \underline{y_i} \\
\Delta x &= \underline{x_f} - 0
\end{aligned}$$

$$\begin{array}{ll}
\Delta x = \underbrace{v_{ix}}\Delta t + 0 & \underbrace{\Delta y} = \underbrace{v_{iy}}\Delta t + \frac{1}{2}\underline{a_y}\Delta t^2 \\
\underbrace{v_{fx}} = \underbrace{v_{ix}} + 0 & \underbrace{v_{fy}} = \underbrace{v_{iy}} + \underline{a_y}\Delta t \\
\underbrace{v_{fx}^2} = \underbrace{v_{ix}^2} + 0 & \underbrace{v_{fy}^2} = \underbrace{v_{iy}^2} + 2\underline{a_y}\,\underbrace{\Delta y}
\end{array}$$

We can see that if we knew Δt then the first equation of our x-set would work. But we don't know Δt. But we can find Δt from the second equation in our y-set!

$$\underbrace{v_{fy}} = \underbrace{v_{iy}} + \underline{a_y}\Delta t$$

so

$$\underbrace{v_{fy}} - \underbrace{v_{iy}} = \underline{a_y}\Delta t$$

$$\frac{\underbrace{v_{fy}} - \underbrace{v_{iy}}}{\underline{a_y}} = \Delta t$$

or

$$\Delta t = \frac{\underbrace{v_{fy}} - \underbrace{v_{iy}}}{\underline{a_y}}$$

then

$$\Delta x = \underbrace{v_{ix}}\left(\frac{\underbrace{v_{fy}} - \underbrace{v_{iy}}}{\underline{a_y}}\right)$$

$$x_f = x_i + \underline{v_i}\cos\underline{\theta_i}\left(\frac{\underbrace{v_{fy}} - \underline{v_i}\sin\underline{\theta_i}}{\underline{a_y}}\right)$$

Putting in values gives

$$\begin{aligned}
x_f &= \left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\cos(60^\circ)\left(\frac{-6.480\frac{\mathrm{m}}{\mathrm{s}} - \left(7.00\frac{\mathrm{m}}{\mathrm{s}}\right)\sin(60^\circ)}{-9.8\frac{\mathrm{m}}{\mathrm{s}^2}}\right) \\
&= 4.4793\,\mathrm{m}
\end{aligned}$$

We call this the "range" of the projectile motion. If you are in the army, or just out hunting, you might want to know this.

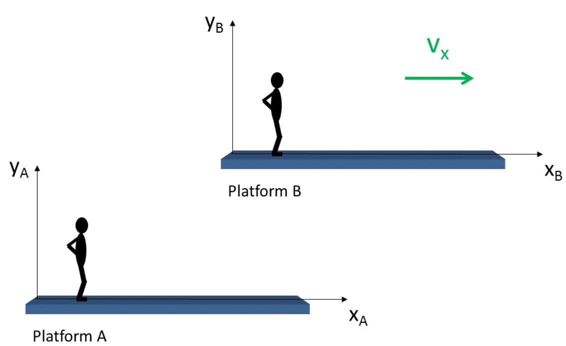
Our problems have become longer, now that we have two-dimensions. But not really harder if we take a systematic approach. We will continue with the topic of two-dimensional motion in our next lecture.

11 Relative Motion and Beginning of Circular Motion

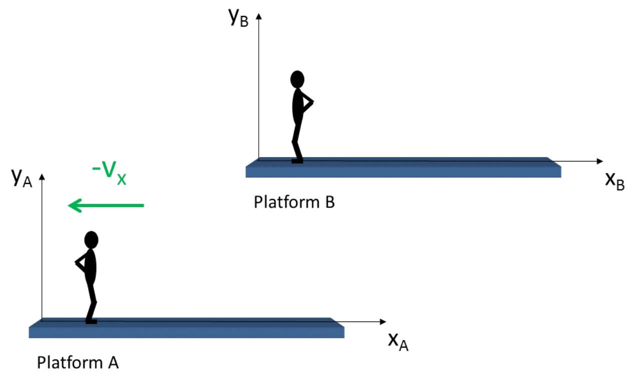
We are well into our study of two dimensional motion, but there is a complication that we have partially ignored. We have considered what would happen to our description of motion if we move the origin of our coordinate system. But what if we have a whole coordinate system that moves? We will take on this problem first. And then we will revisit acceleration for the special case of an object that moves in a perfect circle.

Relative Velocity

Let's consider a, somewhat far-fetched situation. Suppose we have two alien beings traveling on platforms, far from anything else in space. We can call the platforms A and B. Further suppose that each platform has it's own coordinate system attached to the platform. And suppose one of the platforms is moving with speed v_x to the right. We will give a name to these separate coordinate systems. We will call them *reference frames.* So we can call the coordinate system of platform A reference frame A and the coordinate system of platform B reference frame B.



Alien A sees himself as stationary and sees alien B traveling with velocity v_x. But suppose only aliens A and B exist on their reference frame platforms and there are no other objects in the universe to give perspective. Alien B would see himself as stationary, and would see alien A traveling with velocity $-v_x$.I



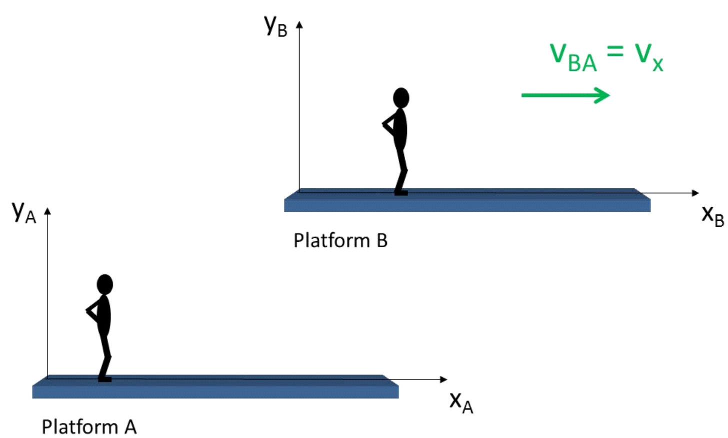
You might ask, who is right, A or B? If there are only these two aliens and their platforms, is there any way to tell which is "really moving?" It might surprise you to find out that the answer is - "no!" Recall that there is no universal zero point that is the center of everything in the universe. There is also no universal point that we know to be not moving. In fact, everything we can see in the universe, stars, planets, galaxies, etc. all seem to be moving closer to each other or away from each other, including our planet, Sun, and solar system. We believe we are moving around the galaxy, and that our galaxy is moving too. So we have no place that we can find that will work as a point that is not moving. since that is the case, it is the *relative* speed v_x that we must consider. That is all we can be absolutely sure of. Alien A really sees B moving and Alien B really sees A moving. And each viewpoint is valid.

We have all tried this experiment in real life, only, in our experiment there were other objects around us. Consider driving South on I-20. There are two lanes of traffic on the Southbound side of the freeway. As you drive, consider the cars next to you, also going south. You are all going south at $70\,\mathrm{mi/h}$ (carefully obeying speed laws). But think of the relative speed. Since you are all going about the same speed, the relative speed between your car and the next is very small. That is why you can wave to the person in the next car and consider if they are someone you would like to meet of if they are likely to be a maniac and force you off the road.

Now consider the cars going north. They whizz past seeming to go terribly fast. What is their relative velocity? If we take a coordinate system that is fixed to our car. The velocity of our car seems to be zero in that coordinate system. But then we see the northbound cars going past us at $70\,\mathrm{mi/\,h} + 70\,\mathrm{mi/\,h} = 140\,\mathrm{mi/\,h}$. The velocity of the two cars add. That is why you want to avoid a head-on collision with the Northbound traffic at nearly all costs!

But is a moving frame of reference useful–a coordinate system that moves with your car? Sure, if you want to hand a drink to a passenger, you don't want to have to consider where that passenger is with respect to Idaho Falls. You only need to know where the passenger is with respect to your position within the car. A more dramatic example might be finding your dining area on a cruse ship (I always wanted to go on a cruse). You don't want to have to consider your motion toward the Bahamas. You just need to know where the dinning area is with respect to your cabin.

Let's adopt a way to express our viewpoint of who is moving relative to whom.



Returning to the aliens, suppose we again join space guy A and observe space guy B. The speed of B as observed from A's reference frame we will call v_{BA}. Note that in our case

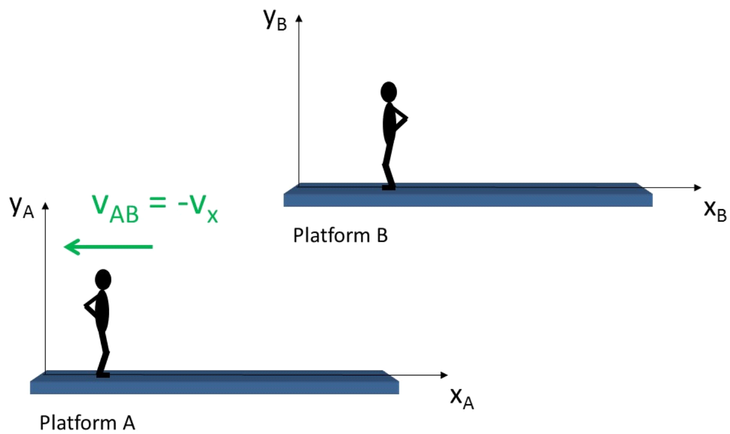
$$v_{BA} = v_x$$

in the positive x_A-direction.

Then if we jumped to the other platform and observed the motion of A from the perspective of B's platform we would see A moving with speed

$$v_{AB} = -v_x$$

The first subscript is always the moving object or mover, the second subscript tells the point of view from which the motion is observed.

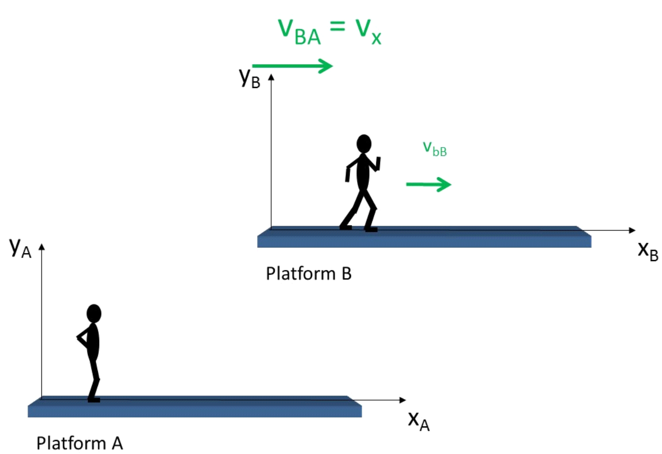


In this notation system, it will always be true that

$$v_{BA} = -v_{AB}$$

since the relative motion is always seen as opposite in direction when we switch view points.

Let's try a somewhat harder problem. Let's consider our spacemen again. and let's observe space guy in frame B from the viewpoint of A. But let's have space guy in frame B walk along his platform with a velocity $\overrightarrow{v}_{bB}$ relative to his own platform. The b stands for space guy b that is in frame B. How fast would the space guy in frame A see guy b go?



It's not too hard to see that guy in frame A would see guy b go

$$v_{bA} = v_{bB} + v_{BA}$$

We would add the platform speed and the walking speed to get how fast guy a in frame A sees guy b going.

Of course we could consider the spaceman in frame A (let's give him the subscript, a) walking on his platform with speed v_{aA}. From the perspective of platform B we would see spaceman a's speed as

$$v_{aB} = v_{bA} + v_{AB}$$

This gives a pair of equations for relative motion

$$\begin{aligned} \overrightarrow{\mathbf{v}}_{bA} &= \overrightarrow{\mathbf{v}}_{bB} + \overrightarrow{\mathbf{V}}_{BA} \\ \overrightarrow{\mathbf{v}}_{aB} &= \overrightarrow{\mathbf{v}}_{bA} + \overrightarrow{\mathbf{V}}_{AB} \end{aligned}$$

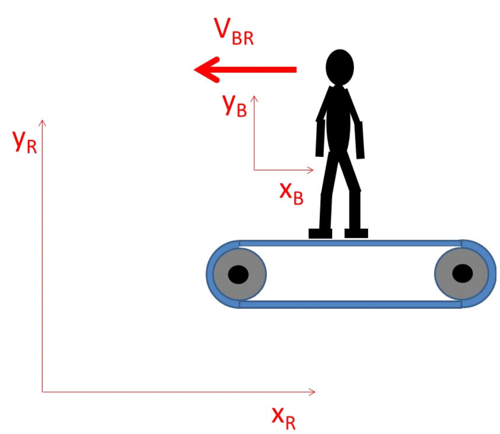
or we could write this as

$$\begin{aligned} \overrightarrow{\mathbf{v}}_{bA} &= \overrightarrow{\mathbf{v}}_{bB} + \overrightarrow{\mathbf{V}}_{BA} \\ \overrightarrow{\mathbf{v}}_{aB} &= \overrightarrow{\mathbf{v}}_{bA} - \overrightarrow{\mathbf{V}}_{BA} \end{aligned} \tag{11.1}$$

This equation set is often called the *Galilean relativity transformation* equation set because they were discovered by Galileo and because they "transform" our velocity from one perspective to another.

It might be tempting to try to do relative motion problems without the subscripts, but don't do it! The subscripts are important for keeping the motions straight.

Let's do an example:



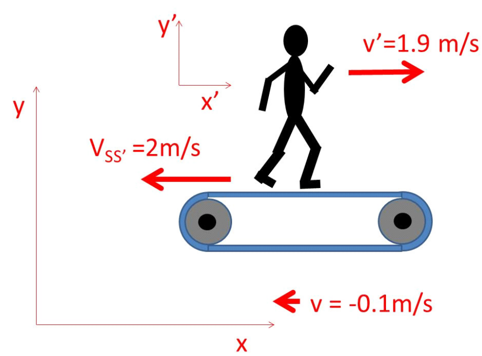
Suppose we have a boy (b) in the gym running on a treadmill. Let's call the room frame R and the top of the treadmill belt frame B (for belt). Then our transformation equations would be

$$\begin{aligned}\overrightarrow{\mathbf{v}}_{bR} &= \overrightarrow{\mathbf{v}}_{bB} + \overrightarrow{\mathbf{V}}_{BR}\\ \overrightarrow{\mathbf{v}}_{aB} &= \overrightarrow{\mathbf{v}}'_{bR} - \overrightarrow{\mathbf{V}}_{BR}\end{aligned}$$

The treadmill track belt has a relative velocity $\overrightarrow{\mathbf{V}}_{BR} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_R$ with respect to the room. A person standing on the treadmill in frame B sees themselves as not moving, and the rest of the room as moving the opposite direction.

The notation $\overrightarrow{\mathbf{V}}_{BR}$ means the velocity of the reference frame B with respect to frame R or in our case the speed of the treadmill belt top with respect to the room $\overrightarrow{\mathbf{V}}_{BR} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_R$.

Now suppose the person is running at speed $\overrightarrow{\mathbf{v}}_{bB} = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_B$ on the tread mill in the tread mill frame B.



What is his/her speed with respect to the room? It seems obvious that we take the two speeds and add them. This is the first of our Galilean transformation equations

$$\overrightarrow{\mathbf{v}}_{bR} = \overrightarrow{\mathbf{v}}_{bB} + \overrightarrow{\mathbf{V}}_{BR}$$
$$\overrightarrow{\mathbf{v}}_{bR} = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_B - 2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_R = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$$

since the $\hat{\imath}_B$ and $\hat{\imath}_R$ directions are the same (look at the figure).

The person is going to fall off the end of the treadmill unless they pick up the pace!

Likewise, if we want to know how fast the person is walking with respect to the treadmill frame, we would use the second of our Galilean transformation equations.

$$\overrightarrow{\mathbf{v}}_{bB} = \overrightarrow{\mathbf{v}}_{bR} - \overrightarrow{\mathbf{V}}_{BR}$$

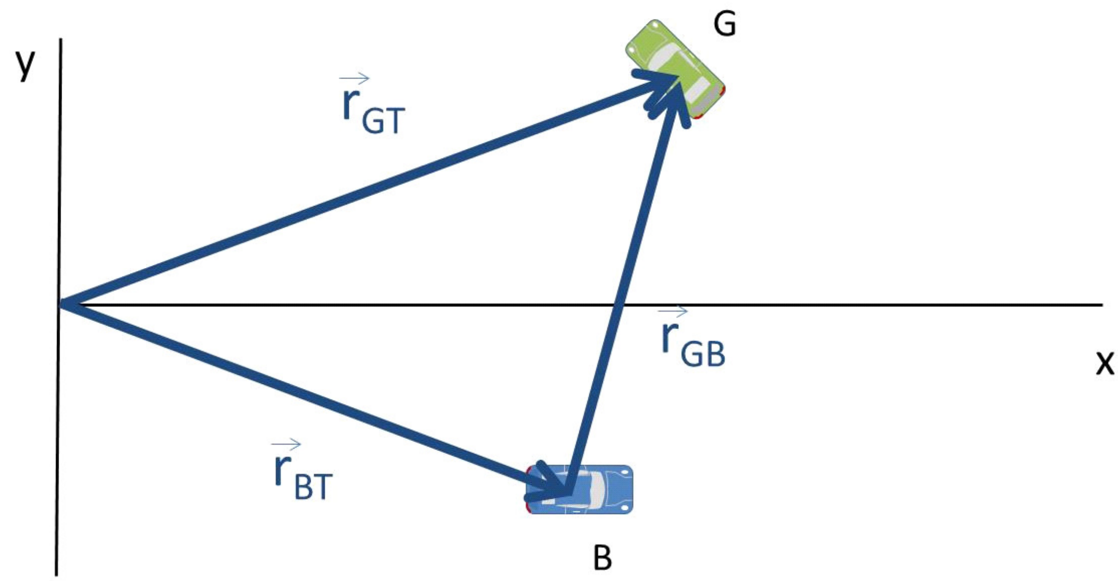
We take the room speed $\overrightarrow{\mathbf{v}}_{bR} = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ and subtract from it the treadmill/room relative speed $\overrightarrow{V}_{BR} = -2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}$ to obtain

$$\overrightarrow{\mathbf{v}}_{bB} = -0.1\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_R - \left(-2\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_R\right) = 1.9\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath}_B$$

Using the Galilean transformation equations in one-dimension is tricky, but not too bad if we keep our subscripts straight. But now we can do two-dimensional problems. And our transformations equations work just fine for two-dimensional relative motion. Let's give this a try.

Relative velocity in two dimensions

The situation is even more complicated in two dimensions. Let's consider two cars. We will label them car G (for green) and car B (for blue). We will label the origin T (indicating that this is coordinate system is fixed with the track) and assume we have a third person at the origin observing our cars.



Then $\vec{r}_{GT}$ is the position of car G as measured by the person standing on the track, T. Likewise,$\Delta\vec{r}_{BT}$ is the position of car B as measured by the person on the track, T. And $\vec{r}_{GB}$ is the position of car G as measured by the person in car B. The first subscript tells us which object is being measured, and the second tells us what viewpoint is being used. Looking at the figure, if we know $\vec{r}_{BT}$ and $\vec{r}_{GT}$ as position vectors. How do we find $\Delta\vec{r}_{GB}$?

$$\Delta\vec{r}_{GB} = \vec{r}_{GT} - \vec{r}_{BT}$$

We just use our subscript notation, and relative motion problems become much simpler.

Suppose we want to know the instantaneous velocity of car G with respect to car B. We know

$$\overrightarrow{v}=\frac{d\overrightarrow{r}}{dt}$$

so let's try

$$\overrightarrow{v}_{GB}=\frac{d\vec{r}_{GB}}{dt}$$

this would be

$$\overrightarrow{v}_{GB}=\frac{d\vec{r}_{GB}}{dt}=\frac{d}{dt}\vec{r}_{GT}-\frac{d}{dt}\vec{r}_{BT}$$

or

$$\vec{v}_{GB}=\vec{v}_{GT}-\vec{v}_{BT}$$

This is just the same as one of our transformation equations, the first one of our set

$$\begin{aligned}\overrightarrow{\mathbf{v}}_{bR} &= \overrightarrow{\mathbf{v}}_{bB}+\overrightarrow{\mathbf{V}}_{BR}\\ \overrightarrow{\mathbf{v}}_{aB} &= \overrightarrow{\mathbf{v}}'_{bR}-\overrightarrow{\mathbf{V}}_{BR}\end{aligned}$$

or

$$\overrightarrow{\mathbf{v}}_{bB}=\overrightarrow{\mathbf{v}}_{bR}-\overrightarrow{\mathbf{V}}_{BR}$$

only with G for b because the moving object now is the green car, and with T in place of R because we are on a track and not in a room. The subscript B now means "car B" instead of "treadmill belt."

$$\overrightarrow{\mathbf{v}}_{GB}=\overrightarrow{\mathbf{v}}_{GT}-\overrightarrow{\mathbf{V}}_{BT}$$

Let's summarize what we have learned about solving relative motion problems.

1. Label each object involved with a letter. Letters that remind you of the object are better. The first subscript is the object label, and the second is the viewpoint label.

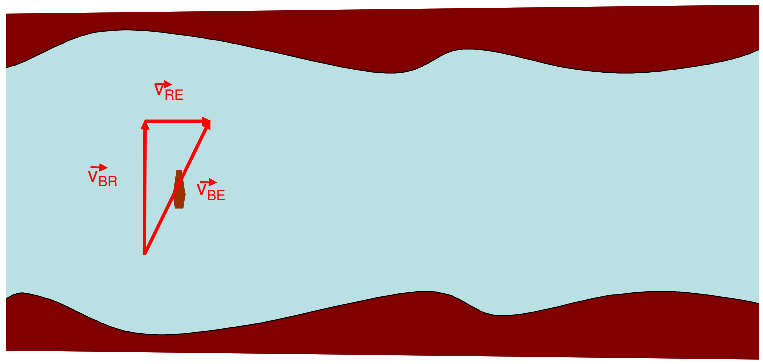
2. Look through the problem for phrases like "the velocity of object A relative to object B." If the velocity of one object is not specifically stated as being relative to another object, is usually the velocity with respect to the Earth, or a room or track that is attached to the Earth.

3. Take the velocities you have identified and arrange them into our transformation equations. If you don't have the velocities, you may need to look at position vectors and take a derivative.

4. Solve for the unknown x and y components of the velocity. Remember that we are still making our two-dimensional problem into two one-dimensional problems!

Example

Problem Statement: You have herd the fishing is great in Idaho and so you rent a boat and take your FHE group out fishing on the river. Your heading is due north and your speed with respect to the water is $v_{BR}=10.0\,\mathrm{km/h}$. The river goes by to the east with a speed with respect to the shore is $v_{RE}=5.00\,\mathrm{km/h}$. How fast would a forest ranger see the boat if the ranger is standing on the shore?



Let's choose the y axis to be positive in the northern direction. The x axis will be positive in the eastern direction.

Variables:

$\vec{v}_{BR} = 10.0\,\mathrm{km/h}\hat{\jmath}$	Velocity of the boat with respect to the river
$\vec{v}_{RE} = 5.00\,\mathrm{km/h}\hat{\imath}$	Velocity of the river with respect to the shore (Earth)
$\vec{v}_{BE}$	The velocity of the boat with respect to the shore

Basic Equations:

We will use our transformation equations

$$\begin{aligned}\vec{\mathbf{v}}_{bA} &= \vec{\mathbf{v}}_{bB} + \vec{\mathbf{V}}_{BA}\\ \vec{\mathbf{v}}_{aB} &= \vec{\mathbf{v}}_{bA} - \vec{\mathbf{V}}_{BA}\end{aligned}$$

but we will also need our vector set

$$\begin{aligned}v_x &= v\cos(\theta)\\ v_y &= v\cos(\theta)\\ v &= \sqrt{v_x^2 + v_y^2}\\ \theta &= \tan^{-1}\left(\frac{v_y}{v_x}\right)\end{aligned}$$

Symbolic Solution:

Using the picture we can write an equation with our vectors:

$$\vec{v}_{BE} = \vec{v}_{BR} + \vec{v}_{RE}$$

this is the first in our transformation set.

Now this is two-deminsional problem, so we will make it into two one-dimensional problems by taking components of the vectors.

$$v_{BEx} = v_{BRx} + v_{REx}$$

and

$$v_{BEy} = v_{BRy} + v_{REy}$$

so

$$\vec{v}_{BE} = (v_{BRx} + v_{REx})\,\hat{\imath} + (v_{BRy} + v_{REy})\,\hat{\jmath}$$

this is the vector, but we want the magnitude and direction. Let's take our direction to be ϕ, the angle between the direction we wanted (north) and the direction we actually go, then

$$\begin{aligned} v_{BE} &= \sqrt{v_{BEx}^2 + v_{BEy}^2} \\ \phi &= \tan^{-1}\left(\frac{v_{BEx}}{v_{BEy}}\right) \end{aligned}$$

Numeric Solution:

We can now plug in the nubers

$$\begin{aligned} \vec{v}_{BE} &= \left(0 + 5.00\frac{\mathrm{km}}{\mathrm{h}}\right)\hat{\imath} + \left(10.0\frac{\mathrm{km}}{\mathrm{h}} + 0\right)\hat{\jmath} \\ &= \left(5.00\frac{\mathrm{km}}{\mathrm{h}}\right)\hat{\imath} + \left(10.0\frac{\mathrm{km}}{\mathrm{h}}\right)\hat{\jmath} \\ v_{BE} &= \sqrt{\left(5.00\frac{\mathrm{km}}{\mathrm{h}}\right)^2 + \left(10.0\frac{\mathrm{km}}{\mathrm{h}}\right)^2} = 11.18\frac{\mathrm{km}}{\mathrm{h}} \\ \phi &= \tan^{-1}\left(\frac{\left(5.00\frac{\mathrm{km}}{\mathrm{h}}\right)}{\left(10.0\frac{\mathrm{km}}{\mathrm{h}}\right)}\right) = 26.565\,^{\circ} \end{aligned}$$

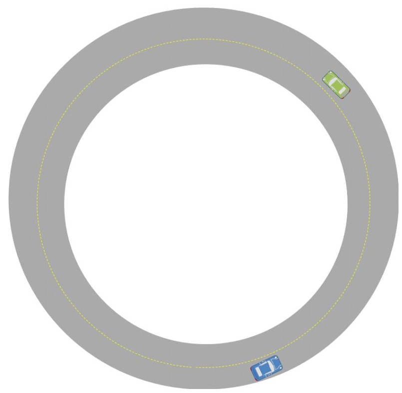
Uniform Circular Motion

We have been studying two-dimensional motion. And we have learned quite a bit. But there is another special case that we need to consider.

Think of a race track. Many tracks consist of straight parts, and we know how to deal with linear motion, so we could predict motion for the straight parts.

We have also dealt a little with the curved parts. But what if we have all curved parts?

Let's consider a completely curved track as a special case. A completely curved trace would be a circle.



This would probably not be a fun tract to drive on. But if we can find a way to express motion on a circle, then we could use this circular motion technique and our linear motion technique combined to deal with the entire track.

Let's start our analysis by considering uniform motion in a circle, that is, moving in a circle but not speeding up or slowing down.

This is a lot like a constant motion problem. We would expect to be able to use an equation like

$$v = \frac{dx}{dt}$$

but there is a problem, We only go purely in the x-direction for a short time as we go around the circle. We need a way to say how far we have gone as we travel around the circle. From Geometry, way back in junior high school, we know how to measure this.

Lets consider how far we would travel in distance if we went all the way around the circle. That would be the entire circumference of the circle. So our distance would be

$$C = 2\pi r$$

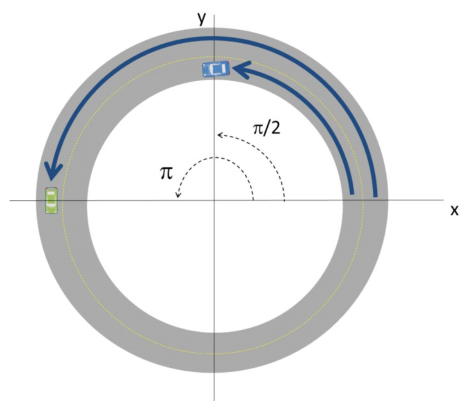
where r is the radius of the circle and where C is for circumference. But suppose we just traveled half way around the circle. The distance would be

$$\frac{C}{2} = \pi r$$

and notice that we swiped out an angle of $180\,^\circ$ (see next figure, green car). How about a quarter way around the circle?

$$\frac{C}{4} = \frac{\pi}{2} r$$

Notice that we swiped out an angle of $90\,^\circ$ (seen next figure, blue car).



Now notice that in radians $90\,^\circ$ is $\pi/2$ and $180\,^\circ$ is π. Then how far we travel around the circle seems to be given by

$$\begin{aligned} s &= \text{(angle in radius) (radius)} \\ &= \phi r \end{aligned}$$

The distance traveled we gave the letter, s from di(s)splacement. And it is called the *arclength* because in geometry the distance we travel in a circle is given this name. We will just use the names right from geometry.

We also have a name for how long it takes to go around the whole circle. That is called a *period*, and it is given the symbol, T. This makes some sense, because a period is a *time* for going around the whole circle. We can finally describe the speed of the car as it travels the circular path

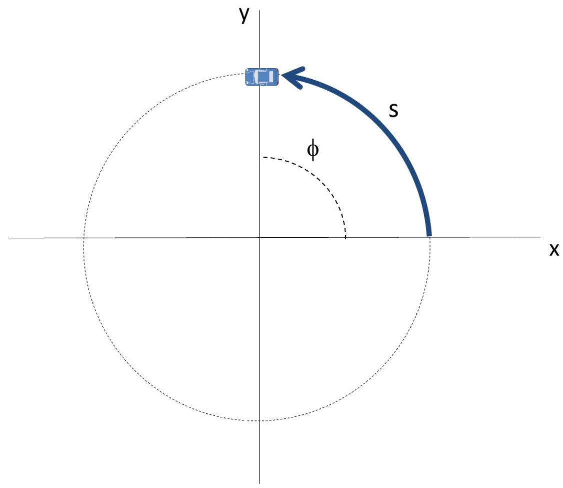
$$v = \frac{C}{T} = \frac{2\pi r}{T}$$

or we could say

$$v = \frac{\Delta s}{\Delta t} = \frac{s_f - s_i}{t_f - t_i}$$

if we don't go all the way around the circle.

We remember from geometry that all the way around the circle gives $365\,^{\circ}$ or $2\pi\,\mathrm{rad}$. And notice that the arclength is proportional to the angle, and the radius. The radius is not changing, so if we know the angle part of the displacement, and we know the car must be on the circular track, then the angle is enough to tell us where the car is.



This really only works for circular motion. But for circular motion knowing the angle is usually enough to know the position. We could write Δs as

$$\Delta s = r\Delta\phi$$

since the r is not changing. Then

$$v_{ave} = \frac{\Delta s}{\Delta t} = \frac{r\Delta\phi}{\Delta t}$$

and we could even write

$$\frac{v_{ave}}{r} = \frac{\Delta\phi}{\Delta t}$$

as a sort of scaled speed. This tells us how much the angle changed in an amount of time. It is a lot like a velocity, but it has units of $\mathrm{rad}/\,\mathrm{s}$. It tells us how fast the angle of the car changes. Since for circular motion, this is equivalent to knowing how the position of the car changes, our new quantity is like a speed. Let's give it a name an a symbol. The name is *angular speed* and the symbol is a Greek letter ω. This is not a "w." It is an omega. But you may call it "w-looking thing" if that helps. But it is not a "w" and we should make the distinction because we will use "w" for something else in physics.

But ω does tell us how fast something spins around. It is how fast the angle changes.

$$\omega_{ave} = \frac{\Delta\phi}{\Delta t}$$

notice that we can also write

$$\omega_{ave} = \frac{v_{ave}}{r}$$

from our definition. This relates how fast the object is going to how fast the angle is changing for circular motion.

If we take a limit, letting Δt get very small, then we will have an instantaneous angular speed

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \phi}{\Delta t} = \frac{d\phi}{dt}$$

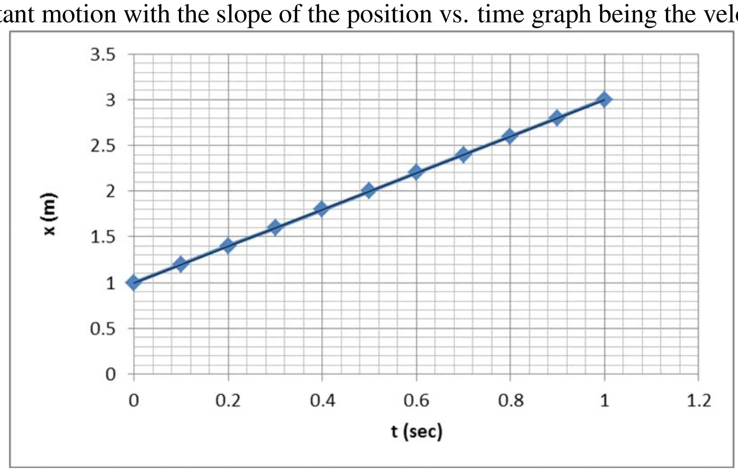
Notice that something wonderful has happened! The equation for angular speed looks just about the same as the equation for linear speed! It turns out we can make an entire set of equations for constant circular motion look very much like the equations for constant linear motion.

Linear	Circular
$\Delta r = r_f - r_i$	$\Delta \phi = \phi_f - \phi_i$
$\Delta t = t_f - t_i$	$\Delta t = t_f - t_i$
$v_{ave} = \frac{\Delta r}{\Delta t}$	$\omega_{ave} = \frac{\Delta \phi}{\Delta t}$
$v = \frac{dr}{dt}$	$\omega = \frac{d\phi}{dt}$

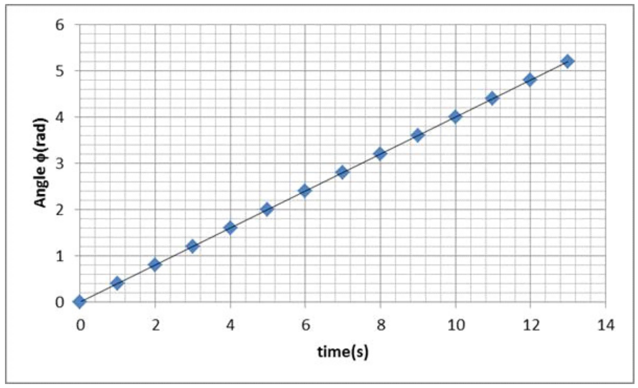
$$\omega = \frac{v}{r}$$

with an additional equation tying the two types of motion together.

This means we can use all the graphing techniques we learned for linear motion for our circular motion equations. For example, we used position vs. time graphs to show linear constant motion with the slope of the position vs. time graph being the velocity.



now we could plot ϕ vs. time to get a plot like this



For linear constant motion we found that

$$x_f = x_i + v\Delta t$$

We can take our equation for angular speed

$$\omega = \frac{\Delta\phi}{\Delta t}$$

and write is using $\Delta\phi = \phi_f - \phi_i$ to get

$$\omega = \frac{\phi_f - \phi_i}{\Delta t}$$

or

$$\phi_f = \phi_i + \omega\Delta t$$

Let's try some problems. First, using the previous figure, what is ω?

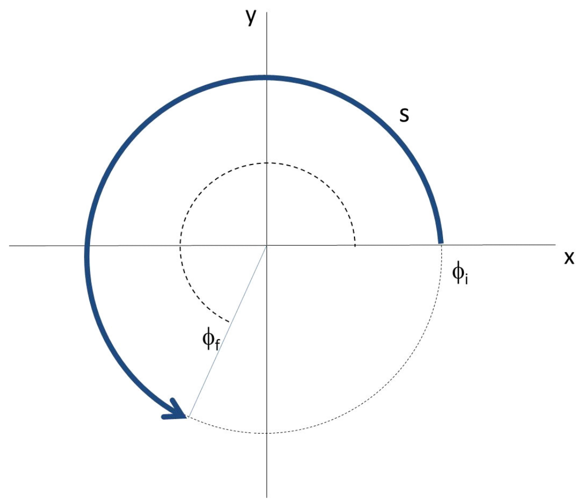
$$\omega = \frac{\phi_f - \phi_i}{t_f - t_i}$$

we recognize this as "rise over run" just like it was for linear constant motion. So the slope of the ϕ vs. time graph will be ω. We an see that at $t = 10\,\mathrm{s}$ we have $\phi = 4\,\mathrm{rad}$ and at $t = 0\,\mathrm{s}$ we have $\phi = 0\,\mathrm{rad}$, so

$$\begin{aligned} \omega &= \frac{4\,\mathrm{rad} - 0}{10\,\mathrm{s} - 0} = \\ &= 0.4\frac{\mathrm{rad}}{\mathrm{s}} \end{aligned}$$

For a second example, suppose a child is on a merry-go-round. The child hops on at $\phi_i = 0.0\,\mathrm{rad}$ and deices she does not like the ride, so she hops off at $\phi_f = 4.5\,\mathrm{rad}$. Suppose she was on the ride for $0.50\,\mathrm{s}$. What is the angular speed of the merry-go-round?

We can identify this as a constant angular speed problem



The final ϕ is about $258\,^{\circ}$. So we draw a diagram showing where the child got on and off.

We know

$$\begin{aligned}\phi_i &= 0\,\mathrm{rad}\\ \phi_f &= 4.5\,\mathrm{rad}\\ \Delta t &= 0.50\,\mathrm{s}\end{aligned}$$

and our basic equation is

$$\phi_f = \phi_i + \omega\Delta t$$

We can solve this for ω

$$\phi_f = \phi_i + \omega\Delta t$$

$$\begin{aligned}\phi_f - \phi_i &= \omega\Delta t\\ \frac{\phi_f - \phi_i}{\Delta t} &= \omega\end{aligned}$$

$$\omega = \frac{\phi_f - \phi_i}{\Delta t}$$

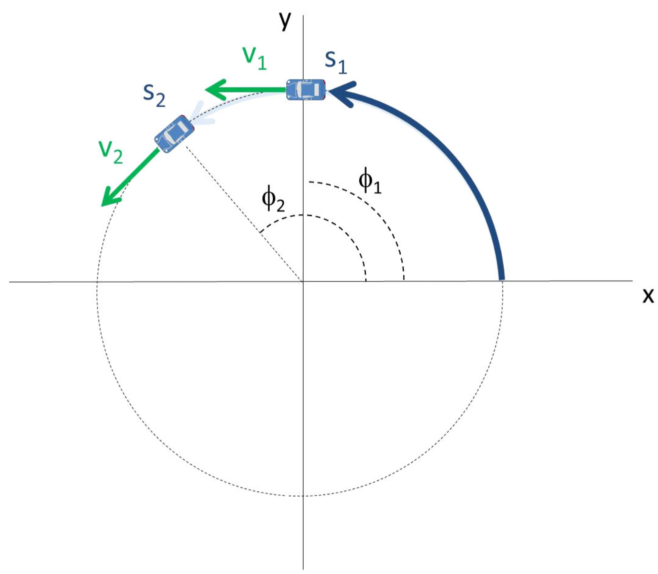
Using our numbers we get

$$\begin{aligned}\omega &= \frac{4.5\,\mathrm{rad} - 0\,\mathrm{rad}}{0.5\,\mathrm{s}}\\ &= \frac{9.0}{\mathrm{s}}\,\mathrm{rad}\end{aligned}$$

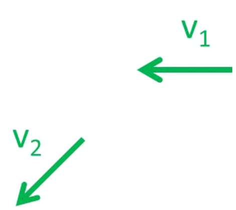
A quirk of history can make rotational problems tricky. sometimes people will talk in terms of "rotations per second" or, worse yet, "cycles per second." A rotation is just $2\pi\,\mathrm{rad}$, and so is a cycle. They both mean, "go all the way around." So if you are given a value in rotations per second or cycles per second, just convert to radians per second.

12 Circular Motion

In our last lecture we learned about uniform circular motion. Let's start there again. Suppose we have a car going on a circular track at a constant speed.

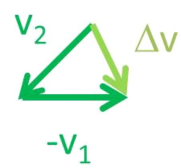


We know the car is accelerating, because it is turning. But it is not speeding up or slowing down. Notice that the velocity vectors always point along a tangent line to the circle. We will say they have a *tangential direction*. Let's find the direction of the acceleration. To do this we recopy our velocity vectors



and find

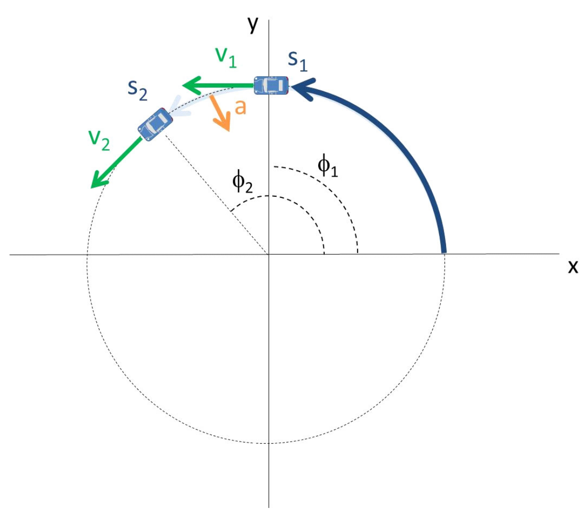
$$\Delta\overrightarrow{v} = \overrightarrow{v}_2 - \overrightarrow{v}_1$$



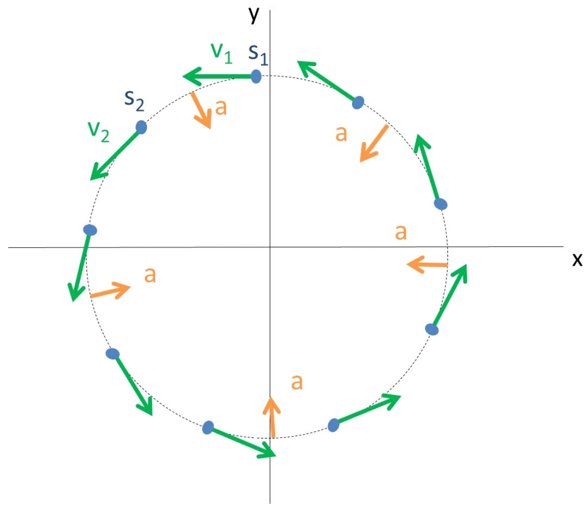
and divide $\Delta\overrightarrow{v}$ by Δt

$$a_{ave} = \frac{\Delta\overrightarrow{v}}{\Delta t}$$

There is nothing new in all this. We have found accelerations before. But now let's put our $\overrightarrow{a}$ back on our original diagram

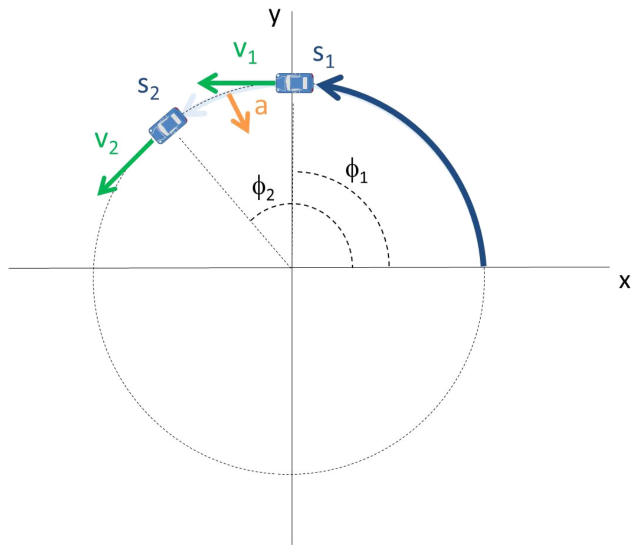


We have placed in between the two points where v_1 and v_2 are marked, because we have found an average acceleration. Note which way the acceleration points. It points into the center of the turn. Let's look at a few more places around the circular track.



Notice that no matter where we are on the track, the acceleration for constant circular motion always points to the center of the turn. We called the velocity "tangential" because it's direction was always along a tangent line. Notice that the acceleration is always along a radial line. So we could call it "radial" and sometimes we will. But the acceleration is not just radial, along a radial line. It specifically points to the *center* of the circular motion. There is a word that means "points to the center" and we will use this word to describe acceleration for uniform circular motion. That word is *centripetal.* like the word "tangential" just described the direction the velocity pointed, the word "centripetal" just describes the direction the acceleration points. This is not some new kind of acceleration. You could think of this like a person gaining a new title, say, "king." It is the same old person, but "king" does describe the direction their life is going!

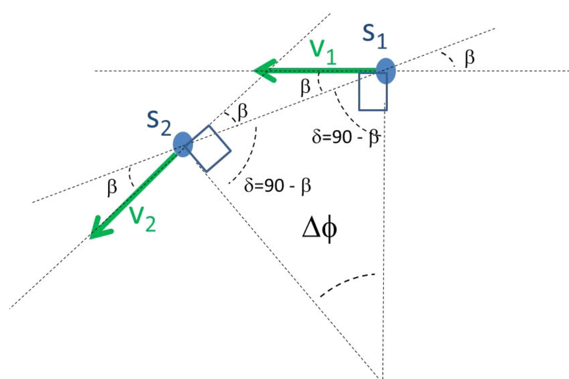
Let's go back to our traveling car going in a circle



Notice that the car sweeps out an angular displacement

$$\Delta\phi = \phi_2 - \phi_1$$

as it goes from s_1 to s_2. We can use this to find a very useful expression for a centripetal acceleration. But it will take some geometry. Here is a diagram of our motion with some extra lines added in.



Recall that the interior angles of a triangle must sum to 180°. So

$$\Delta\phi + \delta + \delta = 180^\circ$$

Also notice that the velocities are along tangent lines, and that the tangent lines and radial lines meet at right angles. Then

$$\beta + \delta = 90^\circ$$

and notice that $180^\circ = 90^\circ + 90^\circ$ (you probably already knew that!) so

$$180^\circ = 2\beta + 2\delta$$

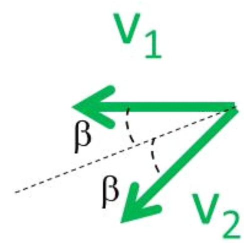
then

$$\Delta\phi + \delta + \delta = 2\beta + 2\delta$$

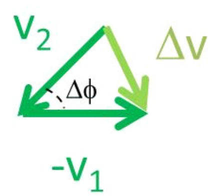
or

$$\Delta\phi = 2\beta$$

Not notice that the angle between v_1 and v_2 is exactly $2\beta = \Delta\phi$



Then in our triangle that defines $\Delta\overrightarrow{v}$ the angle between $-\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ will be $\Delta\phi$.



and these vectors form an isosceles triangle. The length the vector Δv is nearly equal to the arclength

$$\Delta v \approx v_1 \Delta\phi$$

If we let $\Delta\phi$ get very small so it becomes $d\phi$, then we can write

$$dv = v d\phi \tag{12.1}$$

where I dropped the subscript, because the length of v_1 is the same as the length of v_2 so we can write $v_1 = v_2 \equiv v$. And we know the speed

$$v_1 = v_2 = v = \frac{ds}{dt}$$

and

$$v = \frac{ds}{dt} = \frac{rd\phi}{dt}$$

It is not obvious that this will give us anything good, but let's solve for dt

$$dt = \frac{rd\phi}{v}$$

Now let's substitute our new expression for dt for uniform circular motion into our equation for acceleration

$$a_{ave} = \frac{\Delta v}{\Delta t}$$

but we will need to let $\Delta t \to dt$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{dv}{\frac{r d\phi}{v}} \\ &= \frac{v}{r}\frac{dv}{d\phi} \end{aligned}$$

and use equation () for dv again

$$\begin{aligned} a &= \frac{v}{r}\frac{v d\phi}{d\phi} \\ &= \frac{v^2}{r} \end{aligned}$$

This seems like a lot of work to go through for a simple equation

$$a_r = \frac{v^2}{r}$$

but it is a very important equation. It tells us that for constant circular motion the acceleration is equal to the speed squared divided by the radius. If we know the size of the circle, and how fast we are going, we can know the magnitude of the acceleration. Of course the direction of the acceleration is toward the center of the circle.

It is important to remember that we are studying circular motion at a constant speed–we are not speeding up or slowing down. Our centripetal acceleration is just making our object change direction.

Lets try a problem. Suppose we are in our race car and we know from the speedometer that we are going a constant $26\,\mathrm{m/s}$ as we go around a circular track with a radius of $100\,\mathrm{m}$ (about the length of a football field). What is our acceleration?

We are not speeding up or slowing down, so this must be a turning or centripetal acceleration problem. We can use our new equation

$$\begin{aligned} a &= \frac{v^2}{r} \\ &= \frac{(26\,\mathrm{m/s})^2}{100\,\mathrm{m}} \\ &= 6.76\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Angular Acceleration

Last lecture we said we could write the angular speed as

$$\omega = \frac{\Delta\phi}{\Delta t}$$

like we wrote

$$v = \frac{\Delta x}{\Delta t}$$

Recall that before we let v change, speeding up or slowing down. We can also let ω change. This would be "spinning up" or "spinning down." Think of your parents CD player. When you turn it on, the CD is just sitting there. But once you push "play" the disk begins to spin. This is what we mean by spinning up. Once you stop playing your music, the disk stops spinning This is an example of spinning down. Notice that the CD did not gain any normal acceleration. It did not come shooting out of the CD player. Spinning up is different than speeding up. When you throw a frisbee, you give the frisbee both a spin and you speed it up. But spinning and throwing are two different ways of changing motion.

Note that this angular velocity is really a way to describe the motion of an entire spinning object. Think of the CD again. Or think of a potter's wheel. These objects spin, and the entire object spins at once. If the CD didn't all spin as one object, then the CD would be ripped apart in the CD player. Our particle model doesn't handle such spinning objects well. Instead of particle model, we want a way to consider the rotational motion of the entire object, not just at a point, but the entire object. We use angular speed to describe this whole-object rotational motion.

For speeding up and slowing down of particles we defined an acceleration

$$a = \frac{\Delta v}{\Delta t}$$

and we can do that for the rotational motion of whole objects as well. We will say that changing the rotational motion of an object is *spinning up* or *spinning down.*This change in angular speed is like an acceleration (a change in motion), so let's call it *angular acceleration*. Again this is the change in the rotational motion of the whole object (like the whole CD) not just a point on the object. So we won't be confused with normal acceleration, lets use a new symbol, α, for angular acceleration. This symbol is pronounced "alpha" (as in alpha and omega) and is from the Greek alphabet. Then

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

Of course this would be an average angular acceleration

$$\alpha_{ave} = \frac{\Delta\omega}{\Delta t}$$

We could also define an instantaneous acceleration

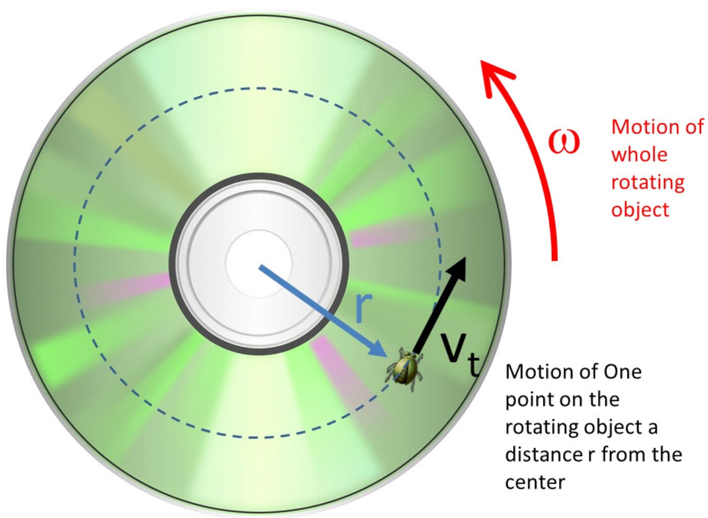
$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Let's suppose we have a small bug on the CD that is we want to spin up. The bug isn't the whole CD. It is just located on part of the CD a distance r from the center. We could find out the speed of the bug. The bug is going in a circle, so it would be a tangential speed. We remember from last lecture that

$$\omega = \frac{v_t}{r}$$

so we could say

$$v_t = r\omega$$



We have related the angular speed of the whole spinning platform (the CD) to the motion of just one spot on that platform (the bug). Our equation tells us that if the bug crawls toward the center of the CD it will lower it's speed. If you have played on a playground merry-go-round you know this is true. If the bug is on the CD before you press "play," then the CD and bug will accelerate. The entire CD will accelerate with an angular acceleration

$$\alpha = \frac{d\omega}{dt}$$

but we could also think of the bug accelerating. It is still moving in a circle, so it's speed is still tangential. We require a tangential acceleration, a_t, to change a tangential speed.

$$a_t = \frac{\Delta v_t}{\Delta t}$$

but surely that acceleration of the bug is tied in some way to the angular acceleration of the entire CD. Let's mathematically investigate this.

$$\alpha = \frac{d\omega}{dt}$$

but since $\omega = v_t/r$

$$\alpha = \frac{d}{dt}\left(\frac{v}{r}\right) = \frac{1}{r}\frac{dv}{dt} = \frac{1}{r}a_t$$

and since r is not changing for our bug,

$$\alpha = \frac{1}{r}\frac{dv}{dt} = \frac{1}{r}a_t$$

that is, if we know the acceleration and the radius of our rotating platform, then the angular acceleration for just one spot on that platform is just

$$\alpha = \frac{1}{r}a$$

Note that we have a complete set of motion variables to describe motion in a circle

One Spot		Whole Platform
$\Delta s = s_f - s_i$		$\Delta\phi = \phi_f - \phi_i$
$\Delta t = t_f - t_i$		$\Delta t = t_f - t_i$
$v_{t_{ave}} = \frac{\Delta s}{\Delta t}$		$\omega_{ave} = \frac{\Delta\phi}{\Delta t}$
$v = \frac{ds}{dt}$		$\omega = \frac{d\phi}{dt}$
	$\omega = \frac{v_t}{r}$	
$a_t = \frac{dv_t}{dt}$		$\alpha = \frac{d\omega}{dt}$
	$\alpha = \frac{a_t}{r}$	

Notice that the equations for the rotational of the whole platform are very like the equations for the motion of just one spot on the platform!

Back when we were working just linear problems we found a set of equations for constant acceleration problem, for example, from $a = \Delta v/\Delta t$ we found

$$v_f = v_i + a\Delta t$$

This is one of our kinematic equations. We could do the same thing for our rotational variables, for example from $\alpha = \Delta\omega/\Delta t$ we would have

$$\omega_f = \omega_i + \alpha\Delta t$$

This looks like a kinematic equation for rotation...and it is! We could do the math to find the rest of the set, but I am just going to state them here

$$\Delta\theta = \omega_i\Delta t + \frac{1}{2}\alpha\Delta t^2 \tag{12.2}$$

$$\omega_f = \omega_i + \alpha\Delta t \tag{12.3}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\phi \tag{12.4}$$

$$\Delta\theta = \frac{1}{2}\left(\omega_f + \omega_i\right)\Delta t \tag{12.5}$$

The most wonderful thing about this is that we know how to use the kinematic equations already, and we can use the same techniques to solve rotational constant angular acceleration problems as we did to solve linear constant acceleration problems. Let's do some examples.

Suppose we have a bicycle wheel and it rotates 5.40 revolutions. How many radians has it rotated?

We know we have 5.40 revolutions,

and our basic equation is

$$\Delta\phi = \phi_f - \phi_i$$

but what is a revolution? The word "revolution" means "goes all the way around." Then every revolution is $2\pi\,\mathrm{rad}$

$$1rev = 360\,^{\circ} = 2\pi\,\mathrm{rad}$$

so

$$\begin{aligned} & 5.40rev\frac{2\pi\,\mathrm{rad}}{1rev} \\ = \quad & 33.929\,\mathrm{rad} \end{aligned}$$

Note that this is bigger than $2\pi\,\mathrm{rad} = 6.2832\,\mathrm{rad}$. And this brings up a major difference between linear displacement and angular displacement. Tradition in physics is that we can return to our zero point and keep going around to every larger numbers for rotational motion. With linear displacement, if we return to our starting point, we have a Δx of zero. But that is not what we do for angular displacement!

Let's continue with our bike wheel and do another example.

If our bike wheel made 5.4 revolutions in $0.5\,\mathrm{s}$, what is the angular speed?

We know

$$\begin{aligned} \Delta\phi &= 5.4rev = 33.929\,\mathrm{rad} \\ \Delta t &= 0.5\,\mathrm{s} \end{aligned}$$

and our basic equation is

$$\omega = \frac{\Delta\phi}{\Delta t}$$

we know all the parts, so we can put in numbers

$$\begin{aligned}\omega &= \frac{33.929\,\mathrm{rad}}{0.5\,\mathrm{s}} \\ &= 67.858\frac{\mathrm{rad}}{\mathrm{s}}\end{aligned}$$

Let's do another example. Suppose we started our bike wheel at $t_i = 0$ with $\omega_i = 0$ and suppose it spins up to speed $\omega_f = 67.858\frac{\mathrm{rad}}{\mathrm{s}}$ in 5 seconds. what is the angular acceleration? We know

$$\begin{aligned}t_i &= 0 \\ t_f &= 5\,\mathrm{s} \\ \omega_i &= 0 \\ \omega_f &= 67.858\frac{\mathrm{rad}}{\mathrm{s}}\end{aligned}$$

our basic equations are the rotational kinematic set

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha \Delta t^2$$
$$\omega_f = \omega_i + \alpha \Delta t$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\phi$$

and our basic rotational motion set

$$\Delta\omega = \omega_f - \omega_i$$
$$\Delta\phi = \phi_f - \phi_i$$

We can use the second of our set of rotational kinematic equations

$$\omega_f = \omega_i + \alpha \Delta t$$

we can solve this angular kinematic equation for α using our basic motion equations

$$\begin{aligned}\alpha &= \frac{\omega_f - \omega_i}{\Delta t} && (12.6) \\ &= \frac{\omega_f - \omega_i}{t_f - t_i} && (12.7) \\ &= \frac{67.858\frac{\mathrm{rad}}{\mathrm{s}} - 0}{5\,\mathrm{s} - 0\,\mathrm{s}} && (12.8) \\ &= 13.572\frac{\mathrm{rad}}{\mathrm{s}^2} && (12.9)\end{aligned}$$

Caution about Angular Acceleration

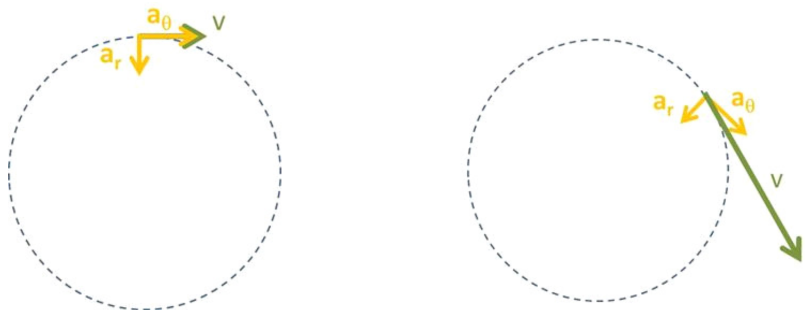
We need to be careful. We now have two kinds of speed, linear and rotational, and two kinds of acceleration, linear and rotational. How do we keep them from being confused? Usually angular acceleration is specifically called "angular acceleration." To get you used to this practice, I will use the terms this way. "Acceleration" without the word "angular" means the kind of acceleration we have been using all along. Only assume angular acceleration if the problem specifically says "angular" in the problem statement.

Constant Tangential Acceleration

We have dealt with the centripetal acceleration, that is, the part that points toward the center of a circle. This acceleration makes the object turn. It is in the radial inward direction. But we know there can be more to acceleration. We could speed up or slow down.

Remember that speeding up or slowing down comes from acceleration that is in the same direction or opposite direction of the motion.

Our centripetal acceleration can never make the object speed up or slow down. But now we know from rotational that we could have a tangential acceleration as well as a centripetal acceleration! The velocity as we go around the circle is always tangential.



so the tangential acceleration will make the object speed up or slow down. Looking at the figure is should be clear that we could add the centripetal and tangential accelerations to get a total acceleration.

$$a = \sqrt{a_r^2 + a_\theta^2}$$

or we could write this as $a_r = a_c$ for centripetal and $a_\theta = a_t$ for tangential.

$$a = \sqrt{a_c^2 + a_t^2}$$

Kinematics around the circle

Our arclength, s, is just a distance (think of it like the distance the bug goes around the circle on the CD), and we can define a change in arclength

$$\Delta s = s_f - s_i$$

and we have defined the speed as we go around the circle as

$$v = \frac{ds}{dt}$$

By now you are probably guessing that I could define a whole set of kinematic equations for the arclength traveled around a circle.

$$s_f = s_i + v_{ti}\Delta t + \frac{1}{2}a_t \Delta t^2$$
$$v_{tf} = v_{ti} + a_t \Delta t$$

$$v_{tf}^2 = v_{ti}^2 + 2a_t \Delta s$$

but notice that we had to be careful. Only the tangential acceleration, a_t, will make the object speed up or slow down as it goes around the circle, so only the tangential acceleration can appear in our equations for arclength kinematic.

Let's try a problem using these equations.

suppose you get in your race car and the initial speed is $v_i = 0$ but in $2.00\,\mathrm{s}$ you are going around a circular track with speed $v_f = 25\,\mathrm{m/\,s}$. What is your tangential acceleration?

We know

$$\begin{aligned} v_{ti} &= 0 \\ v_{tf} &= 25\,\mathrm{m/\,s} \\ \Delta t &= 2.00\,\mathrm{s} \end{aligned}$$

and our basic equations are

$$s_f = s_i + v_{ti}\Delta t + \frac{1}{2}a_t\Delta t^2$$
$$v_{tf} = v_{ti} + a_t\Delta t$$

$$v_{tf}^2 = v_{ti}^2 + 2a_t\Delta s$$

If we underline what we know and use our zeros we have

$$\underline{v_{tf}} = a_t\underline{\Delta t}$$

$$s_f = s_i + v_{ti}\underline{\Delta t} + \frac{1}{2}a_t\underline{\Delta t}^2$$

$$\underline{v_{tf}^2} = 0 + 2a_t\Delta s$$

it looks like the first of these equations will work.

$$\underline{v_{tf}} = a_t\underline{\Delta t}$$
$$\frac{\underline{v_{tf}}}{\underline{\Delta t}} = a_t$$
$$\begin{aligned} a_t &= \frac{25\,\mathrm{m/\,s}}{2.00\,\mathrm{s}} \\ &= 12.5\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Kinematic has allowed us to solve linear, two dimensional, rotational, and arclength motion problems! We have gone a long way with the ideals of displacement, velocity, and constant acceleration. But you are probably wondering how we get things to move in the first place. In our next few lectures we will take on this topic.

13 Changing Motion

We have talked at length about displacement, velocity, and acceleration. But so far we have not discussed how we get motion started. When you get in your car, you know the engine is involved in changing your speed from $0\,\mathrm{m/s}$ to $26\,\mathrm{m/s}$. So there must be an acceleration. But how does this work? How do we start motion? And then, when you get to the parking lot, you want to go from your $26\,\mathrm{m/s}$ back to $0\,\mathrm{m/s}$. How do we change motion once we have it?

Really we have discussed this. We have had spring cannons, and people throwing balls. From the examples we have had, we know the answer. To change motion, we give something a push! The spring in the spring cannon pushes the ball. The person's hand pushes the ball. The tires of the car push against the road, and the road pushes back, making the car go forward.

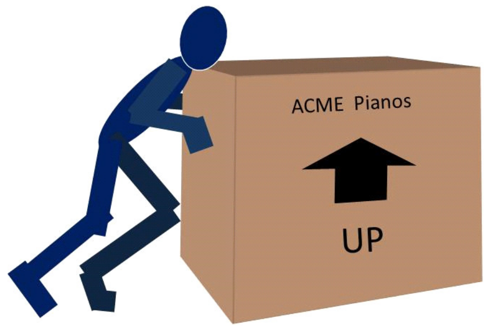
Of course, we could also pull something by tying a rope on it and tugging. This may not sound very scientific, changing motion by pushing or pulling. So let's give a new name to a push or a pull. Let's call a push or a pull a *force*.

To change a motion, there needs to be two objects involved. One that is doing the moving, and the other creating the force. In this course, we will call the moving object the *mover*. The other object makes the force that causes the change in the mover's motion. We will call the object that makes the force the *environmental object*. This environmental object is changing the environment in which the mover moves.

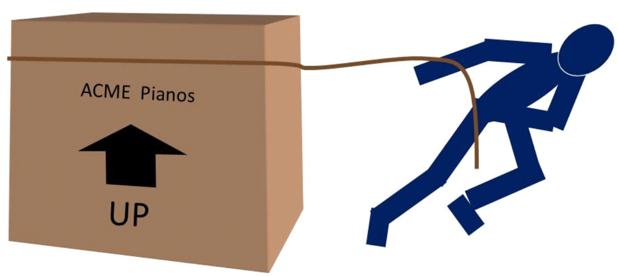
That last sentence was a whopper! Let's see if we can restate it in simpler terms. What we mean is that the second object is doing the pushing or the pulling. It is something in the near vicinity of the mover (the mover's environment) that pushes or pulls on the mover. The volume of space around the mover we will call the *mover environment*. This is why we will call the object in the region around the mover that does the pushing and pulling the "environmental object" so we know it is not the object who's movement we are studying.

Types of forces

Some pushes and pulls are obvious. If I push on a piano, you see me with my hands on the piano, and watch it change its motion. You are justified in concluding that the contact between my hands and the piano have somehow caused the motion of the piano to change. The piano is the mover in this example, and I am the environmental object making the force.



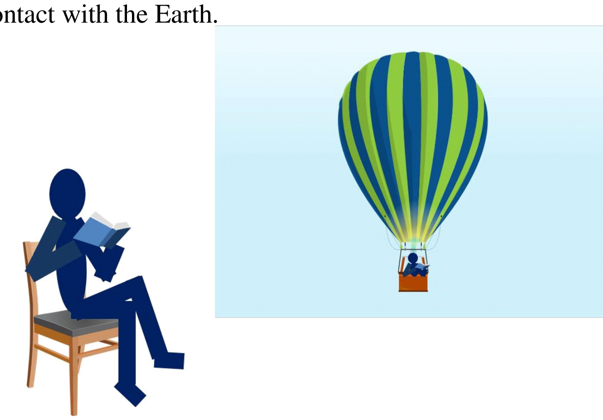
Likewise, if I tie a rope to the piano, and pull, you would see me, the rope, the piano, and the change in motion and conclude with justification that my pull from the rope was responsible for the change in motion. The piano box is still the mover, and now the rope is the environmental object pulling on the box. You might object to this analysis. Isn't it the person pulling that is making the force? of course you are right, ropes don't pull on objects unless there is another object pulling on them. But since the box is the mover, I am only concerned about what is creating the force on the box, not what is creating the force on the rope. So it is correct to say that the rope is the environmental object for the box.



In both of these cases, physical contact was necessary for the mover (the piano box) to change motion. We will call this type of force that requires physical contact a *contact force*.

But there are other ways to change motion. Suppose I give you a magnet and have you walk over to a workbench covered with nuts and bolts. You can predict that the magnet will change the motion of the nuts and bolts even before it makes contact with them. The nut (see the figure) is the mover and the magnet is the environmental object making the force due to magnetism. The magnet doesn't have to touch the nut to move it, so we will call this kind of force a *non-contact force.*

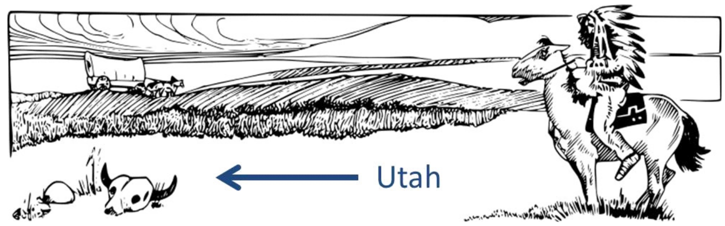
As you sit in your chair reading your physics reading assignment, another non-contact force is acting on you. It is the force of gravity. The Earth is the environmental object making the force due to gravity that is pulling you down. Right now you are probably in contact with the Earth, but the force of gravity would be pulling on you even if you were not in contact with the Earth.



The Earth's gravitational force is a non-contact force.

Forces are vectors

Suppose we hitch a team of oxen to a wagon. The wagon is our mover, and the oxen are the environmental objects pushing on the wagon tongue to change the wagon's motion. But does it matter which way the oxen pull?

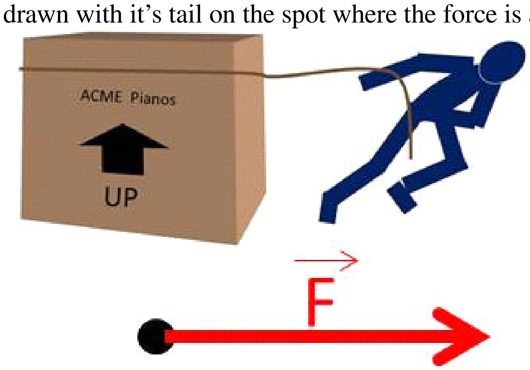


Of course it does! Forces must have direction, and we know what to call a quantity that has a magnitude and a direction. Forces are vectors.

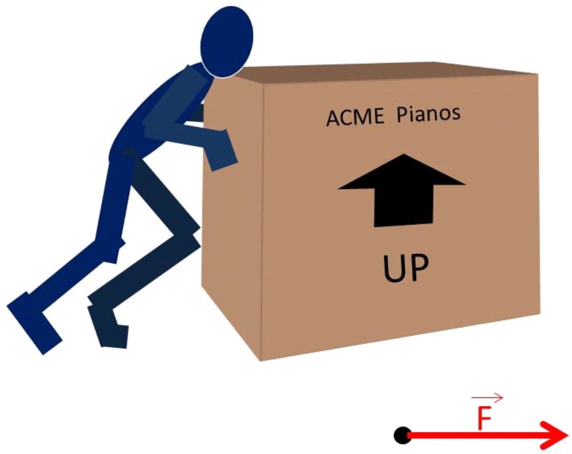
The magnitude of the force is how hard it pushes or pulls, and the direction is which way it pushes or pulls. In our wagon example, we want the force to pull to the west $270\,^{\circ}$ on a compass. If they pull at $90\,^{\circ}$, they get to Boston instead of Salt Lake City. That would make a difference!

Let's review vectors for forces.

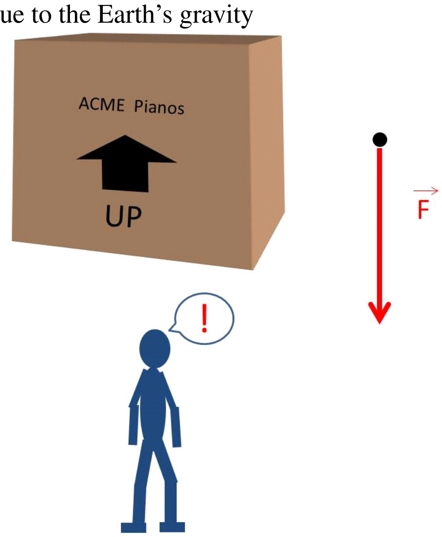
A force vector is drawn with it's tail on the spot where the force is applied.



We still use the particle model. So we draw the whole piano box as just a dot. Then we draw the force vector with the length proportional to how hard the rope is pulling on the box, and make the arrow point the way the box is begin pulled. Here is a vector representation of the box being pushed

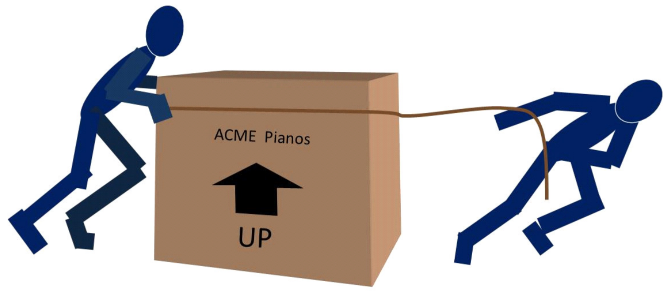


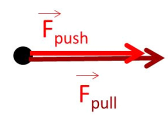
and here is the force due to the Earth's gravity



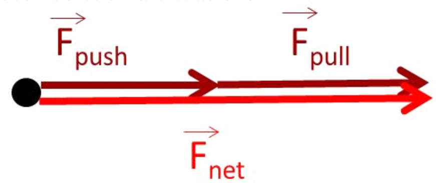
In each case the object is drawn as a article, just a dot, and the vector shows how strong our push or pull is, and in what direction the push or pull is pointing.

Suppose we have more than one push or pull





You know from experience that two pushes or a push and a pull working together are more likely to change the motion of an object. We can use our wonderful vector notation and mathematics to describe such a situation.



Vectors add by placing the tail of the second vector on the tip of the first vector, then drawing a vector from the tail of the first to the tip of the second. This is the sum or net vector.

$$\vec{F}_{net} = \vec{F}_{push} + \vec{F}_{pull}$$

If the piano box was very heavy, it might take a whole group of people to push it. To keep our equations small, we could use summation notation. We give each person a number, so we can tell which person caused which force. Then the net force is just

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6$$

which we can write as

$$\vec{F}_{net} = \sum_{n=1}^{6} \vec{F}_n$$

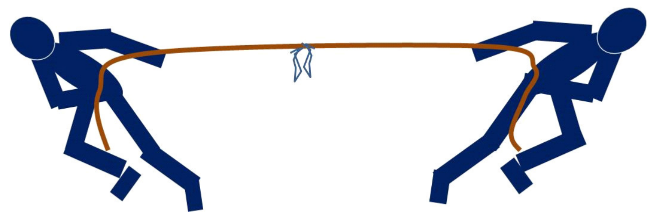
This looks like a smaller equation, but it really has just the same meaning as the equation before it. And note that the summation is a vector summation, That means for two-dimensional problems we really want to take components of the force vectors and separate our problem into x and y-parts

$$F_{net_x} = \sum_{n=1}^{6} F_{xn}$$

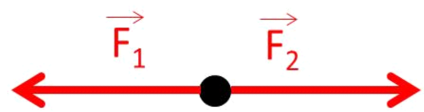
$$F_{net_y} = \sum_{n=1}^{6} F_{yn}$$

Let's try a problem.

Suppose we have two people pulling on a rope, but they pull opposite directions. They each pull with a force of $5\,\mathrm{N}$. What is the net force.



We first draw a vector diagram for the forces.



We can see that all the forces are in the x-direction and that they point opposite directions as expected.

We can write

$$\begin{aligned} F_{net_x} &= \sum_{n=1}^{6} F_{xn} \\ &= F_{1x} + F_{2x} \\ &= F_1 \cos\theta_1 + F_2 \cos\theta_2 \end{aligned}$$

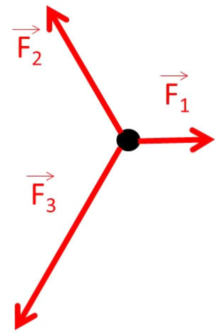
and we can see that $\theta_1 = 180\,^\circ$ and $\theta_2 = 0\,^\circ$. We know all the pieces so we can put in values

$$\begin{aligned} F_{net_x} &= (5\,\mathrm{N})\cos(180\,^\circ) + (5\,\mathrm{N})\cos(0\,^\circ) \\ &= -5\,\mathrm{N} + 5\,\mathrm{N} \\ &= 0 \end{aligned}$$

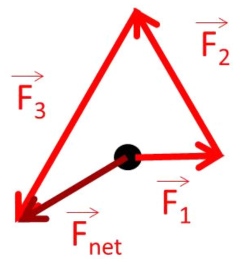
Notice how much direction mattered in this case! Also notice that we have introduced a unit for force, the *Newton*, abbreviated N.

Let's try another problem. Suppose we have three forces applied to an object, all spaced $120\,^\circ$ from each other. Force 1 has a magnitude of $10\,\mathrm{N}$ and is at $0\,^\circ$. Force 2 has a magnitude of $20\,\mathrm{N}$ and is at $120\,^\circ$. Force 3 has a magnitude of $30\,\mathrm{N}$ and is at $240\,^\circ$. What is the net force?

We will need a vector diagram for the forces



And we could use graphical vector addition to see what $\overrightarrow{F}_{net}$ will look like



and we recognize that from the problem statement we know

$$\begin{aligned}F_1 &= 10\,\mathrm{N}\\F_2 &= 20\,\mathrm{N}\\F_3 &= 30\,\mathrm{N}\\\theta_1 &= 0\,^\circ\\\theta_2 &= 120\,^\circ\\\theta_3 &= 240\,^\circ\end{aligned}$$

Let's use our vector component set of equations to solve this:

$$\begin{aligned}v_x &= v\cos\theta\\v_y &= v\sin\theta\end{aligned}$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Separate the two-dimensional problem into two one-dimensional problems using our force component summation equations

$$F_{net_x} = \sum_{n=1}^{6} F_{xn}$$

$$F_{net_y} = \sum_{n=1}^{6} F_{yn}$$

then

$$F_{net_x} = F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3$$
$$F_{net_y} = F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3$$
$$F_{net_x} = \sqrt{F_{net_x}^2 + F_{net_y}^2}$$
$$\theta = \tan^{-1}\left(\frac{F_{net_y}}{F_{net_x}}\right)$$

$$\begin{aligned} F_{net_x} &= (10\,\mathrm{N})\cos(0\,^\circ) + (20\,\mathrm{N})\cos(120\,^\circ) + (30\,\mathrm{N})\cos(240\,^\circ) \\ &= -15.0\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} F_{nety} &= (10\,\mathrm{N})\sin(0\,^\circ) + (20\,\mathrm{N})\sin(120\,^\circ) + (30\,\mathrm{N})\sin(240\,^\circ) \\ &= -8.660\,3\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} F_{net_x} &= \sqrt{(-15.0\,\mathrm{N})^2 + (-8.660\,3\,\mathrm{N})^2} \\ &= 17.321\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-8.660\,3\,\mathrm{N}}{-15.0\,\mathrm{N}}\right) \\ &= 0.523\,6\,\mathrm{rad} \\ &= 30.0\,^\circ \end{aligned}$$

Looking at our diagram we realize that this angle can't be right. We should be in the third quadrant because both F_{net_x} and F_{net_y} are negative. Our inverse tangent gave the angle with respect to the $-x$-axis. We usually want to report an angle with respect to the $+x$-axis. We need to add $180\,^\circ$ to our result to get the angle with respect to the positive x-axis.

$$\begin{aligned} \theta &= 180\,^\circ + 30\,^\circ \\ &= 210\,^\circ \end{aligned}$$

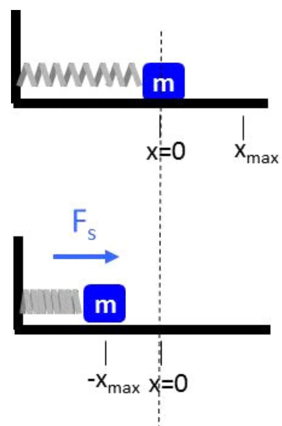
This seems reasonable

Origins of Forces

To practice solving problems with forces, we will need some environmental object acting on mover objects. So we need to know how environmental objects can act. We can't explain every type of force in this section, but we will explain a few forces, enough to start doing problems.

Spring Force

You may have had a toy dart gun as a child. The dart gun works because when you push in the dart, it compresses a spring. The compressed spring pushes on the dart–and that is a force!

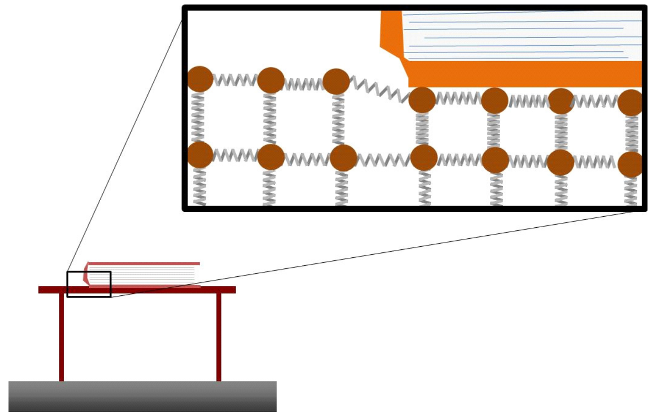


This is also the way our spring cannons work.

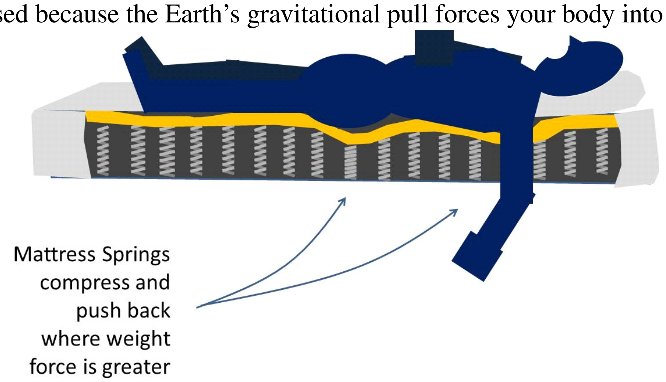
Pushing (Normal Forces)

We have already discussed direct contact and pushing and pulling. In PH220 we are going find that this direct contact is not so simple as we might think. It will involve forces due to the electric charges in the atoms that make up our objects. And those forces are non-contact forces! But that is getting way too far ahead of the physics story. We need just enough here to allow us to study forces.

Suppose we place a book on a table. the book has mass. You probably already know that the book will be pulled downward by the Earth's gravity, so the book has a weight. *Weight* is what we call the force due to gravity that pulls things toward the Earth. But the book can't fall to the Earth because the table atoms are in the way.



This is a little like sleeping on a mattress. The mattress is made of springs. The springs are compressed because the Earth's gravitational pull forces your body into the springs.



where more of your mass is concentrated, the springs are compressed more. Compressed springs push back with a spring force.

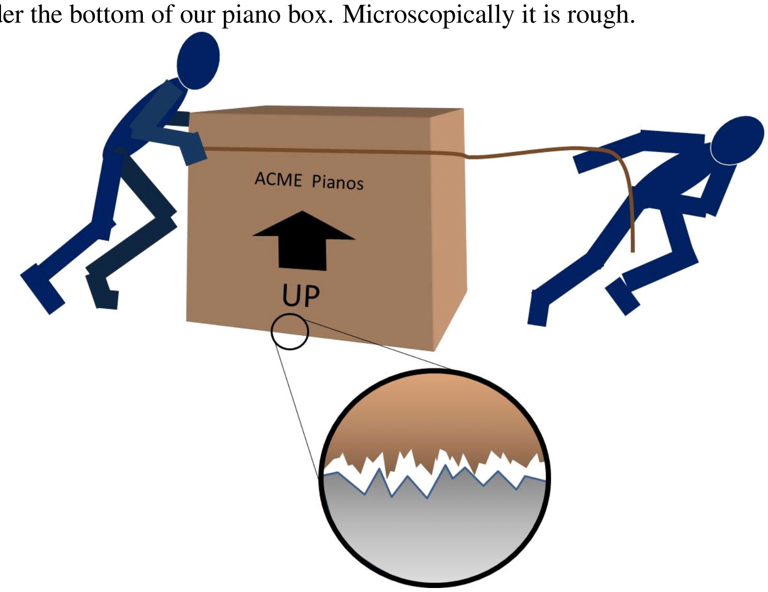
The molecular bonds in the table are a lot like little spring forces. The weight of the book compresses these bonds, and the spring-like bonds make the atoms push back. We call this a *normal force*. The old word "normal" used to mean "perpendicular." And this is where this force got it's name. The springs always push back perpendicular to the surface that is being squashed.

Pulling (Tension Force)

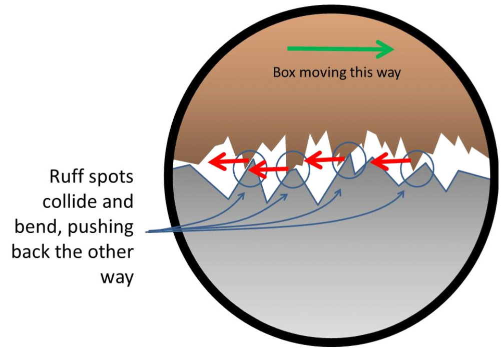
We have also discussed pulling on ropes. But how does a rope manage to get pulled without the fibers of the rope falling apart. To understand how it works, we need to jump ahead and think of the atoms that build up the rope fibers. The atoms are held together by bonding forces that are really due to the electric forces of the subatomic particles. These atomic bonds can stretch a little like the springs we were just discussing. The spring-like forces that hold the molecules are the reason the rope can pull on a box. The fibers stretch, like springs, pulling on the thing they are attached to. This force created by the stretched atomic bonds is called *tension*.

Friction Force

If you have been in Rexburg long, you have experienced surfaces with little friction. Friction is a push, and again it is due to the bonds in molecules that are being stretched. Consider the bottom of our piano box. Microscopically it is rough.



And the floor under the box is also microscopically rough. As the people push and pull the box the rough spots on the floor and the box catch against each other. As the box is pushed the rough spots collide and bend. This bending stretches the spring-like bonds between the atoms in the box and floor materials.



The bent and stretched molecules in the floor material push back against the box molecules. This backward push is what we call friction.

Gravitational Force

We have discussed that the Earth's gravity tends to make things accelerate downward. This is because the Earth's gravity is a force! So far all of our forces have been contact forces where atoms are involved in making the force. But this gravitational force is fundamentally different. It is a non-contact force. It takes general relativity to truly understand gravity (even then there is some uncertainty!) so for now we will just state that the Earth and other objects with mass pull on other objects with mass. This is gravity.

We call the pull or force due to gravity *weight.*

Sometimes we refer to how much matter we have in our body as our weight. This is because the pull of gravity is proportional to our weight. But they are not the same thing. Weight is a force. How much matter we have is a mass.

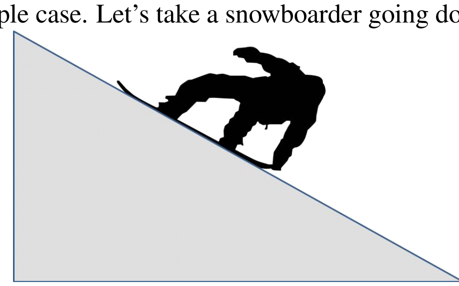
Other force origins

There are other forces like thrust and air resistance that we won't use for a while. As we need these forces, we'll discuss their origins. But for now, we have plenty of forces to get started studying how motion is changed.

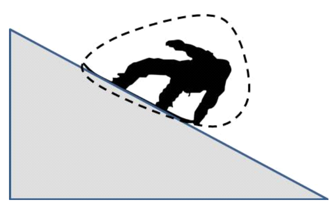
14 Force Diagrams, Mass, and Acceleration

You can guess that with a new topic, we will need to learn new techniques for restating our problems as drawings. A large part of the task of drawing the situation will be to identify what is the moving object, and what is the environment. You may not believe that this could be difficult. But it often is.

Let's start with a simple case. Let's take a snowboarder going down a slope.

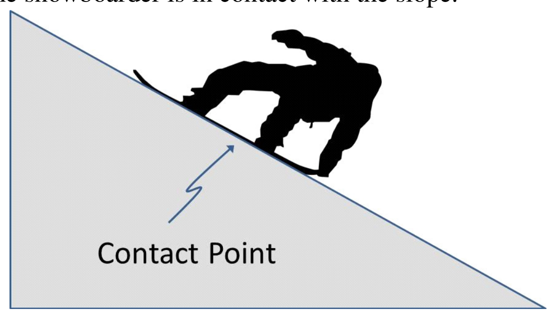


The snowboarder would be the moving object. If you draw a picture of the snowboarder situation as part of your drawing, it might be good to circle the snowboarder to indicate that it is the mover.



We should identify where forces may be acting on our mover object (the snowboarder). We can guess that gravity is pulling down on the snowboarder. Gravity is a non-contact force. And it pulls on just about everything that is near the Earth's surface. The Earth must be an environmental object acting on the snowboarder. The gravitational force will pull the snowboarder in the direction of the Earth's center.

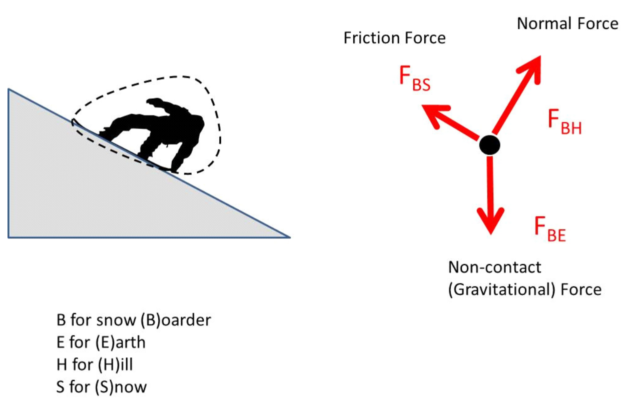
It is unlikely that other non-contact forces will be acting on the snowboarder (the snowboarder is unlikely to have a magnetic suit or an overall electric charge). So let's consider contact forces. The snowboarder is in contact with the slope.



It is likely, then, that the slope is an environmental object acting on the snowboarder. To be sure, we need to identify the mechanism for a slope force acting on the snowboarder. We can recognize that the hill atoms will be compressed by the weight of the snowboarder. The hill atoms will resist being compressed. They will push back on the snowboard, creating a normal force.

It is also true that there is a friction force due to the rough snow. You probably know that you need to wax your snowboard to reduce this friction. But some friction will still act on the snowboard. The friction force always acts in a direction opposite the direction the mover is going. So the friction force would act up the slope.

We put all this information about the forces acting on our mover object (the snowbaorder) into a diagram. Using the particle model, we draw a dot that represents the snowboarder.

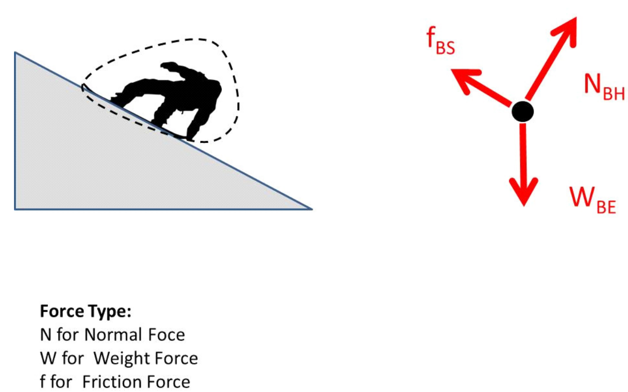


Arrows represent the force vectors. Their tails are on the dot, indicating that the forces act at the location of the mover. Notice that the length of the arrows are different. The force vector lengths give the relative magnitude of the forces. Also notice that we have drawn the vectors in the direction the forces act.

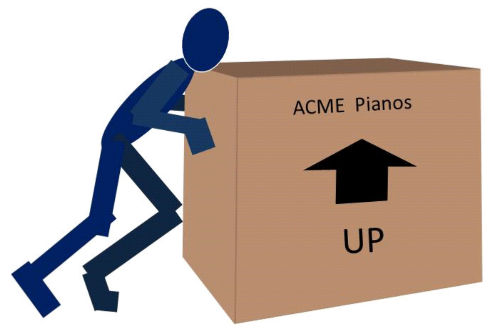
This particle model diagram with force vectors is called a *free body diagram*. In the old days, the word "body" was used instead of the word "object." So maybe we should call it a "mover object diagram." The diagram is for one object only! It just contains forces acting on the mover object. If we have two objects that can move, we need two free body diagrams, one for each mover. Our goal is always to be able to study the motion of an individual object. If I have more than one individual object, I split the problem into parts, one part for each mover object.

Notice that we have again used a set of subscripts. For each force we have identified the two objects involved in the force. The first subscript is for the mover object. In this case it is the snow(B)oarder. The second is the environmental object. In the case of gravity, it is the Earth that is pulling on the snowboarder, so we have used and "E" for "Earth" as the subscript for the gravity force.

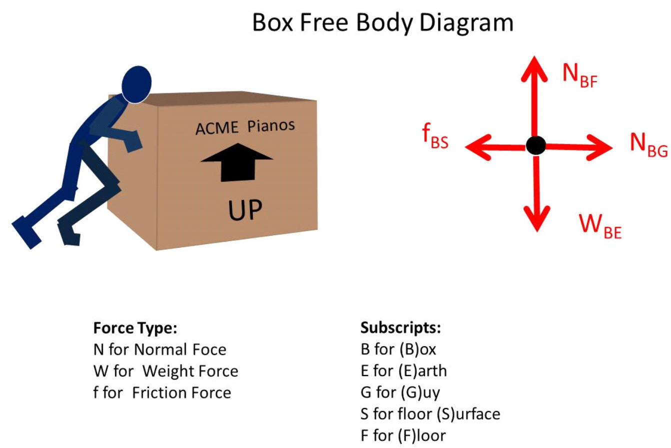
We can further clarify our diagram by labeling different force types with different letters. After all, we know they are forces so the F is not telling us anything we don't already know! We could use a "N" to label a normal force, for example. A small "f" is traditional for friction forces. And for gravitational forces, we could use a "W" for "weight force." Sometimes the letters F_g are use for a force due to gravity, but the subscript "g" gets in the way of our mover-environment subscripts. So for now we will prefer "W." Here is how it might look.



Let's try another example. Consider again a guy pushing a box. What forces act on the box?



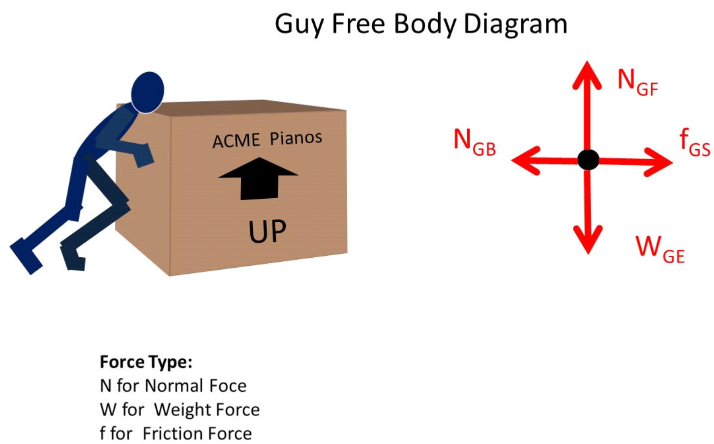
Let's draw a free body diagram.



Notice that each force is identified by it's letter symbol. The guy's atoms press against the box, so the guy's push is due to the guy's compressed atoms. It is a normal force! We recognize the normal force due to the floor holding up the box, the friction force, and the gravitational weight force.

The subscripts all start with "B" for "box." The box is our mover. The rest of the subscripts tell us what environmental objects act on the box. We can see that the Earth is pulling on the box as expected. We know the guy is pushing on the box. We know the floor is pushing up on the box. We could use the "F" for floor in both the normal force and the friction force. But I have chosen to indicate that the forces are fundamentally different by saying the floor rough surface is the source for the friction force, using an "S" for this aspect of the floor, and leaving the floor's compression strength to be indicated by the "F."

Now let's draw the diagram for the guy! We have two moving objects. We chose the box for our lasts diagram, but we could draw a diagram for the guy as well.

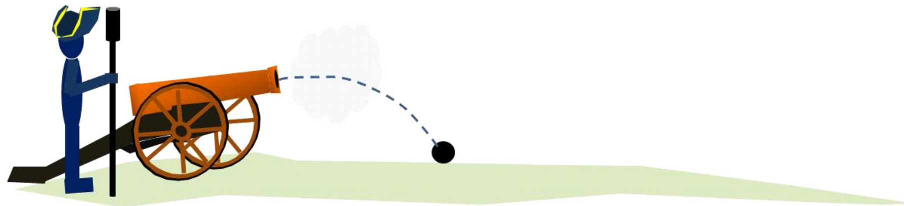


Note that for the guy's diagram each subscript set begins with a "G" for "guy." The guy pushes on the box, but the squashed atoms of the box push back. We see that in the normal force N_{GB}. The guy has a weight, W_{GE}, and is held up by a normal force due to the floor, N_{GF}. If the floor were perfectly smooth, the guy would not be able to move himself or the box. So there must be a friction force. The guy pushes his feet against the floor, and the rough floor pushes back. This friction force is why the guy moves forward. We label it f_{GS}.

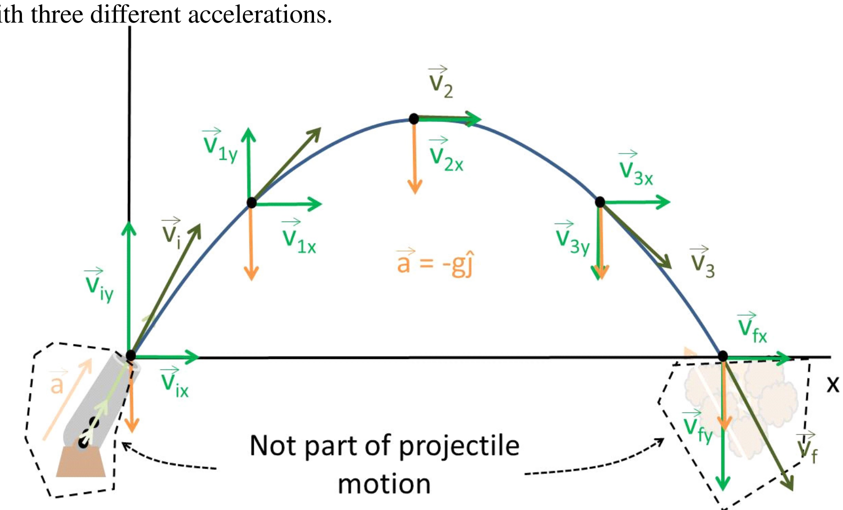
Force and Acceleration

We said last lecture that forces change motion. Changing motion is acceleration. So forces must cause acceleration. This should make intuitive sense. If I push on an object, it begins to move. my push accelerated the object. From experience you might guess that the harder we push, the larger the acceleration. Let's take an example to see that this is true.

Suppose you are a soldier in the French Revolution. You have been assigned to use a cannon. You load the powder and cannon ball in the cannon, but this is the result:

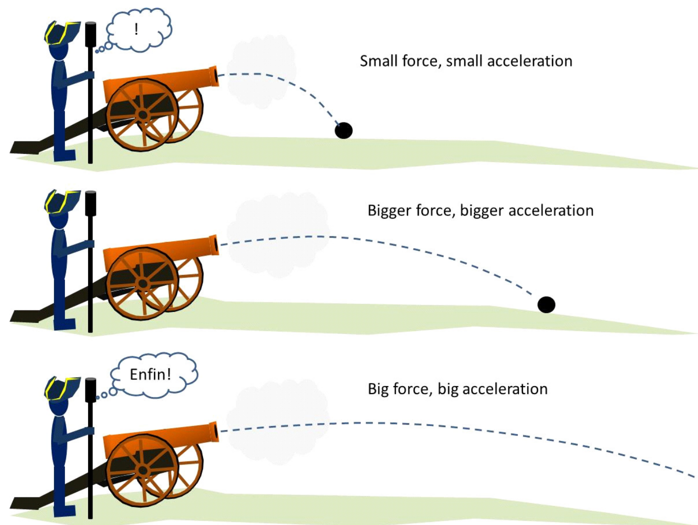


Last time we studied the flight of a cannon ball, we divided the problem into three parts with three different accelerations.



Back then, we only studied the second part of the motion, the projectile motion part. We are now experts in projectile motion, so let's use what we know about projectile motion to study the first part of the cannon problem, the part where the cannon ball is in the cannon and the exploding gunpowder is exerting a force on the ball.

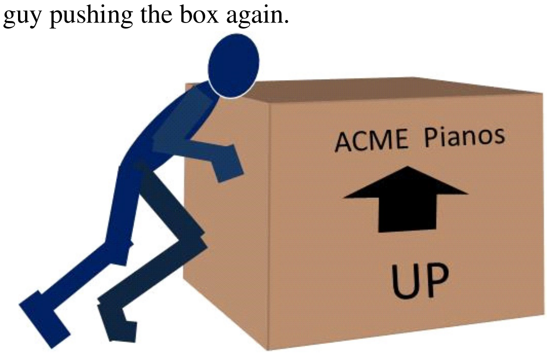
Think about our projectile motion study. We know that the acceleration of the cannonball after it leaves the cannon is free-fall acceleration, $-g$. We know that how far the cannon ball goes depends on the initial speed of the cannon ball once it leaves the cannon. That initial speed depends on how much acceleration the ball has while it is inside the cannon. The power provides a force that causes this acceleration when the powder ignites. What would you think if you received the above result? Not enough powder! There was not enough force to accelerate the ball enough make a good initial velocity for the free-fall part of the cannon problem. How would we fix this? Add more powder! That is, we will make a bigger force. A bigger force makes a bigger acceleration, providing a larger initial velocity for the free fall part. By trial and error you would eventually find the right force to make the cannon effective.



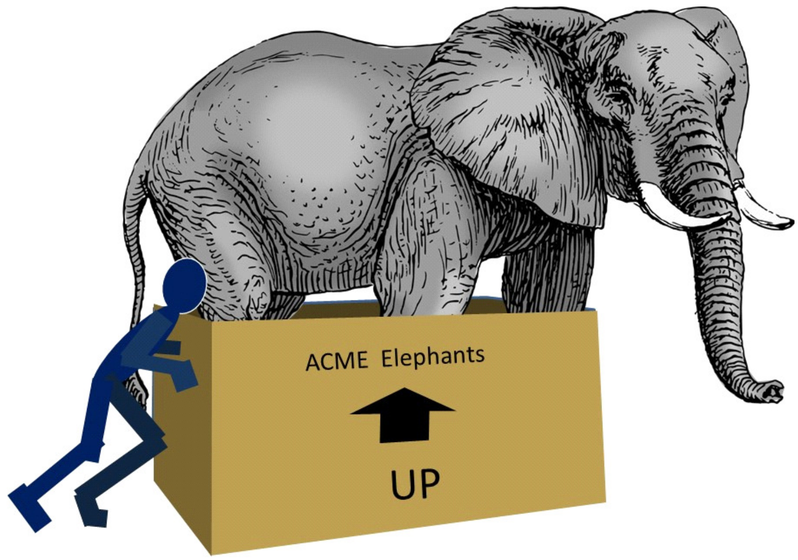
Let's summarize what we have said so far. It looks like the acceleration is proportional to the force that creates it.

$$a \propto F$$

The harder we push the more acceleration we get. But there must be more to this. Suppose we take our guy pushing the box again.



but we substitute something larger for the box, say an elephant.



You might guess that the elephant won't accelerate as much as the box (unless you get it angry). The amount of material in the object seems to matter in how effective a force is in causing acceleration. We could say this mathematically by writing

$$a \propto \frac{1}{m}$$

that is, we get less acceleration when we have a more massive object. We can combine these two effects into one equation

$$a = \frac{F}{m}$$

which tells us that our object will accelerate more if we push hard, or if we somehow reduce the mass of the object.

Sometimes you will see this written as

$$F = ma$$

which says that the harder we push the more acceleration we get, but that if the mass is larger the push has to be larger to get the same acceleration. Of course, forces and accelerations are vector quantities, but mass is scalar. So let's use vector notion for our equation.

$$\overrightarrow{F} = m\overrightarrow{a}$$

Now we can see that the force and acceleration must be in the same direction as well.

Units

We can see from our discussion above that the units of force must be the units of acceleration times the units of mass

$$\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2} \tag{14.1}$$

We have given a name for this combination of units. It is called the *Newton*.[6] It is abbreviated N.

$$1\,\mathrm{N} = \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2} \tag{14.2}$$

We will continue to consider forces in our next lecture. Forces are so important that we have that the properties of forces have been condensed into a set of three force laws. We will begin our study these laws next time. But what is a scientific law? Recall that it is most often an equation that shows how our mental model of the universe works. The force laws are old enough that only one of them fits this definition. The second law can be stated as an equation, and it is the one we just found!

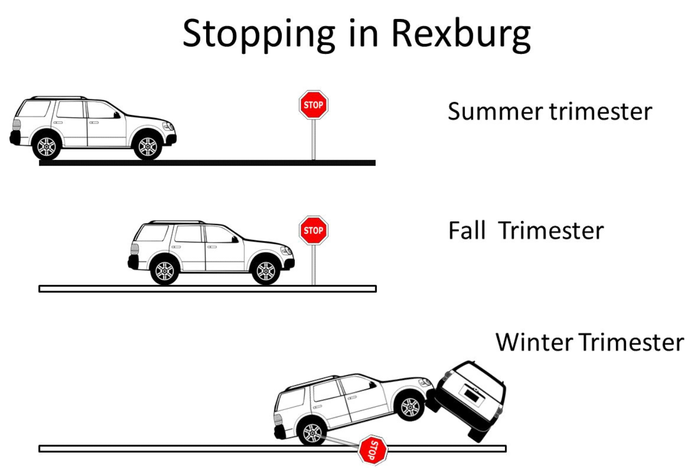
$$\overrightarrow{F} = m\overrightarrow{a}$$

The other two are less, mathematical, but equally important. we will start with the first and second laws.

[6] After Sir Isaac Newton, who was an early researcher that studied forces.

15 Newton's First and Second Laws and Equilibrium

Let's start with a BYU-I Rexburg area public service announcement. Suppose you are driving in your car and you come to a stop sign. You should stop, of course. Let's start our experiment in summer. There is no problem, you press on the break peddle and the car stops



This is because the tires stop spinning. That is what your breaks do, stop the tires. And since there is a frictional force due to the road on the tires of the car, then stopping the tires from spinning leads to the car stopping.

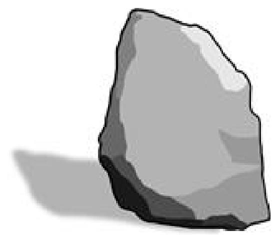
During the fall trimester, however, there is some snow and it get's cold. The friction between your tires and the road is reduced (the little teeth are full of ice). So it takes longer to stop. During winter trimester, the city resurfaces the ice to smooth it out, making the ice less rough. This makes the force due to friction smaller. Now, with little to no force acting on your car, it is nearly impossible to stop!

This situation is an example of a profound understanding of how our universe works. An object that is in motion stays in motion unless a force is acting on it. In our car case.

when the ice (and city) remove the force due to friction, there is no force acting on the car to make the car stop. So the car does not stop.

Another way to express this is to say that it takes a force to change motion, something that we already know.

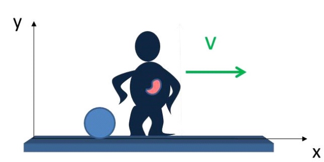
Let’s take another case. Here is a rock.

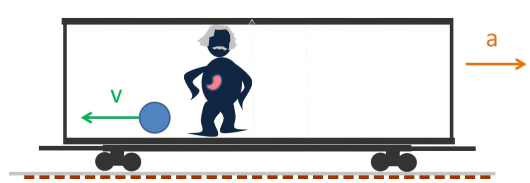


The rock is not moving. We would be surprised if the rock started to move on its own. Again this is a case of an object that not changing its motion unless a force is acting on it. In this case the motion is no motion with respect to the Earth. This has to be another case of the same profound thought. Forces are required to change motion.

Newton wrote down this profound viewpoint with some flair, so he gets credit for this thought. We call it *Newton’s first law.* But this law only tells us that without forces, motion does not change. We need the idea called *Newton’s second law* to explain the origin of all the motion we experience. And motion can get quite complicated. Since this is a PH121 (first semester physics class) let’s limit our discussion a little bit.

Let’s consider aliens on platforms again:





Notice that if we have a slightly larger space alien on his/her/its platform reference frame, he/she/it is free to believe that his/her/its reference frame has no motion. As we now know, that is because there is no net force acting on the ball, or the space alien's stomach, or anything else in the reference frame. We could write this as

$$\overrightarrow{F}_{net} = 0$$

But if we take a similar situation, say, a man in a train, but we make the train accelerate, we can see that the man will recognize that something is changing. The man's skeletal structure will move with the train, but his stomach, floating free of the skeletal structure, is not pushed by the train. It tries to stay at the same velocity (Newton's first law!). Likewise, the ball will try to stay in place. But since the man is viewing the ball and his stomach from the reference frame moving with the train car, he will "see" the ball move toward the back of the car and feel his stomach move backward relative to his skeleton. Both tell us that there is an acceleration, and therefore both tell us that there must be a net force.

$$\overrightarrow{F}_{net} \neq 0$$

Our idea of a reference frame needs a modification. We will call the alien's reference frame an *inertial reference frame*. This means that the reference frame, itself, is not accelerating. The train car we will call an *accelerated reference frame.* And we will leave accelerated reference frames for another physics class.[7] In this class we will only deal with intertial reference frames.

Now on to Newton's second law!

Newton's Second Law and Equilibrium

Our first of Newton's laws talks about an absence of forces, but we know there are forces. Newton's second law tells us how motion changes when forces act. Newton's second law states:

> *The acceleration of an object is directly proportional to the net force acting on an object and is inversely proportional to the mass of the object.*

Mathematically this is written as

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\mathbf{F}}_{net}}{m} \tag{15.1}$$

the variable $\overrightarrow{\mathbf{F}}$ stands for force. Notice that both $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{F}}$ are vectors. Remember that

[7] Specifically, a physics class that deals with Einstein's Theory of General Relativity.

because forces are vectors, they add like vectors. Then

$$\overrightarrow{\mathbf{F}}_{net} = \sum_{i=1}^{N} \overrightarrow{\mathbf{F}}_i$$

The symbol Σ means to sum or add up all the forces, so $\Sigma\overrightarrow{\mathbf{F}}$ is a way to write the net force. To find the net force, we add up all the forces as vectors. Also remember that forces are vectors, so we must use vector addition. Another way you may see this written is

$$\overrightarrow{\mathbf{F}}_{net} = m\overrightarrow{\mathbf{a}} \tag{15.2}$$

This Newton's second law applies to only one object at a time. The object who's mass is m is the object that is being pushed by $\overrightarrow{\mathbf{F}}_{net}$ and which has acceleration $\overrightarrow{\mathbf{a}}$. This means that *we will have to write out a separate Newton's second law equation for every moving object in our problems!*

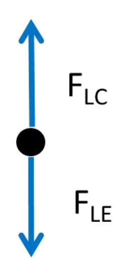
Suppose you are going to the temple and you observe the beautiful chandelier in one of the sealing rooms. The chandelier is not accelerating. This is a special case of equation ()

$$0 = \frac{\overrightarrow{\mathbf{F}}_{net}}{m}$$

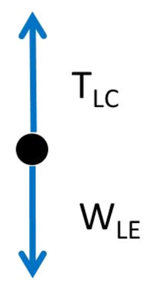
And it is an important special case. It says the net force is zero. It does not say that there are no forces acting on the object. But it says the forces balance, like two equally matched tug-o-war teams. So the object is not accelerating. We will call this situation *equilibrium.*

$$\overrightarrow{\mathbf{F}}_{net} = 0 \qquad \text{equilibrium}$$

The net force is the vector sum of all the forces. Let's find those forces and then sum them up to see if it makes sense that they would be zero. We start, of course, with a diagram



I used L for"light" to identify the object. I used C for "chain" to identify the environmental object for the upward force. I used E for "Earth" to identify the environmental object for the downward force (due to gravity). We should also note that the upward force is a tension force, and the downward force is a non-contact gravitational force. We could even modify our drawing to show this.



where the symbol "T" is used for tension forces and we know that "W" is used for weight forces (forces due to gravity).

We see that all our forces acting on the chandelier are in the y-direction, so we have

$$F_{net_x} = 0$$

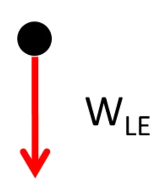
simply because none of the forces have x-components.

But

$$F_{net_y} = T_{LC} - W_{LE}$$

We should check, We only write out equations for one mover object at a time. So only forces with the first subscript, L, should be in our equation. Any force with a different subscript would be a force acting on some other object, and could not be acting on our chandelier. Since we only have two objects (the chain and the light), it's not surprising that we got this right. But it might be more difficult if we had more objects (coming up soon). Notice that we could have a Newton's second law equation set for each free-body diagram we draw. That is, we could have a separate Newtons' second law equation set for every mover object in our problem.

We know about gravitational, weight, forces. Let's get a mathematical expression for such a force. Consider the case where the chandelier is not on a chain. In that case there would only be one force on the chandelier (ignoring air drag). Then we would have just one force in our force diagram.



Then

$$F_{net_y} = -W_{LE}$$

and we know from Newton's second law that

$$F_{net_y} = ma_y$$

and since this would be a free fall situation we know

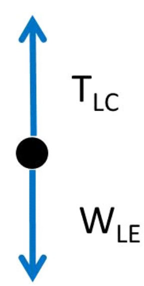
$$a_y = -g$$

thus

$$\begin{aligned} F_{net_y} &= -mg \\ F_{net_y} &= -W_{LE} \end{aligned}$$

So $W_{LE} = mg$ where m is the mass of the object, and g is the free-fall acceleration.

Let's go back to our chandelier hanging on a chain.



We could write our Newton's second law equation as

$$0 = T_{LC} - mg$$

or even

$$T_{LC} = mg$$

which tells us that the chain tension must support the weight of the chandelier. This seems reasonable!

Usually we solve for forces using components. Conveniently, the equation for net force can be resolved into components.

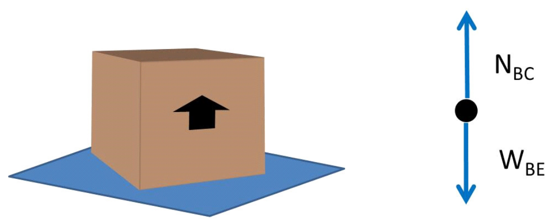
$$F_{netx} = ma_x \tag{15.3}$$

$$F_{nety} = ma_y \tag{15.4}$$

$$F_{netz} = ma_z \tag{15.5}$$

so we can turn two-dimensional force problems into two one-dimensional problems just like we did for projectile motion! Once again we turn a two or three-dimensional problem into two or three one-dimensional problems.

Let's try another force problem. Suppose you have been asked to help a neighbor move. The neighbor is a physics major and has conveniently marked each box with it's weight. You see a box marked $400\,\mathrm{N}$. What is the normal force that the floor must provide to support this box?



We would be surprised if the box accelerated on it's own, so we can identify this as an equilibrium problem. Since the box is not even moving (in our reference frame) we will give it another name, *static* (which means, "not moving"). Together, we call a problem where the net force is zero and the object is not moving *static equilibrium*. So this is a static equilibrium problem. The chandelier problem was also a static equilibrium problem. We know that

$$W_{BE} = 400\,\mathrm{N}$$

and we know that for static equilibrium

$$\overrightarrow{\mathbf{F}}_{net} = 0$$

so we can see that

$$\begin{aligned} F_{net_x} &= 0 \\ F_{net_y} &= 0 \end{aligned}$$

The net force is the sum of all the forces. So we could write

$$F_{net_x} = \Sigma_i F_{xi}$$

but this reduced to

$$0 = 0$$

because there are no x-components of the forces. In the y-direction

$$\begin{aligned} F_{net_y} &= \Sigma_i F_{yi} \\ &= N_{BF_y} + W_{BE_y} \end{aligned}$$

so that

$$0 = N_{BF_y} + W_{BE_y}$$

and we can see that

$$\begin{aligned} W_{BE_y} &= -W_{BE} \\ N_{BF_y} &= N_{BCF} \end{aligned}$$

so

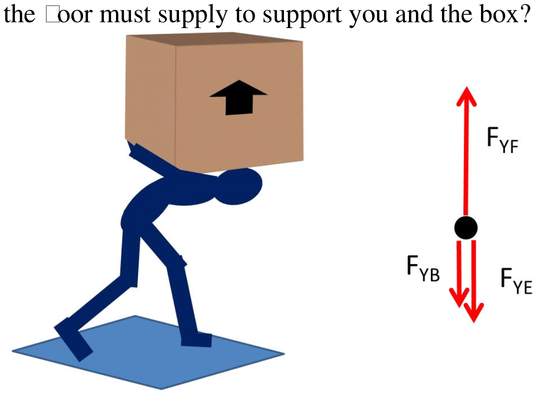
$$0 = N_{BF} - W_{BE}$$

and therefore

$$\begin{aligned} N_{BF} &= W_{BE} \\ &= 400\,\mathrm{N} \end{aligned}$$

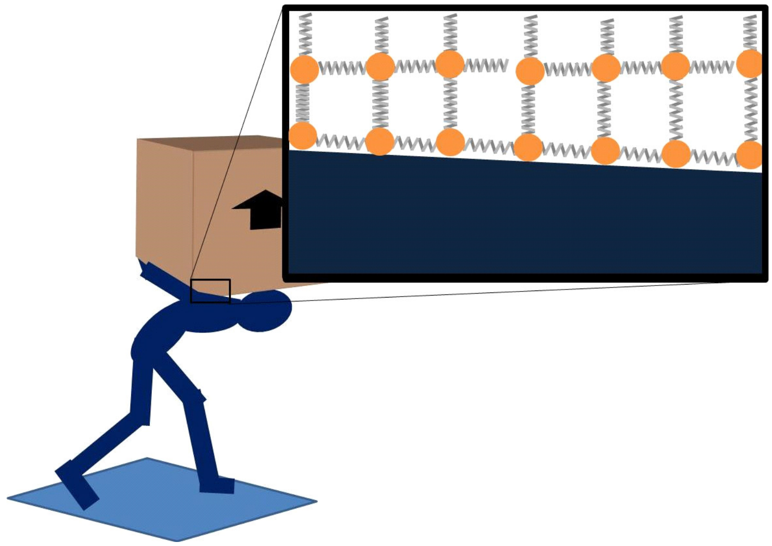
which says that the floor's normal force must be equal to the weight of the box. This seems to make sense.

But now you pick up the box. You know your own weight to be $667.2\,\mathrm{N}$. What is the normal force that the floor must supply to support you and the box?

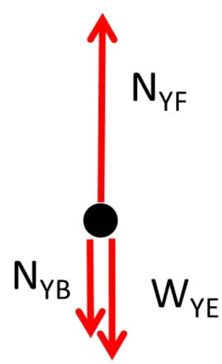


The picture shows our new situation. It seems reasonable that we have to have more forces than in the last case. After all now we have the box and you as objects. So at a minimum we must have your weight added into the situation. Since we are a little self-centered, let's consider you to be the mover (after all, you don't care about the box crashing through the floor nearly as much as you care about you crashing through the floor!). You would very much like to be in equilibrium, not accelerating downward. We recognize that there will be a force due to the floor pushing you up. We also recognize that there is the Earth pulling you down. And we do need to account for the box pushing down on you. The Earth's pull is a weight, so we can write $F_{YE} = W_{YE}$ and the floor's push is a normal force, $F_{YF} = N_{YF}$. But what kind of force is the box's push?

It might be tempting to call this force the weight of the box. But we must not do this! The weight of the box is W_{BE} and this force is the force *on the box* due to *the Earth's gravity*. This is *not* a force on you! So it can't go on your force diagram. So what is the box really doing to you?



Let's look at the molecules that form the box. They have those spring-like molecular bonds. And as the Earth pulls the box down toward it, the molecules come into contact with your shoulder. The molecular bonds get squashed by your shoulder, and they push back. This is the force on you due to the box. And we recognize this type of force. This is a normal force! So $F_{YB} = N_{YB}$. We can complete our diagram.



We know

$$\begin{aligned} W_{BE} &= 400\,\mathrm{N} \\ W_{YE} &= 667.2\,\mathrm{N} \end{aligned}$$

and our basic equation is still

$$\overrightarrow{\mathbf{F}}_{net} = 0$$

or

$$\begin{aligned} F_{net_x} &= 0 \\ F_{net_y} &= 0 \end{aligned}$$

We still have no x-part (or z-part) to this problem so let's just use our y-equation

$$F_{net_y} = 0$$

then

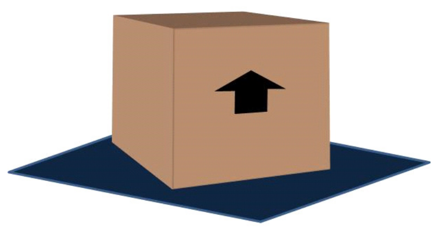
$$0 = N_{YF} - N_{YB} - W_{YE}$$

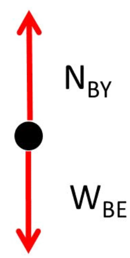
and we can solve for N_{YF}

$$N_{YF} = N_{YB} + W_{YE}$$

but we don't know N_{YB}! What can we do?

Let's consider our last problem. We found that the floor had to support the weight of the box for the box to not accelerate downward. So the normal force on the box had to be equal to the weight of the box. This will be true for our case as well. Note! we are considering a second drawing for a second mover object!





where N_{BY} is the normal force on the box due to the squashed atoms in your shoulder and W_{BE} is the force on the box due to the Earth's pull.

We should ask ourselves, which atoms get more squashed, your shoulder atoms or the box atoms? The answer is *neither!*. Think if we put the box on a large stack of Jello®. The Jello atoms would not push back as hard on the box atoms as the box pushes on them, and the box would sink through the Jello. This does not happen with your shoulder! So we can reasonably assume that The box $N_{BY} = N_{YB}$. And we know from our analysis of the last problem that

$$N_{BY} = W_{BE}$$

so

$$N_{YB} = W_{BE}$$

You might feel a little cheated. Didn't I say that the force pushing down on you from the box was not the weight of the box? And that is right. But the force, a different force than the gravitational pull of the Earth on the box, a force that comes from little spring-like molecular bonds, *has the same magnitude as* of the weight of the box. Note, they are not the same force at all. They have different causes, one gravity and one compressed atomic bonds. But they do have the same magnitude. So now we can write

$$N_{YF} = W_{BE} + W_{YE}$$

$$\begin{aligned} N_{YF} &= 400\,\mathrm{N} + 667.2\,\mathrm{N} \\ &= 1067\,\mathrm{N} \end{aligned}$$

and we can see that the floor now supports both your wight and the box weight. This makes sense.

But we have learned something important along the way. We have to be very careful to correctly identify the source of each force. In this case, knowing how the actual forces allowed us to consider a sturdy box instead of a Jello box of the same mass. Engineers have to know these forces to design actual mechanical systems (most actual mechanical systems are not made of Jello).

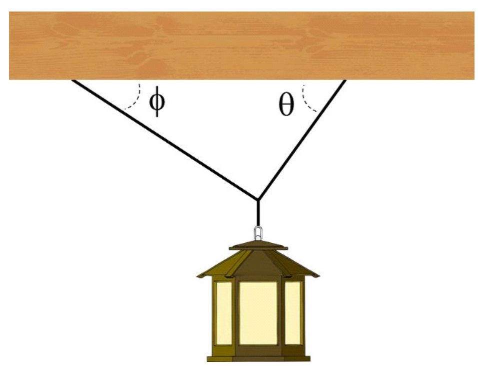
We also learned that we may have to draw a free-body diagram for a second or third object to solve for the forces we want. If we incorrectly identify the source of the forces, we will eventually miss a force, and then the problem will be impossible until we correct our mistake. Next lecture we will take on a more difficult problem that will illustrate all this.

16 Static, Dynamic, and Non-Equilibrium

We practiced some equilibrium problems last lecture. Let's take on a difficult problem as a review.

A static equilibrium example

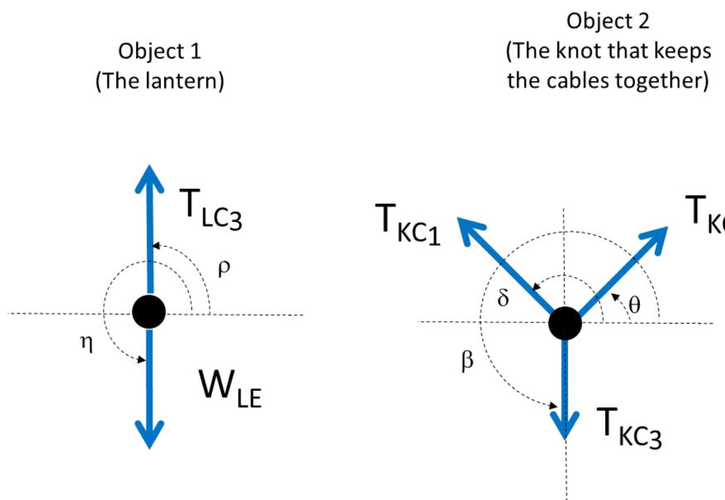
Suppose we are prospectors, and suppose we hang a lantern from two wires supported by a beam to see in our mine. We connect the wires together, and a bit of wire drops from the junction down to support the lantern. The situation is shown in the next figure.



Suppose the lantern weights $50.0\,\mathrm{N}$. The upper cables make angles of $35\,°$ and $52\,°$ with the horizontal. Find the tension-type force on each cable.

The picture above is a good picture, because we need to understand the problem statement. But it is not an easy picture to use to solve the problem. We need to draw force diagrams (free-body diagrams) for each object that has forces acting on them. Sometimes those objects are hard to identify, but picture all the forces, and see where those forces act. I found four objects, the lantern, and the knot that holds the cables together, the beam, and the Earth. It is sometimes hard to tell which objects will need force diagrams. The Earth only participates in our problem through gravity. I am going to guess that I won't need to draw the Earth's diagram. I also don't think I will need the beam's diagram, though it does have two of the tension forces acting on it. If I guess wrong, I can go back and draw the missing diagrams. I choose to draw diagrams for the lantern and the knot. Here are my pictures.

As usual, we can label our forces with a letter that reminds us of the type of force we have.



You might notice that I changed from the angle ϕ, which I know from the problem statement (and a little geometry) to the angle δ that is measured from the positive x-axis. You may be good at trigonometry, and may be able to skip this step. But I want to demonstrate how to do this so those who struggle with trig a little can always use our set of equations for vector components. We can see from the diagram

$$\delta = 180^\circ - \phi$$

I know that

$$\begin{aligned}
\phi &= 35^\circ \\
\delta &= 180^\circ - \phi \\
\theta &= 52^\circ \\
\beta &= 270^\circ \\
\rho &= 90^\circ \\
\eta &= 270^\circ \\
W_{LE} &= 50N
\end{aligned}$$

Next I want a basic equation, and that would be Newton's second law!

$$\vec{a} = \frac{\vec{F}_{net}}{m}$$

we also need the definition of net force

$$\vec{F}_{net} = \sum_i \vec{F}_i$$

We also might write as a basic equation.

$$W = mg$$

This is a two-dimensional problem. We don't really want a two dimensional answer. We want the magnitude of the tension forces. But to get there we still need to split our two-dimensional problem into two one-dimensional problems. So I will write Newton's second law as and x-equation and a y-equation.

$$\begin{aligned} a_x &= \frac{F_{net_x}}{m} \\ a_y &= \frac{F_{net_y}}{m} \end{aligned}$$

I'm probably going to have to do this for both objects. Again we need subscripts. From the diagram for the lamp.

$$\begin{aligned} F_{netL_x} &= T_{LC_3} \cos(\rho) + W_{LE} \cos(\eta) \\ F_{netL_y} &= T_{LC_3} \sin(\rho) + W_{LE} \sin(\eta) \end{aligned}$$

$$\begin{aligned} F_{netK_x} &= T_{KC_3} \cos(\beta) + T_{KC_1} \cos(\delta) + T_{KC_2} \cos(\theta) \\ F_{netK_y} &= T_{KC_3} \sin(\beta) + T_{KC_1} \sin(\delta) + +T_{KC_2} \cos(\theta) \end{aligned}$$

As I write out Newton's Second Law in components, I want to ask "is the object I am working on accelerating?" It might be moving, but I want to know if it is accelerating. If there is acceleration then there *is a net force*. Notice that neither of our objects is accelerating. So $\overrightarrow{a} = 0$ for both of them! This is always wonderful because I get to set a whole term equal to zero! This is the situation we called static equilibrium.

$$\begin{aligned} a_{xL} &= \frac{F_{net_{Lx}}}{m} = 0 \\ a_{y_L} &= \frac{F_{net_{Ly}}}{m} = 0 \\ a_{xK} &= \frac{F_{net_{Kx}}}{m} = 0 \\ a_{yK} &= \frac{F_{net_{Ky}}}{m} = 0 \end{aligned}$$

We can use this in our Newton's second law equations.

Let's do the lantern first.

$$\begin{aligned} T_{LC_3}\cos(\rho) + W_{LE}\cos(\eta) &= 0 \\ T_{LC_3}\sin(\rho) + W_{LE}\sin(\eta) &= 0 \end{aligned}$$

we are being a bit through here, we can see that both $\cos(\rho) = 1$, $\cos(\eta) = -1$. and $\sin(\rho) = \sin(\eta) = 0$. Let's use these here

$$\begin{aligned} T_{LC_3} - W_{LE} &= 0 \\ 0 + 0 &= 0 \end{aligned}$$

the second tells us that zero is equal to zero, which we already knew. But the first tells us that

$$T_{LC_3} = W_{LE}$$

or the tension in the cable supporting the lamp is equal to the weight of the lamp. Then

$$T_{LC_3} = mg$$

and we have one third of our goal!

Let's check that this works. The magnitude of the upward force due to the cable must equal the force due to gravity on the lamp if the lamp is not accelerating. This seems reasonable.

Now let's take on the second object, the knot. Remember that

$$\begin{aligned} a_{xK} &= 0 \\ a_{yK} &= 0 \end{aligned}$$

so the net force must be equal to zero for the knot as well.

$$\begin{aligned} 0 &= T_{KC_3}\cos(\beta) + T_{KC_1}\cos(\delta) + T_{KC_2}\cos(\theta) \\ 0 &= T_{KC_3}\sin(\beta) + T_{KC_1}\sin(\delta) + T_{KC_2}\sin(\theta) \end{aligned}$$

We should pause to think of what we know. We know T_{KC_3}. This must be very nearly the same as T_{LC_3}.We would expect the stretched bonds in the wire to pull the same amount both directions. We also know the angles

$$\begin{aligned} 0 &= \underline{T_{KC_3}}\cos\left(\underline{\beta}\right) + T_{KC_1}\cos\left(\underline{\delta}\right) + T_{KC_2}\cos\left(\underline{\theta}\right) \\ 0 &= T_{KC_3}\sin\left(\underline{\beta}\right) + T_{KC_1}\sin\left(\underline{\delta}\right) + T_{KC_2}\sin\left(\underline{\theta}\right) \end{aligned}$$

and let's play the same trick as we did before. If the cosine terms or sine terms are either zero or ± 1, let's use them. We know that

$$\begin{aligned} \cos\left(\underline{\beta}\right) &= 0 \\ \sin\left(\underline{\beta}\right) &= -1 \end{aligned}$$

$$\begin{aligned} 0 &= 0 + T_{KC_1}\cos\left(\underline{\delta}\right) + T_{KC_2}\cos\left(\underline{\theta}\right) \\ 0 &= -T_{KC_3} + T_{KC_1}\sin\left(\underline{\delta}\right) + T_{KC_2}\sin\left(\underline{\theta}\right) \end{aligned}$$

let's put in expressions for δ and T_{KC_3}

$$\begin{aligned} 0 &= 0 + T_{KC_1}\cos\left(180\,^\circ - \underline{\phi}\right) + T_{KC_2}\cos\left(\underline{\theta}\right) \\ 0 &= -\underbrace{W_{LE}} + T_{KC_1}\sin\left(180\,^\circ - \underline{\phi}\right) + T_{KC_2}\sin\left(\underline{\theta}\right) \end{aligned}$$

We have two equations and two unknowns. So we can solve this. Let's solve for T_{KC_1} first

$$T_{KC_1}\cos\left(180\,^\circ - \underline{\phi}\right) = -T_{KC_2}\cos\left(\underline{\theta}\right)$$
$$T_{KC_1} = \frac{-T_{KC_2}\cos\left(\underline{\theta}\right)}{\cos\left(180\,^\circ - \underline{\phi}\right)}$$

now let's put this result in our second equation

$$0 = -W_{LE} + \frac{-T_{KC_2}\cos\left(\underline{\theta}\right)}{\cos\left(\underline{180\,^\circ - \underline{\phi}}\right)}\sin\left(\underline{\delta}\right) + T_{KC_2}\sin\left(\underline{\theta}\right)$$

This is pretty ugly, so let's see if we can make it look better using some algebra

$$W_{LE} = \frac{-T_{KC_2}\cos\left(\underline{\theta}\right)}{\cos\left(180\,^\circ - \underline{\phi}\right)}\sin\left(\underline{\delta}\right) + T_{KC_2}\sin\left(\underline{\theta}\right)$$
$$W_{LE} = T_{KC_2}\left(\frac{-\cos\left(\underline{\theta}\right)}{\cos\left(180\,^\circ - \underline{\phi}\right)}\sin\left(\underline{\delta}\right) + \sin\left(\underline{\theta}\right)\right)$$

and solve for T_{KC_2}

$$T_{KC_2} = \frac{W_{LE}}{\left(\frac{-\cos(\underline{\theta})}{\cos\left(\underline{180\,^\circ - \underline{\phi}}\right)}\sin\left(\underline{\delta}\right) + \sin\left(\underline{\theta}\right)\right)}$$

As bad as this still looks, we know all the parts. So

$$\begin{aligned} T_{KC_2} &= \frac{(50\,\mathrm{N})}{\left(\frac{-\cos(52\,^\circ)}{\cos(180\,^\circ - 35\,^\circ)}\sin\left(180\,^\circ - 35\,^\circ\right) + \sin\left(52\,^\circ\right)\right)} \\ &= 41.014\,\mathrm{N} \end{aligned}$$

We are two thirds done!

To find the final tension, we go back to a previous equation, but now we know T_{KC_2}

$$T_{KC_1} = \frac{-\underline{T_{KC_2}}\cos\left(\underline{\theta}\right)}{\cos\left(180\,^\circ - \underline{\phi}\right)}$$

so

$$\begin{aligned} T_{KC_1} &= \frac{-\left(41.014\,\mathrm{N}\right)\cos\left(52\,^\circ\right)}{\cos\left(180\,^\circ - 35\,^\circ\right)} \\ &= 30.825\,\mathrm{N} \end{aligned}$$

These tensions all seem to be about right. they should be not much different than the weight of the lantern.

Once again we find that this type of problem is not terribly difficult, but is kind of tedious to run through the entire process. I prefer tedious to difficult. So our method seems like a good one.

We will spend the next few lectures practicing problems like this one. Each time we will add in new forces from our list of force types. These problems are not hard, but a little tedious–unless you try to skip steps. Then they can be quite difficult. The diagrams are essential! The subscripts really help. So we will practice doing all the preliminary work up-front. then the problems will be a lot like projectile motion problems, long, but not hard.

Dynamic Equilibrium

We could ask an important question here. Would any of our previous equilibrium problems' answers change if we found that they were in a moving inertial frame?

Say, we had a temple on wheels (the ancient Kahns did this!) or we were really moving our neighbor friend from one state room on a cruise ship to another as the ship moved at a constant rate, or our lamp was hung on the beam of freight car moving at a constant rate. Would the answers be different?

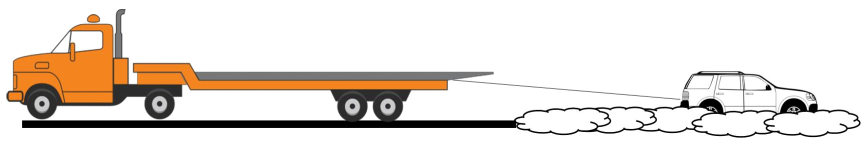
Clearly the answer is no. So long as we have constant motion, $\overrightarrow{a} = 0$ and Newton's second law works just fine. You probably wouldn't think about the cruse ship's motion at all as you moved the box. And that is because the constant velocity of the ship, box, rooms, and people just does not matter to our physics problem. We could call this type of problem where everything in the problem is moving at a constant rate a *dynamic equilibrium* problem. But we don't do anything different for dynamic equilibrium problems. The acceleration is still zero. So we treat them just the same. We just have to be a little careful to notice that $\overrightarrow{a} = 0$ even though the whole system is moving.

Dynamics and Newton's Second Law

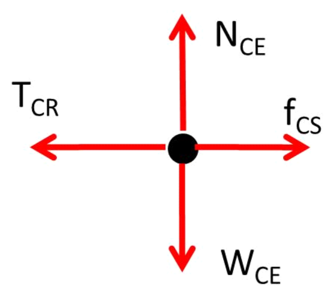
Of course, its not always true that forces sum to zero for all objects. But that's not bad. If we have an acceleration we just use our equations for constant acceleration problems. We might have to use our Newton's second law techniques to find the acceleration, however. Lets try this type of problem.

Suppose your car is off the road in the snow. Since the car is just sitting there, it has a velocity $v_i = 0$. But you would like it towed out of the snow, back on the road. So you wish to change the car's motion. You are a distance $\Delta x = 10.0\,\mathrm{m}$ from the road. The deep snow causes a constant friction force of $f_{CS} = 1000.0\,\mathrm{N}$. Your car has a mass of $1615.4\,\mathrm{kg}$ (weight of $W_{CE} = 15831\,\mathrm{N}$). The tow truck wench pulls with a force of $2000.0\,\mathrm{N}$. How long does it take to get the car back on the road?

We can see that this is an acceleration problem, we don't want the forces to balance. We may even suspect that this is a constant acceleration problem. But we don't know that constant acceleration. If we did we could us kinematics to solve the problem. But now that we have Newton's second law, we have a way to find the acceleration! Let's try it out.



Our free-body diagram for the car might look like this.



Notice that this is the free-body diagram for the car. I didn't draw the diagram for the truck (or the snow).

We know

$$
\begin{aligned}
f_{CS} &= 1000\,\mathrm{N} \\
T_{CR} &= 2000\,\mathrm{N} \\
m &= 1614.4\,\mathrm{kg} \\
W_{CE} &= 15831\,\mathrm{N} \\
\Delta x &= 10.0\,\mathrm{m}
\end{aligned}
$$

and we know that

$$\begin{aligned} F_{net_x} &= ma_x \\ F_{net_y} &= ma_y \end{aligned}$$

and if the acceleration is constant, we can use our kinematic set.

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + 0 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + 0 \\ v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 0 \end{array}$$

We can realize that we know many more things based on a choice of coordinate system.

$$\begin{aligned} v_{ix} &= 0 \\ v_{iy} &= 0 \\ x_i &= 0 \\ y_i &= 0 \\ y_f &= 0 \end{aligned}$$

We should realize that we actually know $a_y = 0$ (or at least we hope it does!) but we don't know a_x.

$$\begin{array}{ll} \underline{\Delta x} = \underline{v_{ix}} + \frac{1}{2}a_x\Delta t^2 & \underline{\Delta y} = \underline{v_{iy}}\underline{\Delta t} + \frac{1}{2}\underline{a_y}\Delta t^2 \\ v_{fx} = \underline{v_{ix}} + a_x\Delta t & \underline{v_{fy}} = \underline{v_{iy}} + \underline{a_y}\Delta t \\ v_{fx}^2 = \underline{v_{ix}^2} + 2a_x\underline{\Delta x} & \underline{v_{fy}^2} = \underline{v_{iy}^2} + 2\underline{a_y}\underline{\Delta y} \end{array}$$

and using all the zeros, we have

$$\begin{array}{ll} \underline{\Delta x} = 0 + \frac{1}{2}a_x\underline{\Delta t}^2 & \Delta y = v_{iy}\Delta t + 0 \\ v_{fx} = 0 + \underline{a_x}\Delta t & v_{fy} = v_{iy} + 0 \\ v_{fx}^2 = 0 + 2\underline{a_x}\Delta x & v_{fy}^2 = v_{iy}^2 + 0 \end{array}$$

It looks like our best bet is to use the first x-equation.

$$\underline{\Delta x} = \frac{1}{2}a_x\underline{\Delta t}^2$$

$$\frac{2\Delta x}{a_x} = \Delta t^2$$

$$\sqrt{\frac{2\Delta x}{a_x}} = \Delta t$$

But we don't know a_x. We need a strategy to find it. We know that the net force has our acceleration in it. Let's try Newton's second law.

Let's start with the forces to see this. Remember that the net force is the sum of the forces, so

$$\begin{aligned} F_{net_x} &= (T_{CR} - f_{CS}) \\ F_{net_y} &= (N_{CE} - W_{CE}) \end{aligned}$$

and Newton's law tells us that

$$\begin{aligned} F_{net_x} &= ma_x \\ F_{net_y} &= ma_y \end{aligned}$$

so putting these two together tells us

$$\begin{aligned} ma_y &= (N_{CE} - W_{CE}) \\ ma_x &= (T_{CR} - f_{CS}) \end{aligned}$$

We can guess from our experience that $N_{CE} = W_{CE}$ so $a_y = 0$. The car is not lifting off the surface or burrowing into the ground! but T_{CR} and f_{CS} are not the same.

$$\begin{aligned} a_x &= \frac{(T_{CR} - f_{CS})}{m} \\ a_x &= \frac{1}{1614.4\,\mathrm{kg}}(2000\,\mathrm{N} - 1000\,\mathrm{N}) \\ &= 0.619\,43\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

This *is* a constant acceleration. So our guess that we could use kinematics is justified! Let's finish it up. We know

$$\Delta t = \sqrt{\frac{2\Delta x}{a_x}}$$

$$\begin{aligned} \Delta t &= \sqrt{\frac{2\,(10\,\mathrm{m})}{0.619\,43\frac{\mathrm{m}}{\mathrm{s}^2}}} \\ &= 5.682\,2\,\mathrm{s} \end{aligned}$$

The problem was not too hard! It was a little longer because we had to use two different problem types to solve this single problem, but it really was not too bad.

Gravitation

Now that we know more about forces and motion, we should take another look at the gravitational force, We need to know more than just

$$W = mg$$

Newton not only studied motion, he also studied gravitation (the legend is that this started when an apple fell on his head). Gravitation is not a contact force. This was very mysterious in Newton's day. But Newton was able to describe this force mathematically. In words, his *universal law of gravitation* states:

> *Every particle in the universe attracts every other particle in the universe with a force that is proportional to the masses of both particles and in-*

versely proportional to the square of the distance between the particles.

The mathematical expression is

$$F_g = G\frac{m_m m_E}{r_{mE}^2} \tag{16.1}$$

where the subscript m is for "mover" and the subscript E is for "environmental object." The distance r_{mE} is the distance from the center of the mover object to the center of the environmental object. Notice that there are two objects involved in Newton's equation, the moving object and the object causing the force. That is just what we should expect from out subscript system!

Notice that this equation does not give the direction of the force, only the magnitude. The direction is along the line connecting the centers[8] of the two particles. The constant G is a factor included to keep us in units we can use and to make the expression exact (instead of just a proportionality). It's value is

$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}} \tag{16.2}$$

Example: "The Jupiter Effect"

A few years ago, there was a major health scare because the planets were about to align. People were worried that the gravitational force would cause disasters and would affect personal health. They rushed to their doctors to see what they could do. What would you tell them?

Let's calculate the gravitational pull of Jupiter on an average human on earth. First we need some data:

$$\begin{array}{ll} M_J = 1898 \times 10^{24}\,\mathrm{kg} & \text{Mass of Jupiter} \\ m = 91\,\mathrm{kg} & \text{mass of a typical person } (\sim 200\,\mathrm{lb}) \\ G = 6.67 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}} & \text{Universal Gravitational Constant} \\ R_J = 779 \times 10^{9}\,\mathrm{m} & \text{Distance from the Sun to Jupiter (on average)} \\ R_E = 149 \times 10^{9}\,\mathrm{m} & \text{Distance from the Sun to the Earth (on average)} \\ r = R_J - R_E & \text{Distance between a person and Jupiter} \end{array} \tag{16.3}$$

Then

$$r = R_J - R_E = 6.3 \times 10^{8}\,\mathrm{km} \tag{16.4}$$

8 Really, the center of mass, but we have not gotten to that yet.

and

$$F_g = G\frac{M_J m}{r^2} \quad (16.5)$$
$$= 2.9026 \times 10^{-5}\,\mathrm{N} \quad (16.6)$$

Is this dangerous? Let's calculate the gravitational force from your refrigerator. Again we need some data

$m_f = 363\,\mathrm{kg}$	Mass of refrigerator ($\sim 800\,\mathrm{lb}$)
$m = 91\,\mathrm{kg}$	mass of a typical person ($\sim 200\,\mathrm{lb}$)
$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}$	Universal Gravitational Constant
$r' = 0.1\,\mathrm{m}$	Distance between a person and your fridge if they reach for the milk

(16.7)

Then

$$F_g = G\frac{m_f m}{(r')^2} \quad (16.8)$$
$$= 2.2033 \times 10^{-4}\,\mathrm{N} \quad (16.9)$$

Even if we have several times this (because during the Jupiter Effect, several planets aligned), the gravitational pull of your fridge is larger. So either we can ignore Jupiter, or we should start putting warning labels on appliances (Warning: this object contains mass and may be hazardous to your health!). OK, I guess you won't hear radio warnings for second hand mass any time soon.

Weight

We mentioned that weight was a force, now we can be very specific about that force. The magnitude of the gravitational force acting on an object of mass m_m near the Earth's surface is called the *weight* of the object. Mathematically we write this as

$$W = m_m g \quad (16.10)$$

where g is the acceleration due to gravity. Now, since we know Newton's formula for gravity, we can say (near the Earth's surface) that

$$W = G\frac{M_E m_m}{r_{mE}^2} \quad (16.11)$$

where M_E is the mass of the Earth. Given this, we can solve for g

$$m_m g = G\frac{M_E m_m}{r_{mE}^2} \quad (16.12)$$
$$g = G\frac{M_E}{r_{mE}^2} \quad (16.13)$$

Notice that g is not a constant! it depends on r_{mE}^2 (look in the denominator of equation

).So g has one value at sea level, and another value in a weather balloon at high altitude. What does this tell us about weight ($w = mg$). We see that weight is not constant for our mass, m. It varies with position relative to the Earth's center.

This may seem strange, but really you already knew this.

Think of an astronaut. Suppose she (our astronaut is female) steps on a scale before she gets on the Space Shuttle. She will find a weight of, say, $445\,\mathrm{N}$ (about $100\,\mathrm{lbf}$, our astronaut is very fit!). She takes her scale with her on the Space Shuttle and tries to step on the scale there. What will the scale read? Well, nothing (actually the scale will float around, because in orbit the effects of gravity seem to be eliminated). We know she is "weightless" in space. But our astronaut did not go on an infinitely reducing diet! she has the same mass. Only her weight has changed. Weight is a force. It is not an inherent property of the astronaut.

$$W = G\frac{m_1 m_2}{r^2} \tag{16.14}$$

This is the first of our inverse square laws that we will study (this just means that the force differs as the inverse square of the distance between the two objects involved)

But we have learned something new! Our old friend g changes with altitude. We remember that in Rexburg it is about $9.80004\frac{\mathrm{m}}{\mathrm{s}^2}$ and at sea level it is $9.81\frac{\mathrm{m}}{\mathrm{s}^2}$. You can see that g is getting smaller as we go up. Near the surface, this is not a big effect $\left(0.01\frac{\mathrm{m}}{\mathrm{s}^2}\right)$ but if we go high enough, it will make a big difference.

Weight and orbits

Back to our astronaut. On the Space Shuttle she appears weightless. But let's see if that would be true. Suppose somehow the Space Shuttle could park in just one spot above the Earth and hover there. Further suppose our astronaut is on a space walk next to the Shuttle. The force on her would be

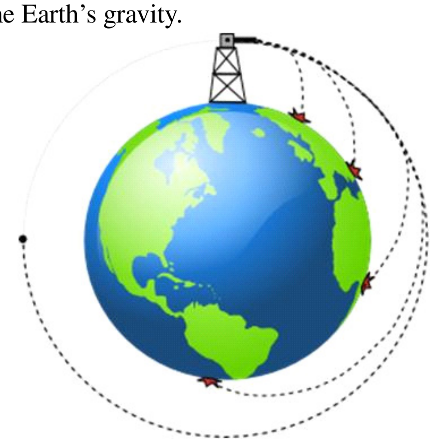
$$F = G\frac{M_E m_a}{(R_E + h)^2}$$

where $M_E = 5.98 \times 10^{24}\,\mathrm{kg}$ is the mass of the Earth and $m_a = 45.362\,\mathrm{kg}$ is our astronaut's mass. $R_E = 6.38 \times 10^3\,\mathrm{km}$ is the radius of the Earth and h is the altitude of the space shuttle flight (usually about $250\,\mathrm{km}$). Then

$$\begin{aligned} F &= \left(6.67 \times 10^{-11} \frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}\right) \frac{\left(5.98 \times 10^{24}\,\mathrm{kg}\right)(45.362\,\mathrm{kg})}{\left(6.38 \times 10^{3}\,\mathrm{km} + 250\,\mathrm{km}\right)^2} \\ &= 411.62\,\mathrm{N} \end{aligned}$$

This is less than the $445\,\mathrm{N}$ that she was on the surface of the Earth, but not very much less (a percent difference of around 7%). This is far from weightless! Is the space program all a fraud? No, there is more to this. The Space Shuttle does not hover. If it tried, it would fall. The reason it can stay up is that it orbits the Earth.

Let's pretend we could build a very tall tower, several hundred kilometers high. On this tower we place a cannon. If we shoot a ball it will travel a distance and hit the earth, pulled downward by the Earth's gravity.



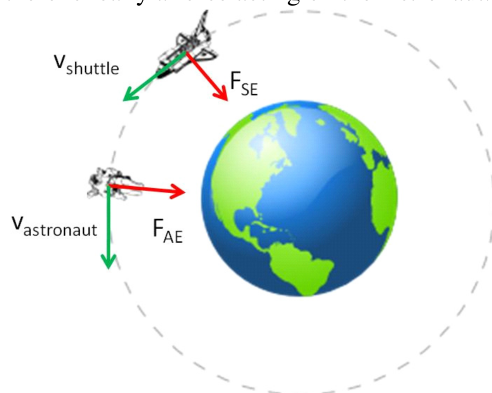
If we increase the muzzle velocity of the cannon, our ball will travel farther. Because the earth is round the ball will travel even farther than if the cannon were on a flat surface, because the round surface of the Earth falls away from the ball's path. If we continue increasing the muzzle velocity, the ball will travel farther and farther around the globe. Eventually, if we make the muzzle velocity high enough, the ball will miss the Earth entirely! It will travel at just the right speed such that the centripetal acceleration keeps in going in the same circle. It will never get nearer the Earth. This is what we call an *orbit*.[9]

Our astronaut is in just such an orbit. She is falling because the Earth's acceleration is constantly pulling her downward, but her velocity is so large (thousands of meters per second) that she continually misses the Earth, making a perfect circle around the planet.

But still why does she seem weightless? This is because the Space Shuttle, her tools, her fellow astronauts, and everything else around her is also in orbit. The acceleration

[9] This explanation of orbits is attributed to Newton, himself.

is the same for the Shuttle and the astronaut. Since they accelerate the same way, their motions are exactly the same. There is no relative motion to press the shuttle surfaces against her and the crew like there would be if the shuttle were parked on Earth. So there is no visible cue that there is really a force acting on the Astronaut.



Can we find a place with no gravity? If we went infinitely far away from all other objects in the universe, then $r \approx \infty$ so

$$\begin{aligned} F &\approx G\frac{m_1 m_2}{(\infty^2)} \\ &\approx 0 \end{aligned}$$

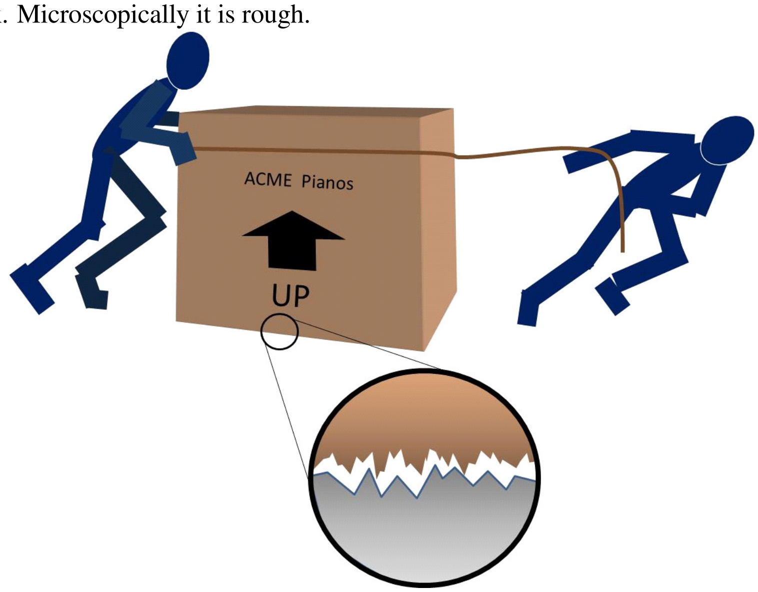
which would be truly weightless. Some scientists object to this definition of weight. They define weight as the reading on a scale. So the astronaut, by their definition, would be weightless in space (preserving what NASA says in the historic videos). But this definition can be confusing. There really is a force acting on our astronaut as she orbits. The feeling of weighlessness is purely due to not having visual cues telling her that she is really falling (but falling around the Earth).

17 Friction and Drag

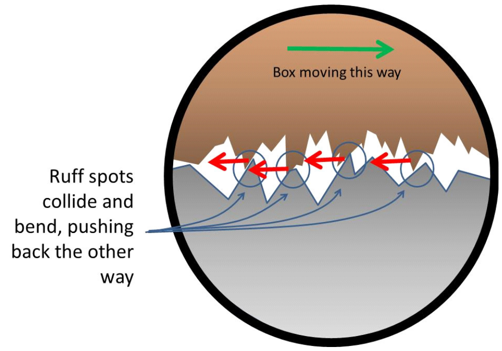
We have studied the origin of some forces, but it would be good to go into more depth on the origin of each type of force. And some forces we have not yet studied at all. In this lecture let's go into a deeper study of friction and Drag forces.

Friction Force

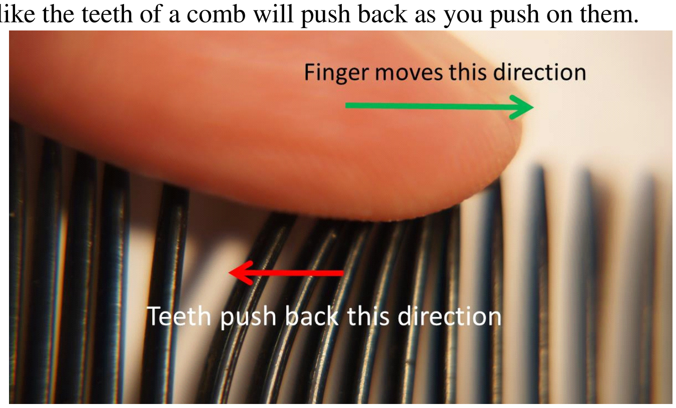
Let's first review what we know about friction. We found that friction is a push, and it is due to the bonds in molecules that are being stretched. Consider the bottom of our piano box. Microscopically it is rough.



And the floor under the box is also microscopically rough. As the people push and pull the box the rough spots on the floor and the box catch against each other. As the box is pushed the rough spots collide and bend. This bending stretches the spring-like bonds between the atoms in the box and floor materials.



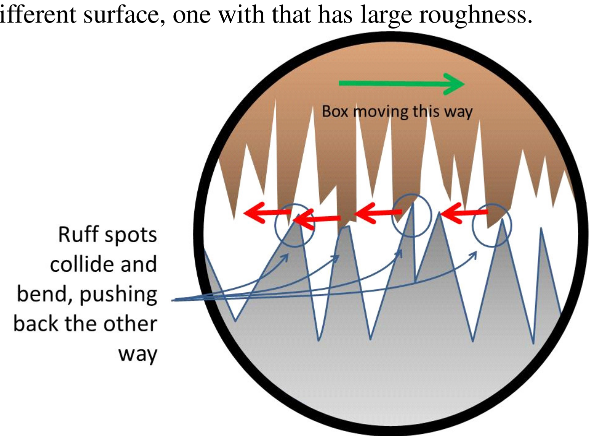
The bent and stretched molecules in the floor material push back against the box molecules. We used the analogy of pushing on the teeth of a comb. The roughness “teeth” push back, like the teeth of a comb will push back as you push on them.



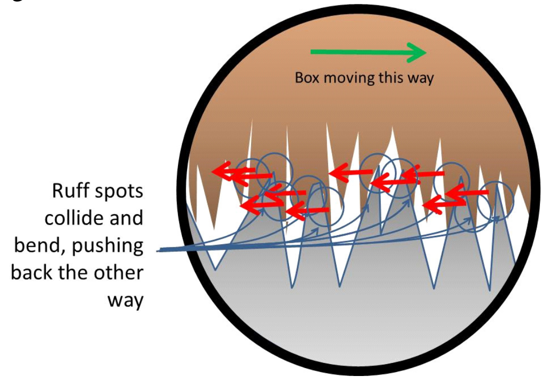
This backward push is what we call friction.

Details of Friction

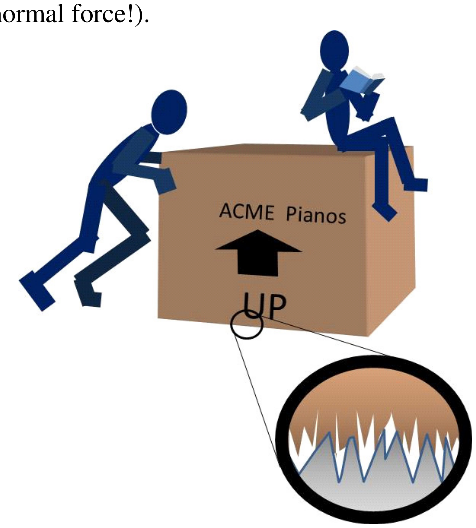
But let’s take a different surface, one with that has large roughness.



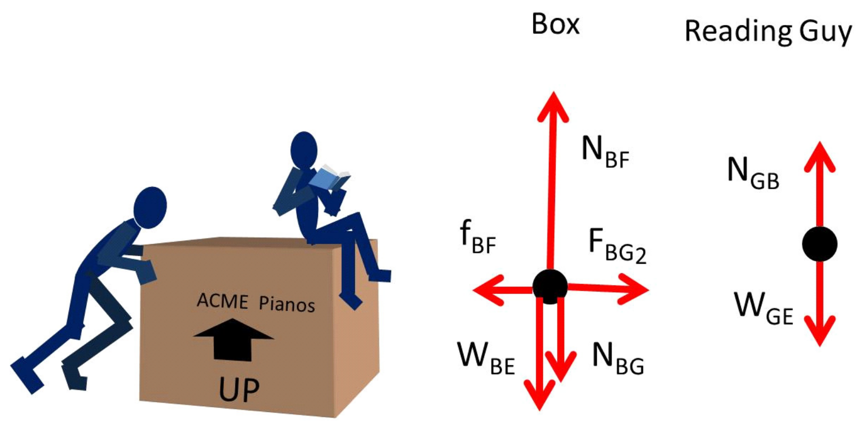
Only a few of the rough peaks hit each other. There would be friction. But we could make more friction if we made the roughness teeth collide more, making the surface area of the interaction larger and the amount of material who's bonds are stretched larger



But how can we make the rough "teeth" mesh more? One way is to push down on the object. Here are our guys moving a box. But one guy is pushing down on the box by sitting on it (with a normal force!).



The downward normal force due to the guy on the box will cause the box to accelerate downward until the force of the teeth pushing back up is matched with the force pushing down. Of course the force of the teeth pushing up must also match the force due to the box or the box would accelerate up or down. That is very unlikely to happen! Let's use Newton's second law to investigate this situation.



This is very like a problem we did last time. Notice that from the figure we can say that $N_{GB} = W_{GE}$ where the subscript G is for "guy" and the subscript B is for "box" and E is for "Earth."

$$F_{Gnet_y} = 0 = N_{GB} - W_{BE}$$

so

$$N_{GB} = W_{GE}$$

this means that the guy pushes down on the box with a force equal to his weight. We know this is true because we can see that the guy is not crushing his way through the box, and the box is not cutting it's way through the guy. Using Newton's third law we can say

$$N_{GB} = N_{BG}$$

Then from the box free-body diagram

$$F_{B_{net_y}} = 0 = N_{BF} - N_{BG} - W_{BE}$$

or

$$N_{BF} = N_{BG} + W_{BE}$$

but we know that $N_{BG} = N_{GB} = W_{GE}$ so

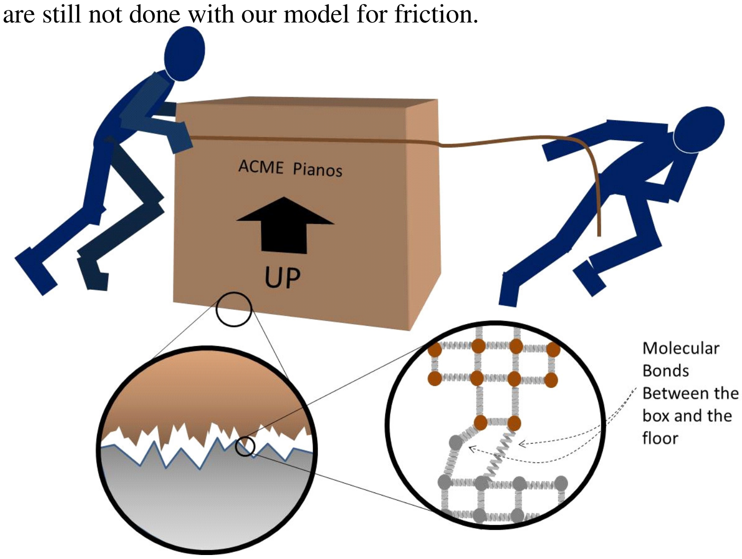
$$N_{BF} = W_{GE} + W_{BE}$$

This is not surprising. We know the floor must support the weight of both the guy and the box. But we can use what we have found to our advantage. The two things that are pushing the roughness teeth together are the wight of the box and the weight of the guy. Those two forces together are equal to the normal force pushing back up. The roughness teeth are sandwiched in between with forces above and below. They will be forced together. So since how much friction we should have depends on the two weights, and the normal force *is equal to* the two weights, we can say that if the normal force gets larger for an object, the friction gets larger too! We can say that

$$f_{BF} \propto N_{BF}$$

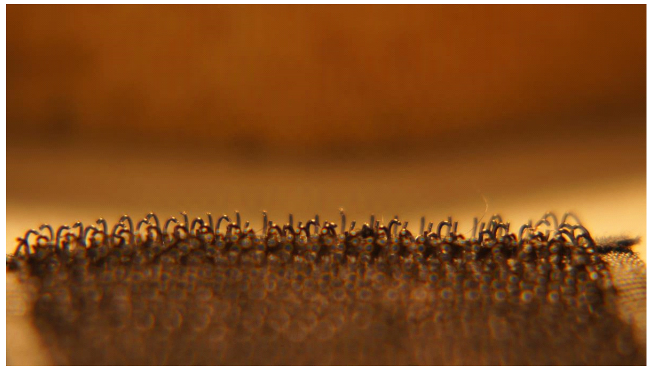
But let's be clear about what we mean. The weight of the box and the guy are doing the job of creating normal forces that are pushing the box down into the floor. The normal force is numerically equal to the sum of these weights. So it is fine to say that the force of friction on the box is numerically proportional to the normal force on the box. The downward force is due to the weight of the box and guy sort of like the guy pulling on a rope is the origin of the tension force in the rope. But it is the normal force that actually pushes the roughness teeth together to increase friction!

But we are still not done with our model for friction.



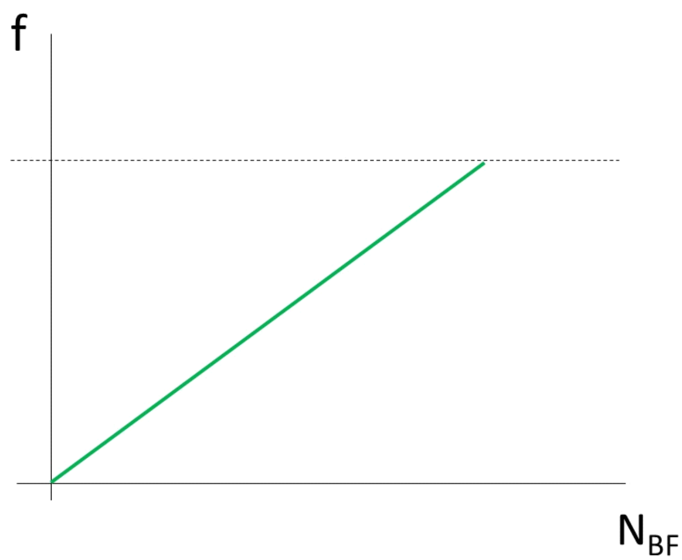
You may have been out on a hot day and noticed that your shoes would stick to the black top. Sticky materials actually form molecular bonds between themselves and other objects. Once these cohesive bonds are formed, they must be broken to make our object move. It's a little like adding a tension force into our frictional force.

Some materials are more sticky than others, and some are more likely to form bonds than others. And some objects are designed to be rougher than others. Think of Velcro®, for example.



it is designed to be so rough that the little Velcro® hooks grab and hold on, creating tension forces as you move the Velcro® parts.

Let's envision our guys pushing the box again. If the box is at all sticky (and most things are, at least at the molecular level). The sticky bonds will stretch before they break. So by pushing on the box, we expect the backward push from the sticky teeth. We could plot the situation as shown below.



The harder we push, the more stretched the sticky bonds are, so the more friction we have.

We know that our friction force is proportional to the normal force

$$f_{BF} \propto N_{BF}$$

but we would like to make this an equation with an equal sign. To do this, we can include a constant, μ_s that contains all the details of the surface of the substances (in our case, the box and floor substances) that are interacting. Are they very rough? Are they very sticky? etc.

$$f_{BF} = \mu_s N_{BF}$$

The constant μ_s would be different for every material, and also different for every roughness of the specific material.

Static friction

This constant μ_s, is called the *coefficient of static friction*. And really is must be different for nearly every item.

This equation for static friction is a little misleading still. Of course, if we push down harder on our box, the friction force can be larger. But look at our last graph. What if we don't push on the box. Suppose the box is just sitting on the floor. Then the little roughens teeth don't bend, and the molecular bonds don't stretch. In that case there is no frictional force even though we have a normal force! If we begin to push on the box, the teeth begin to stretch and push backward. But they won't push back as hard as they can.

If we push even harder on the box, the teeth push back harder, until we reach the point where the bonds start to break. That is the point where the equal sign works, $f_{BF} = \mu_s N_{BF}$. But if we push less on the box then

$$f_{BF} < \mu_s N_{BF}$$

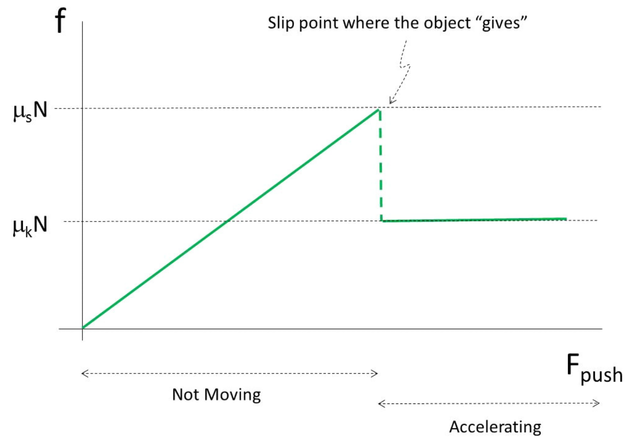
So the way we should write this equation is to say that

$$f_{BF} \leq \mu_s N_{BF} \tag{17.1}$$

meaning that the frictional force could be up to $\mu_s N_{BF}$ if we push hard on the object (in our case, the box), but could be less that $\mu_s N_{BF}$ if we push less hard on the box.

Kinetic friction

Suppose we do push very hard on the box so hard that the bonds brake. Think about pushing a box. You push the box, and at first it does not move. The harder you push, the harder the floor roughness teeth push back. But then something seems to "give." That is the bonds breaking. Now the box scoots along the floor. So long as we keep the box moving, it can't form more bonds with the floor and it can't sink down, meshing the roughness teeth with the roughness of the floor. The breaking point, when the box begins to move is when $f_{BF} = \mu_s N_{BF}$. Beyond this point, the friction force has much less bonding and less teeth meshing, so the friction force is greatly reduced.



We can write a similar equation for the case when the box is finally moving

$$f_{BF} = \mu_k N_{BF}$$

It is still true that there is more friction if we push down harder on the box (making N_{BF} larger). But now we won't have much stickiness and will have less meshing of the roughness teeth. So our coefficient will be much less. To indicate that we have a different coefficient once the box is moving, let's give the new coefficient a new name. We call this the *coefficient of kinetic friction.*

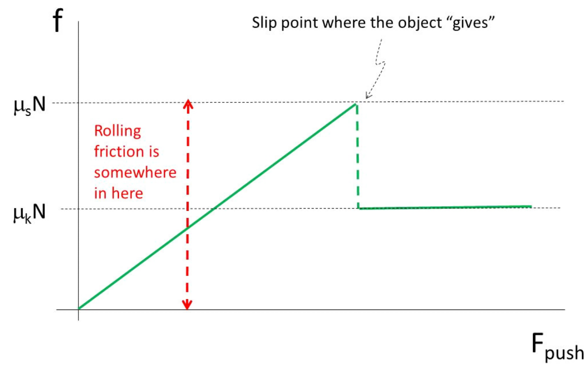
Here are some example coefficients of friction.

Material	μ_s	μ_k
Rubber on concrete	1.00	0.80
Steel on steel	0.74	0.57
Wood on wood	$0.25 - 0.50$	0.20
Waxed wood on snow	$0 - 0.14$	$0.04 - 0.1$
Ice on ice	0.10	0.03

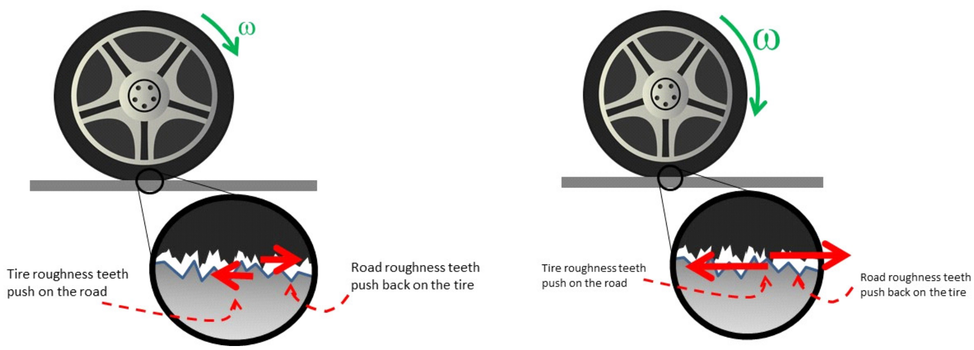
but recall that I could have wood that is more rough than other wood. That would change our table value for "Wood on wood" or "Waxed wood on snow." So these coefficients of friction are useful to get the general idea of how much frictional force we might get for a substance, but should be used with some caution.

Rolling friction

It is becoming popular to define another coefficient of friction, one for a rolling wheel or tire. But this is really just a special case of static friction. Think of a tire. You want your car tire to roll along without slipping. That means that the part of the tire that is on the ground is somewhere in the static friction area of our graph.



The rolling friction depends on how hard we work at spinning the tire (how big F_{push} is). Since the push force can change with how we rev our engine, so can the rotational friction force.



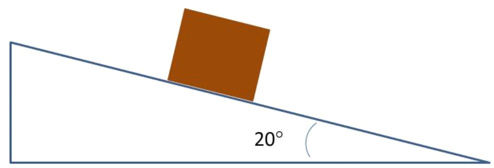
Some books define a coefficient of rolling friction, but we won't. We will realize that rolling friction is just a case of static friction and use

$$f_{BF} \leq \mu_s N_{BF}$$

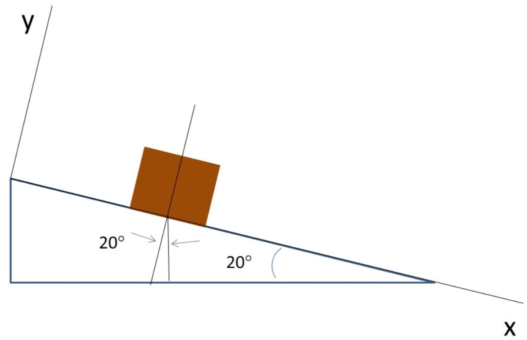
for rolling objects.

An example of friction

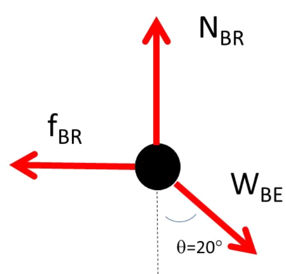
Suppose our moving guys have pushed the box to a ramp. They want to slide the box down the ramp. The ramp has an angle of 20° and somehow you know that the coefficient of kinetic friction $\mu_k = 0.12$ for the box sliding on the ramp material. What is the acceleration of the box as it slides down the ramp?



It will be easier if we use a rotated coordinate system for this problem. Let's make the x-axis be along the ramp. Then the y-axis would be perpendicular to the ramp.



and we can draw a free-body diagram in this coordinate system



This is a friction problem, and since we want acceleration, it is also likely a Newton's second law problem.

We know

$$\begin{aligned}\theta &= 20^\circ\\ \phi &= 270^\circ + \theta\\ \mu_k &= 0.12\end{aligned}$$

and we know Newton's second law

$$\begin{aligned}a_x &= \frac{F_{net_x}}{m}\\ a_y &= \frac{F_{net_y}}{m}\end{aligned}$$

and we can use our new kinetic friction equation.

$$f_{BRk} = \mu_k N_{BR}$$

The box is not likely to fly off the ramp or to burrow into the ramp, so we can also identify

$$a_y = 0$$

then

$$F_{net_y} = 0 = N_{BR}\sin(90^\circ) + f_{BR}\sin(180^\circ) + W_{BE}\sin(\phi)$$

and the x-part would be

$$F_{net_x} = ma_x = N_{BR}\cos(90^\circ) + f_{BR}\cos(180^\circ) + W_{BE}\cos(\phi)$$

We have said that we should always use any value that is zero right away because it takes out whole terms. And we added to this using ± 1 values from sine and cosine functions.

$$F_{net_y} = 0 = N_{BR} + W_{BE}\sin(270^\circ + \theta)$$
$$ma_x = 0 - f_{BR} + W_{BE}\cos(\phi)$$

using our new friction equation, $f_{BRk} = \mu_k N_{BR}$, we can write this as

$$N_{BR} = -W_{BE}\sin(270^\circ + \theta)$$

$$ma_x = -\mu_k N_{BR} + W_{BE}\cos(\phi)$$

and substituting N_{BR} from the first equation into the second

$$ma_x = -\mu_k(-W_{BE}\sin(270^\circ + \theta)) + W_{BE}\cos(\phi)$$

finally, we know that $W_{BE} = mg$

$$ma_x = \mu_k(mg\sin(270^\circ + \theta)) + mg\cos(\phi)$$

the masses cancel, and we can take out a g

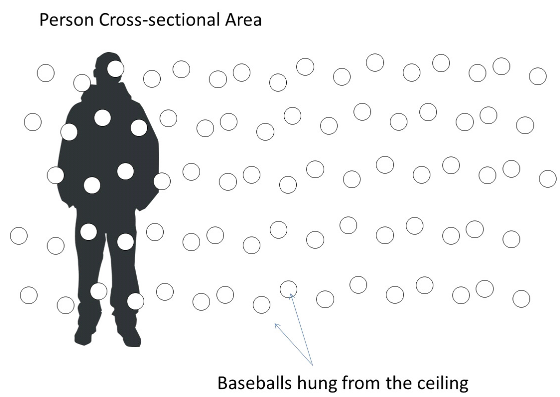
$$a_x = g(\mu_k(\sin(270^\circ + \theta)) + \cos(\phi))$$

$$\begin{aligned} a_x &= \left(9.8\frac{\text{m}}{\text{s}^2}\right)(0.12(\sin(270^\circ + 20^\circ)) + \cos(270^\circ + 20^\circ)) \\ &= 2.2467\frac{\text{m}}{\text{s}^2} \end{aligned}$$

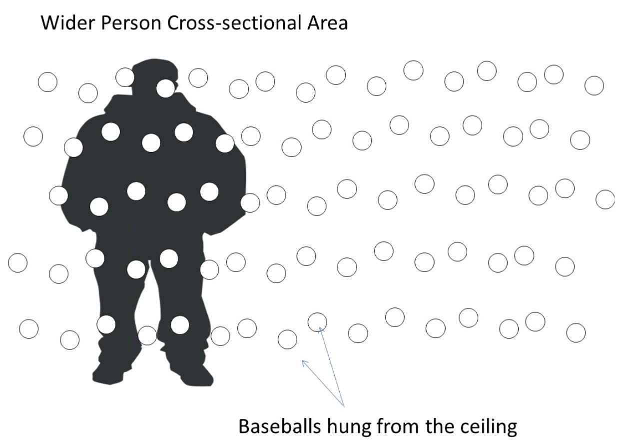
We would not expect to have the full acceleration due to gravity because we have a slope and we have friction reducing the acceleration. So this seems reasonable.

Drag Force

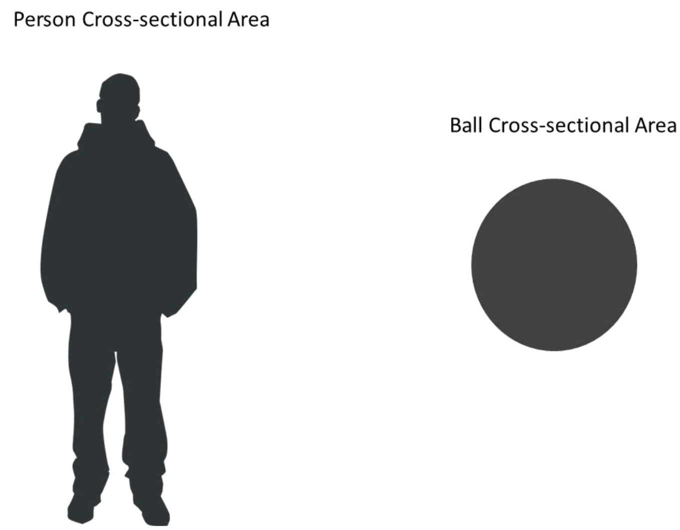
In this class we really don't yet understand enough to see exactly how drag forces like air resistance work. We will set the groundwork for understanding the origins of drag force in PH123. But we can have some intuitive feel for an air resistance with some conceptual reasoning. Suppose we tied baseballs to strings and hung them all over our class room. Every time a student entered into the room the student would collide with the baseballs. Not only would this hurt, but it would slow the progress of the student. The baseballs would exert a force on the student with every collision. The collective resistance to the motion of the student due to all the baseballs is what we would call a drag force.



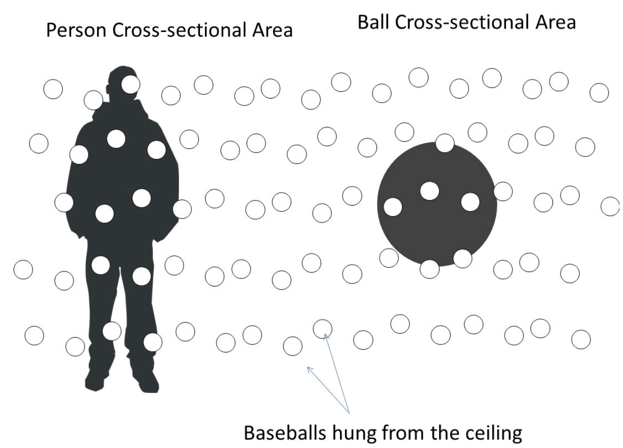
As the student tries to traverse the baseball laden room, we can see that the wider the student the more baseballs the student will strike.



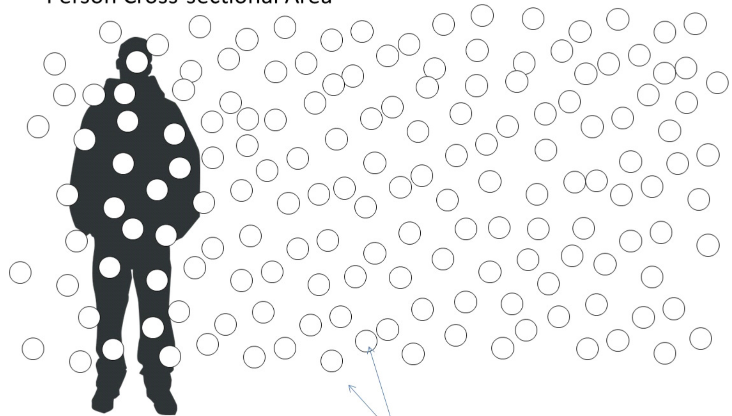
It doesn't matter much how long the person is, just how wide. A horse wouldn't not hit more baseballs due to it's long body, for example. We will call the area filled in from an outline of the student the cross sectional area.



Our drag force must be proportional to this cross-sectional area.



It must also be true that the number of collisions would be proportional to the density of baseballs in the room. The more tightly packed the balls, the more balls we will hit.



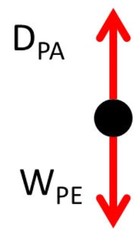
It's also true that the faster we try to run through the room, the more the baseballs will resist our motion. For one thing, the collisions would hurt more! That means the force due to the collision would be higher. This reasoning works for any shape that passes through the baseballs. We could try to push a large beach ball through the room, for example. The beach ball would be slowed down based on how big it's cross sectional area is, and how fast it is going.

So here is our result (borrowed from PH123).

$$D = \frac{1}{2} C \rho A v^2$$

This is the magnitude of the drag force, and the direction is opposite the motion of the object. The value A is the cross-sectional area. The quantity ρ is the air density (think of this like a baseball density in our example). Of course v is the speed, and C is a a coefficient changes with the shape of the object. It's called the *drag coefficient* and it is smaller for pointed things like rockets and larger for blunt things like people

Suppose we have a person that wishes to parachute out of a plane. The person-parachute system will experience a drag force.



where the subscript P is for "person" and A is for "air" and E is for "Earth."

But notice something strange about drag forces. They increase as the speed increases. This makes sense, think of running vs. walking through or room with the baseballs. But for our falling person, it means that the drag force increases the faster the person falls. At some point the parachutist's drag force will be equal to his/her weight force. The parachutist will be in dynamic equilibrium!

$$F_{net} = D_{PA} - W_{PE} = 0$$

The person has stopped accelerating! which is the whole point of using a parachute. We can find out how fast the person will be going when he or she stops accelerating by using our borrowed formula for drag

$$D_{PA} - W_{PE} = 0$$

becomes

$$\frac{1}{2}C\rho Av^2 - mg = 0$$

or

$$\frac{1}{2}C\rho Av^2 = mg$$

so that we can write

$$v^2 = \frac{2mg}{C\rho A}$$

and finally

$$v = \sqrt{\frac{2mg}{C\rho A}}$$

We call this speed *terminal velocity*. Suppose we know that for a falling person on a parachute $C = 1.1$, $m = 89\,\mathrm{kg}$, (person plus parachute) the air density is $\rho = 1.3\frac{\mathrm{kg}}{\mathrm{m}^3}$ and suppose that for our person/parachute system our cross-sectional areal is about $A = \pi r^2$ with the radius about $5.5\,\mathrm{m}$ so $A = \pi\,(5.5\,\mathrm{m})^2 = 95.033\,\mathrm{m}^2$. Then

$$\begin{aligned} v &= \sqrt{\frac{2\,(89\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{(1.1)\left(1.3\frac{\mathrm{kg}}{\mathrm{m}^3}\right)(95.033\,\mathrm{m}^2)}} \\ &= 3.5828\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Which is moving, but not all that fast. It is about eight miles per hour.

We'll deal with Drag force later in the course, and many of you will see it again in PH150 and PH123.

18 Newton's Second Law, Newton's Third Law and the idea of Systems

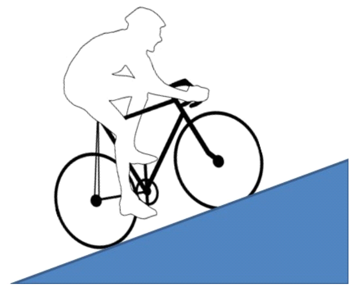
Let's start this lecture with a more in-depth look at drawing force diagrams. our goal today is to determine what we can and can't include in our particle when using particle model.

Strategy for Drawing Force (free-body) Diagrams

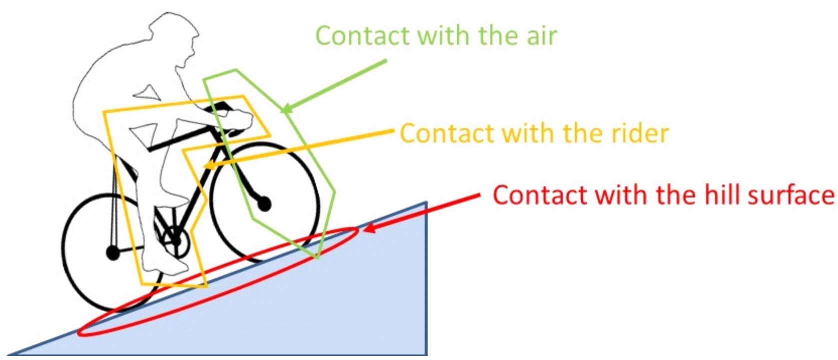
Let's pause and think about how to make sure we draw our free-body or force diagrams correctly. What we have learned is that we must draw a separate diagram for each object that we will study. One object per diagram. And we have learned that we draw forces that act on that object, and only forces that act on that object. We have leaned to use subscripts to show what types of forces we have, and what object is causing the force. Our types of forces (so far) are given in the next table.

Force Type	Symbol	Causes (How this force is made)	Contact of Non-contact
Weight	W	Gravitational Attraction	Non-contact
Normal	N	Squashed or compressed atoms	Contact
friction	f	Molecular bonds or bending of roughness teeth	Contact
Tension	T	Stretched molecular bonds	Contact
Drag	D	Collisions with gas or liquid molecules	Contact

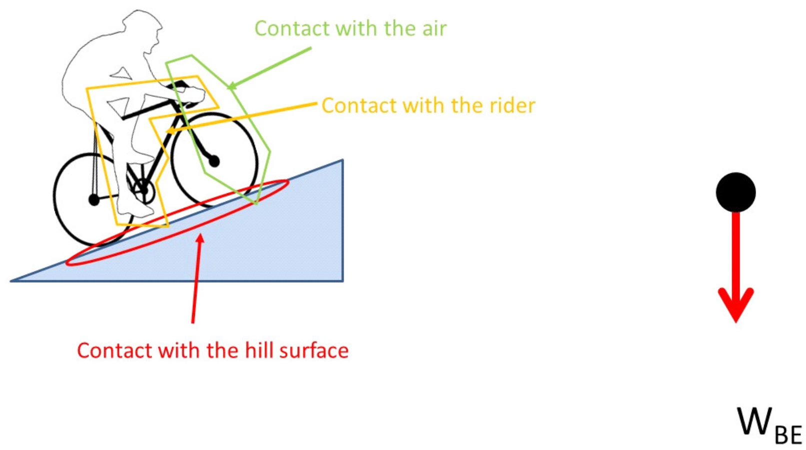
When we wish to make a force diagram we need to draw arrows for all the forces acting on the object, then go through our list to identify what type of force we have. It helps to look at where the object comes into contact with the surrounding environment. Only gravity (so far) is a non-contact force. So all forces that we know must come about by other objects making contact with our object. This also helps us identify the second subscript, the one that indicates what other object is causing the force. Let's try a complicated example, a bicycle going up a hill.



There are really five objects involved with this situation, but we only want the free-body diagram for the bicycle. We can start with the weight force. We know the bicycle has mass, so it has weight. And this is a great place to start because it takes care of our only non-contact force. The rest of the forces will have to be caused by contact with other objects. So we next look for where the bicycle is in contact with other objects.

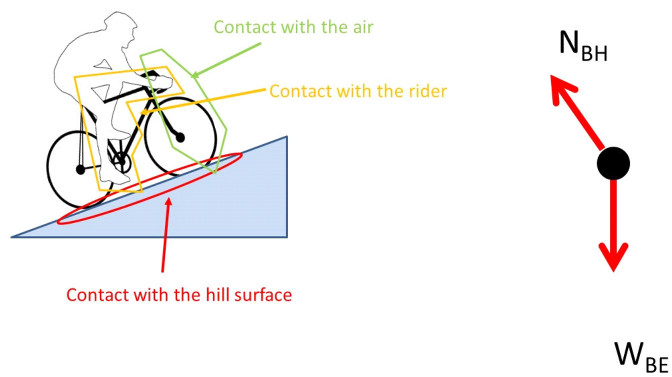


We can see that the bike tires are in contact with the hill. If we look at our table of possible forces we realize that this contact could cause two types of forces, a normal force (from compressing or squashing the hill's atoms, so they push back) and a friction force (because the tires push on the hill's surface roughness teeth, so the hill's roughness teeth push back). We can also see that the bike is in contact with the rider. The rider is being pulled down by the Earth's gravity, so his or her atoms are being pulled into the bike seat, the handle bars, and even the peddles. The rider's atoms don't like being squashed, so they push back. This creates a normal force. The bike is also in contact with the air, and as the bike moves it collides with the air molecules. This would make a drag force. We can place these forces on our diagram, but we have to be careful to think about what is causing the forces to get the directions right. Let's start with the weight force. It should be directed from the center of the bike to the center of the Earth. Usually this means straight down in our diagrams.

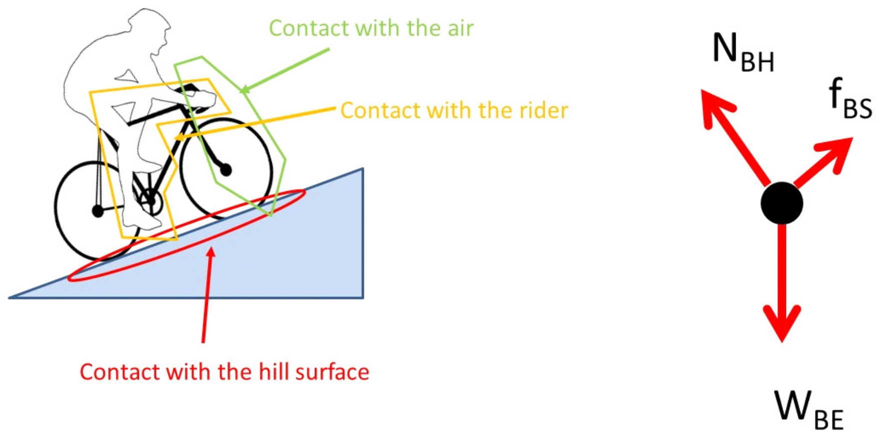


Notice that the first subscript is “B” for “bike.” The second subscript is “E” for “Earth.”

The normal force from the hill’s atoms resisting being squashed will be perpendicular to the hill.



Now let’s place the friction force due to the hill surface. To do this we need to realize that the tires push against the roughness teeth of the road and they push downhill!



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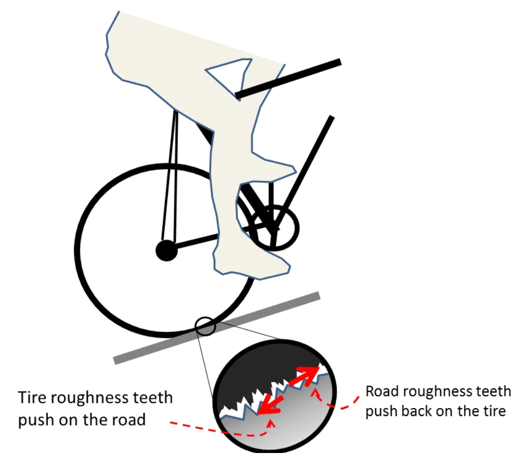
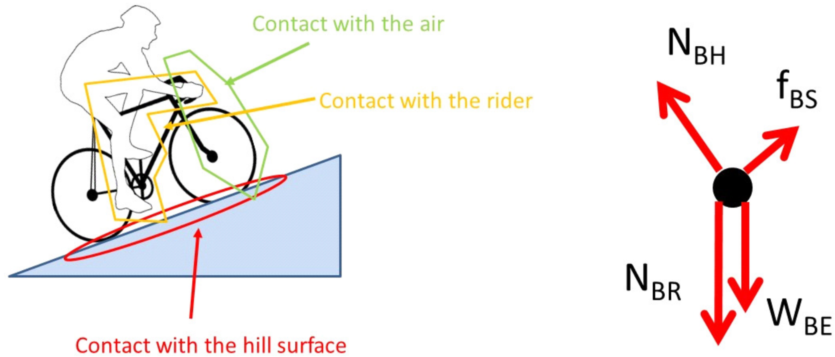
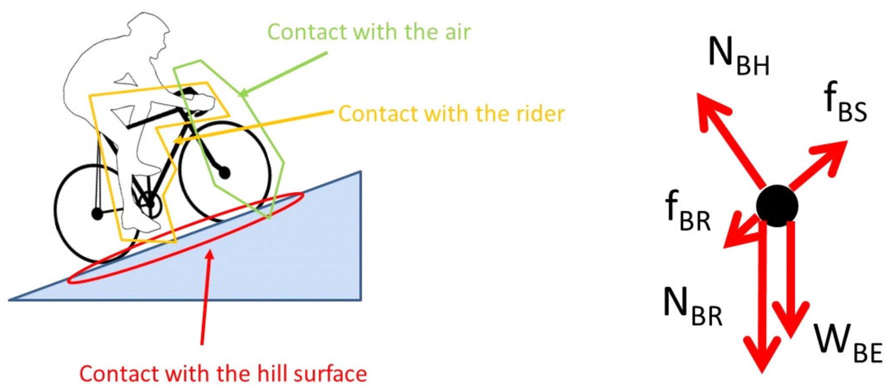


That means the hill surface roughness teeth push uphill! We can add our friction force to the free-body diagram.Notice that the subscript is “S” for hill (S)urface roughness teeth. It is not just the hill, but the surface of the hill that matters for friction. After all, if the hill was perfectly smooth and not at all sticky, there would be no friction. But there would still be a normal force. It is the surface that matters for friction forces.

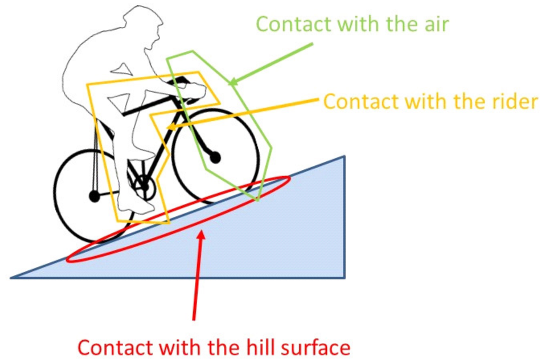
Let’s take on the interaction between the bike and the rider next. The rider’s atoms are resisting being squashed as the Earth pulls the rider into the bike. So there is another normal force due to the rider’s atoms.

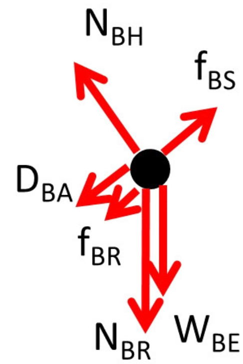


The next force might not be obvious. But think, if we held the bike in place and the rider just sat on the bike. Wouldn't the rider slide off if the seat were frictionless? After all, the bike is on a hill, so the seat is tipped. There must be some friction between the rider and the bike. If we think that the rider would tend to slide down hill, the seat must be pushing the rider back up hill to keep the rider in place. But we want the force on the bike due to the rider. The rider must be pushing down hill on the bike seat.



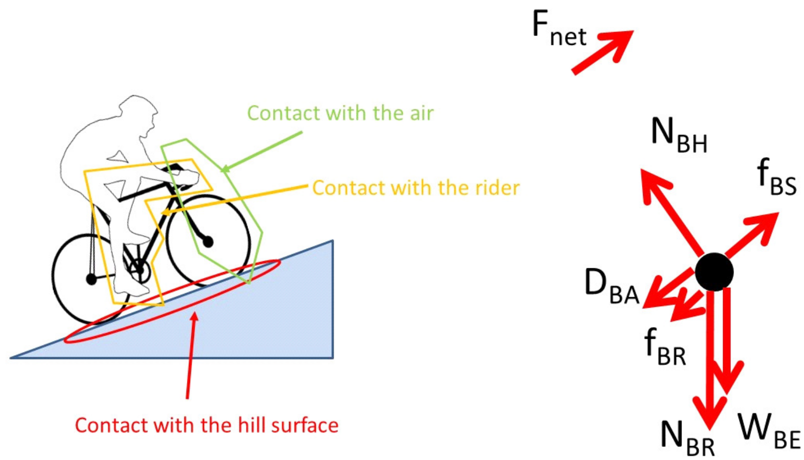
We identified a drag force. From our study of drag forces, we know that the drag force must be opposite the direction the object is going. The bike is going uphill, so the drag force must be down hill.





From our experience so far we can see that we need to get the forces right in our free-body diagram in order to find the motion of an object. So taking time to practice getting all the forces and labeling them so the mover object and the environmental object are obvious, and that the type of force is obvious help us to make sure we got the forces in the right directions. The diagrams may seem trivial, but really if you can't get the diagram right, you likely won't get the problem right. So it is worth taking time to carefully draw free-body diagrams. So let's review what we did

1. Identify the object who's diagram we will draw.
2. Identify any non-contact forces (for PH121 this is usually just a weight force).
3. Identify all environmental objects that make contact with our object
4. Identify what contact forces are caused by the environmental objects by using our table of forces and their causes.
5. Using the force causes from the table, determine the force direction and draw it on the diagram.
6. Label the force with the correct type symbol. Do this for each force.
7. Add the subscripts to indicate what object is being studied (in our case the bike) and what environmental object is making the force (e.g. the Earth for the weight force or the hill for one of the normal forces).
8. Don't put it on the diagram, but it probably a good idea to think of which direction the net force will be. If our cyclist is accelerating up the hill, then the net force will be uphill.



It is important to not put the net force as though it were acting on the dot representing the bike. Remember we will use our force diagram to form $\overrightarrow{F}_{net}$ which is the sum of all the forces, so we can't include $\overrightarrow{F}_{net}$ as part of that sum! If you want to mark $\overrightarrow{F}_{net}$ you can have it float near the free-body diagram, but don't make it part of the diagram.

Models vs. Laws

You might be saying to yourself at this point that friction seems a little less certain an idea than, say, Newton's law of gravity. And you would be right. We need to make a distinction between a way of thinking about how a complicated system works, and a physical law, meaning an equation that describes a physical relationship.

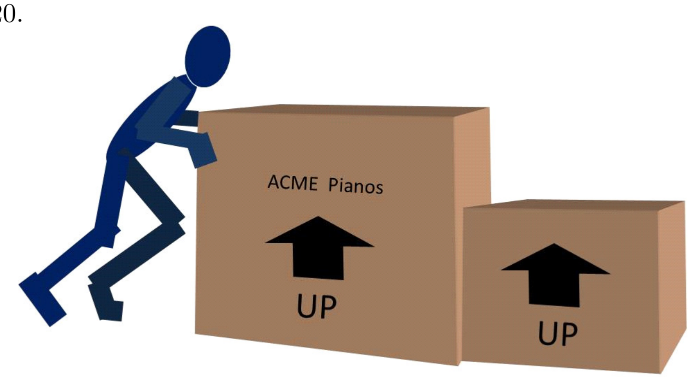
Our model for friction would be

$$\begin{aligned} f_{BFs} &\leq \mu_s N_{BF} \\ f_{BF_k} &= \mu_k N_{BF} \end{aligned}$$

but there is a lot of leeway in our coefficients of friction. And you might further ask, does the amount of area on the object that is in contact with the floor matter? If I have two sets of Velcro®, but one set is much larger than the other, wouldn't it take more force to slide the two sides of the big set across each other than it would to slide the two sides of the smaller pieces across each other? For Velcro®, it does matter! But it turns out that for most surfaces (not like Velcro®) the area is not very important. But it could be for a designer surface that is intended to be extra rough. Our mental model works for most surfaces. But it would not be to surprising to find that a measured coefficient of friction for, say, wood on wood, might not match our table value if the wood had saw marks, or was varnished, etc. We might say that our equations for friction are a good mental model of how things work, but they are not a fundamental theory, describing the basic nature of the universe. The fundamental theory would be involved in the creation of the atomic bonds, etc. our friction model is like a summary of the effects of all the fundamental theories as they act on a particular object.

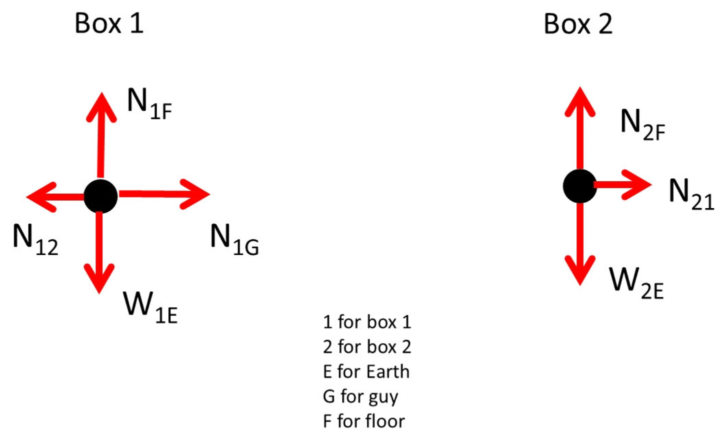
Systems

Suppose you are helping with a neighbor's move. There are two big boxes to move, and you decide you will push both boxes at once. One box has a weight of $50.0\,\mathrm{N}$ and the other has a weight of $40.0\,\mathrm{N}$. The coefficient of sliding friction between the box and the floor is 0.20.



What will be the acceleration of the smaller of the two boxes if you push with a force of $30.0\,\mathrm{N}$?

This is a Newton's second law problem, but a dynamic one. Newton's second law problems require us to draw free-body diagrams, so let's draw a diagram for each box.



We know

$$\begin{aligned} W_{1E} &= 50.\,\mathrm{N} \\ W_{2E} &= 40.\,\mathrm{N} \\ N_{1G} &= 30.\,\mathrm{N} \\ g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

And we will need our Newton's second law equations

$$\begin{aligned} \overrightarrow{\mathbf{a}} &= \frac{\overrightarrow{\mathbf{F}}_{net}}{m} \\ \overrightarrow{\mathbf{F}}_{net} &= m\overrightarrow{\mathbf{a}} \end{aligned}$$
$$W = mg$$

We need Newton's second law for both boxes. From our figure for box 1 we have

$$m_1 a_{1x} = N_{1G} - N_{12}$$
$$m_1 a_{1y} = N_{1F} - \underline{W_{1E}}$$

and we hope that $a_{1y} = 0$ because we don't expect the box to fly up or burrow into the ground.

$$0 = N_{1F} - \underline{W_{1E}}$$
$$N_{1F} = \underline{W_{1E}}$$

Now let's do box 2

$$m_2 a_{2x} = N_{21}$$
$$m_2 a_{2y} = N_{2F} - \underline{W_{2E}}$$

and again $a_{2y} = 0$

$$0 = N_{2F} - \underline{W_{2E}}$$
$$N_{2F} = \underline{W_{2E}}$$

Let's summarize what Newton's second law has taught us about this problem:

$$N_{1F} = \underline{W_{1E}}$$
$$N_{2F} = \underline{W_{2E}}$$
$$m_1 a_{1x} = \underline{N_{1G}} - N_{12}$$
$$m_2 a_{2x} = N_{21}$$

We have six things we don't know, and only two equations. It looks hopeless. But really we know more things. The boxes must accelerate together! if that were not true, one box would launch ahead of the other, or one would collapse as the other accelerated through it. Neither of these things are happening. So we can say $a_{1x} = a_{2x} = a$. This is what we call a "constraint." A constraint is a piece of information that comes from the physics of our situation. We also know that unless one box is crushing the other, $N_{12} = N_{21} = N$, This is another constraint. And each constraint added another equation!

We also know

$$\underline{W_{1E}} = m_1 \underline{g}$$
$$\underline{W_{2E}} = m_2 \underline{g}$$

This brings our equation count to six. We should be able to solve a set of six equations and six unknowns!

Using our constraints we could write our third equation from our Newton's second law set as

$$m_1 a = \underline{N_{1G}} - N$$

and the forth would be

$$m_2 a = N$$

let's solve this last equation for a

$$a = \frac{N}{m_2}$$

and substitute this into the previous

$$m_1 \frac{N}{m_2} = \underline{N_{1G}} - N$$

or

$$m_1 N = m_2 \underline{N_{1G}} - m_2 N$$

and some rearranging

$$m_1 N + m_2 N = m_2 \underline{N_{1G}}$$
$$N\left(m_1 + m_2\right) = m_2 \underline{N_{1G}}$$

gives

$$N = \frac{m_2 \underline{N_{1G}}}{(m_1 + m_2)}$$

We can substitute this into our equation for a

$$a = \frac{1}{m_2} \frac{m_2 \underline{N_{1G}}}{(m_1 + m_2)}$$

but don't yet know m_1 or m_2 But we have two equations relating m_1 and m_2 to the box weights

$$\underline{W_{1E}} = m_1 \underline{g}$$
$$\underline{W_{2E}} = m_2 \underline{g}$$

so we can find our masses

$$m_1 = \frac{\underline{W_{1E}}}{\underline{g}}$$
$$m_2 = \frac{\underline{W_{2E}}}{\underline{g}}$$

and substitute them into our equation for a

$$a = \frac{1}{\frac{W_{2E}}{g}} \frac{\frac{W_{2E}}{g} \underline{N_{1G}}}{\left(\frac{W_{1E}}{g} + \frac{W_{2E}}{g}\right)}$$

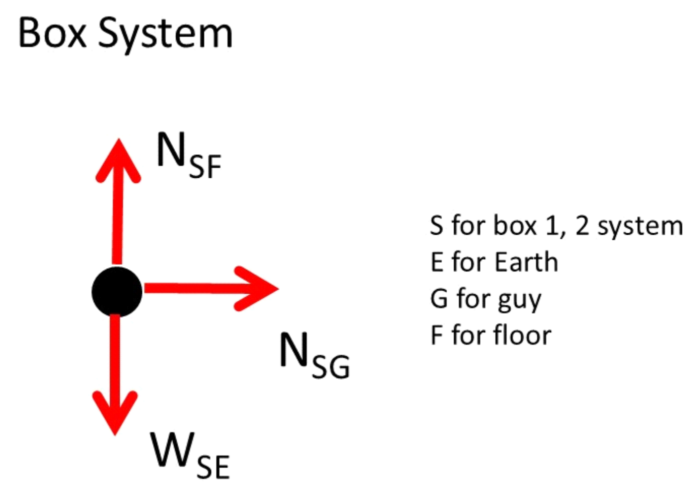
Notice that some of the g terms cancel. Then we have just

$$a = \frac{g}{W_{2E}} \frac{W_{2E} \underline{N_{1G}}}{(W_{1E} + W_{2E})}$$

and we know all the pieces, so we can plug in numbers.

$$\begin{aligned} a &= \frac{9.8 \frac{\mathrm{m}}{\mathrm{s}^2}}{40\,\mathrm{N}} \frac{(40\,\mathrm{N})\,\underline{(30\,\mathrm{N})}}{(50\,\mathrm{N} + 40\,\mathrm{N})} \\ &= 3.266\,7 \frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

But you might think that this was more work than we need to do. If it is really true that the accelerations are the same for both blocks, it should be that we can treat the two boxes together as though they were one object. We do this all the time. A car is one object, but it is made of many parts that all move together. We have used our particle model to find the motion of whole cars at once, so is should be true that we can treat both boxes together as one particle



The we can write out Newton's second law in the x and y-directions for the system consisting of box 1 and box 1

$$\begin{aligned} F_{net_x} &= m_S a_x \\ &= N_{SG} \\ F_{net_y} &= m_S a_y \\ &= N_{WF} - W_{WE} \end{aligned}$$

and

$$m_S = m_1 + m_2$$

then from the x-equation we have

$$m_S a_x = N_{SG}$$

and our acceleration is

$$a_x = \frac{N_{SG}}{m_1 + m_2}$$

We don't know the masses, but we have an expression for them from above, so

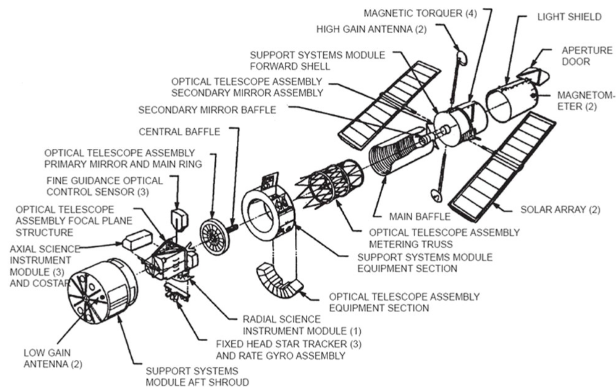
$$a_x = \frac{N_{1G}}{\left(\frac{W_{1E}}{g} + \frac{W_{2E}}{g}\right)}$$

$$a_x = \frac{gN_{1G}}{(W_{1E} + W_{2E})}$$

$$\begin{aligned} a_x &= \frac{\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(30\,\mathrm{N})}{(50\,\mathrm{N} + 40\,\mathrm{N})} \\ &= 3.266\,7\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

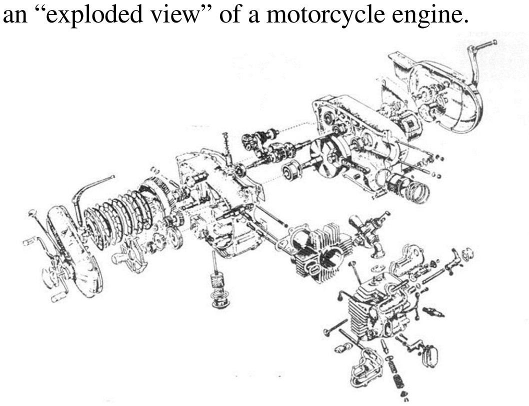
We got the same result, and it was so much easier!

We can give a name to a group of objects that are somehow bound together so that they move together.



Noticed that we call a group of objects that move together a *system*. Notice that in our first analysis of the two box system we found that there were forces between the two boxes. This is part of what makes a system a system. Something must keep the parts of the systems together. Bolts, welds, screws, and glue keep the Hubble Telescope together. But these welds, screws, bolts, and glue all really rely on molecular forces that can stretch, but hold together. For our boxes the forces that keep the boxes together are the normal forces N_{12} and $N_{21.}$

Notice that they figure prominently in our first solution, but don't show up at all in the second solution! These forces are not from outside the two-box system. Rather, they are *internal* forces. If we treat the whole system as a particle, we will never see these forces. The next figure is an "exploded view" of a motorcycle engine.

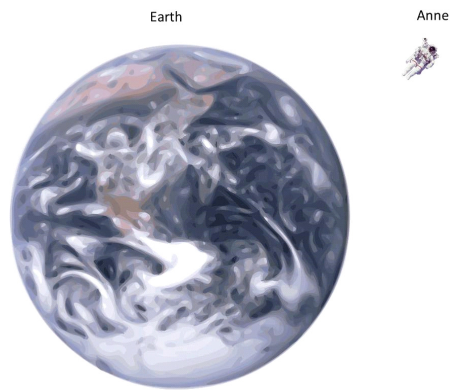


As the motorcycle operates, there will be lots of internal forces among all these parts. But for most practical purposes, we will treat the whole motorcycle, engine parts and all, as one object. We call this one object the "motorcycle system."

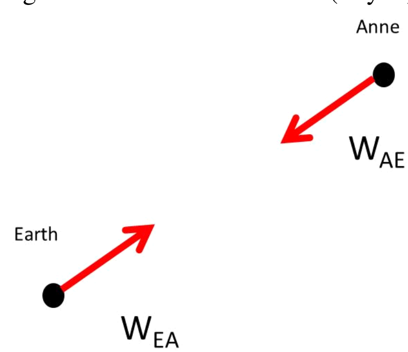
This is both a comfort, and a potential danger. Those internal forces really exist. And as we search for all the forces acting on an object, we will have to be sure that we don't include internal forces in our net force calculations. Internal forces won't accelerate an object. On the other hand, if you have ever had the misfortune of breaking a motor mount, you will recognize that making sure the internal forces are not too large is critical for a mechanical design!

Notice that internal forces come in matched sets. The normal forces N_{12} and N_{21}, for our two box example. In our subscript scheme, the internal forces have matched, but reversed subscripts. This is a dead giveaway that these forces are internal forces.

Almost any group of interacting objects could be considered a system. Suppose we take the Earth and you (or an astronaut) as a system.



Here are the free body diagrams for both the astronaut (or you) and the Earth.



The internal forces will be W_{EA} and W_{AE}. Notice that they are the same type of force, both gravitational forces. Note the subscripts are the same, but reversed. You might also notice that their directions are opposite. Also notice that it took two free-body diagrams to draw both forces. That is because each of the forces is on a different object.

We call forces that match like this *force pairs*. All interactions between objects come as force pairs. But the parts of force pairs can never act on a single object. there must always be two objects for there to be a force pair.

Our normal forces between the boxes are a force pair, N_{12} and N_{21}. Notice that N_{12} and N_{21} are on different diagrams! Also notice that they are opposite in direction, but the same type of force (a normal force). It can be tricky to identify force pairs. So we will spend more time with this next lecture.

Newton's Third Law

Newton' also realized that forces don't act alone in nature. They act in pairs. Let's start with the example of hammering a nail

Nail Demo

Notice how the hammer bounces! This means the motion of the hammer head has changed directions. We can see that there must be an acceleration of the hammer head. We now know that this acceleration is evidence of a force. We provided a force on the nail by hitting it. We now see that the nail provided a force back on the hammer!

We can now state Newton's third law:

> *If object 1 and object 2 interact, the force $\overrightarrow{\mathbf{F}}_{12}$ exerted on object object 1 by object 2 is equal to the force $\overrightarrow{\mathbf{F}}_{21}$ on object 2 by object 1.*

You ,may also hear this expressed as "for every action, there is an equal and opposite reaction," but this almost sounds like one force happens before the other. And that is not true. The forces happen at the same time.

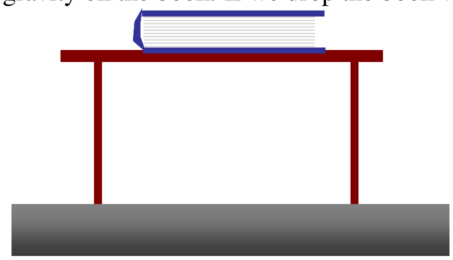
So in the case of our nail, we could write the force of the hammer striking the nail as $\overrightarrow{\mathbf{N}}_{NH}$ and the force of the nail on the hammer as $\overrightarrow{\mathbf{N}}_{HN}$ and state that

$$N_{NH} = N_{HN}$$

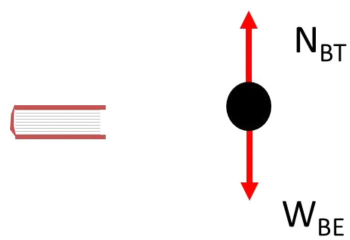
and that the direction of the two forces are opposite, that is

$$\overrightarrow{\mathbf{N}}_{NH} = -\overrightarrow{\mathbf{N}}_{HN}$$

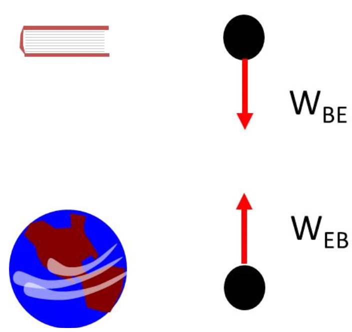
Let's take on a hard problem. Take your text book. It weights a lot. Now place it on your desk. Explain, now, why the book does not push the table to the floor. Certainly there is a force due to gravity on the book. If we drop the book we can see this.



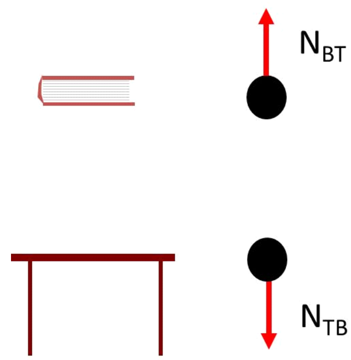
So if the book stays in one place when we put it on the table, there must be a force that opposes its motion so the net force is zero. We already know that we should call this force $\overrightarrow{\mathrm{N}}$ for "normal force" (meaning it is perpendicular to the surface). Let's look for force pairs.



The gravitational force is between the book and the Earth. So there is a force pair as shown below



This accounts for the gravitation, but not the normal force. There is another pair



The table pushes up on the book with force $\overrightarrow{\mathbf{N}}_{BT}$, and the book pushes down on the table with force $\overrightarrow{\mathbf{N}}_{TB}$. Notice that forces $\overrightarrow{\mathbf{N}}_{BT}$ and $\overrightarrow{\mathbf{N}}_{TB}$ both have the magnitude W_{BE}. Note also that the forces acting on the book are $\overrightarrow{W}_{BE}$ and $\overrightarrow{\mathbf{N}}_{BT}$. The other forces, $\overrightarrow{\mathbf{N}}_{TB}$ and $\overrightarrow{W}_{EB}$ are are exerted *by* the book on other objects (the table and the Earth).

We can see that for the book $\overrightarrow{\mathbf{N}}_{BT}$ must equal $\overrightarrow{W}_{BE}$. The book is not even moving. So it clearly is not accelerating From Newton's second law we have

$$\Sigma F_{By} = ma_{By} = 0 = N_{BT} + W_{BE}$$

we know

$$W_{BE} = -mg$$

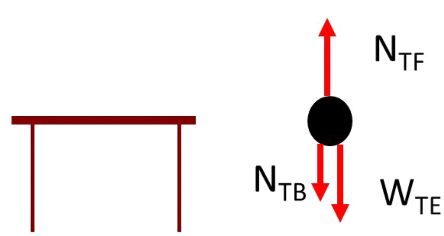
and have deduced that they are oppositely directed in the y direction so

$$0 = N_{BT} - mg$$

or

$$N_{BT} = mg$$

Now you may say that this seems incomplete (it did to me when I first learned this). We are left with the table having a net force of $\overrightarrow{\mathbf{N}}_{TB}$ acting on it. This should accelerate the table downward! But of course the table is sitting on the floor (which we will take to be part of the earth) so we really have another reaction pair that keeps the table in place. Here is the more complete table diagram.



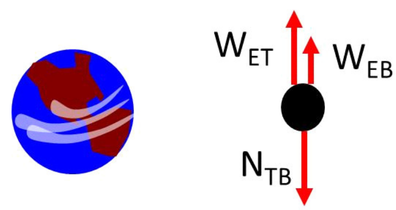
We can see that

$$\Sigma F_{Ty} = ma_{Ty} = 0 = N_{TF} - N_{TB} - W_{TE}$$

So that

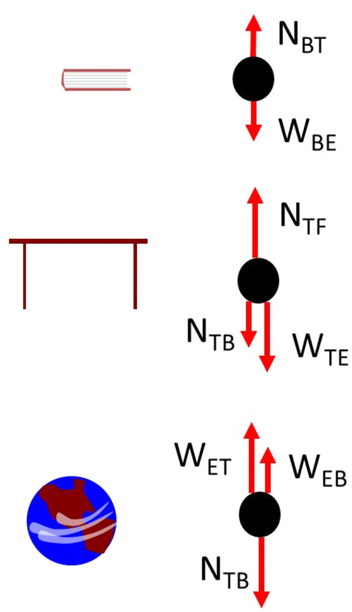
$$\begin{aligned} N_{TF} &= N_{TB} + W_{TE} \\ &= W_{BE} + W_{TE} \end{aligned}$$

The normal force due to the floor must support the weight of the table and the book. This is no surprise. We could also draw a diagram for the Earth.



Notice that the table atoms must also resist the pull of the book and the table on the Earth!

But we were looking for force pairs. Notice that there are four force pairs W_{BE} and W_{EB}, W_{TE} and W_{ET}, N_{TF} and N_{FT}, and N_{BT} and N_{TB}. Notice that no two forces in a force pair are on the same diagram! This is important. Force pairs can never act on the same object. Also notice that each force in a force pair has the same subscripts but in reverse order.

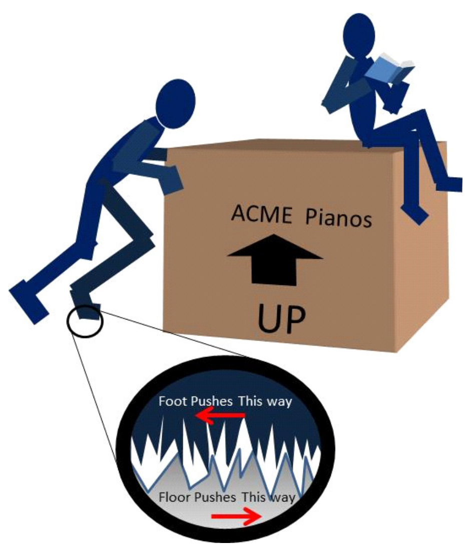


also notice that each of the forces in a force pair are opposite in direction. These are the characteristics of a force pair:

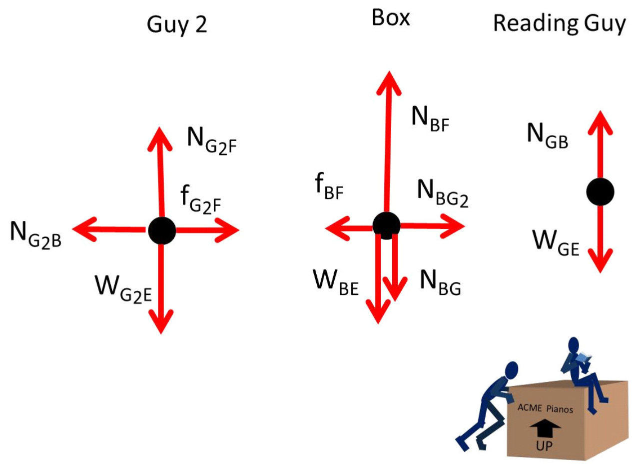
1. Equal magnitudes
2. Opposite Direction
3. Same type of force
4. On two different objects (same subscripts, but reversed)

Propulsion

So far, we have worked rather hard to avoid the details of how we actually get something moving. We know, for example, that if the floor of the apartment were frictionless, our moving guys would not be able to push on the box. Their feet would slide out from under them.

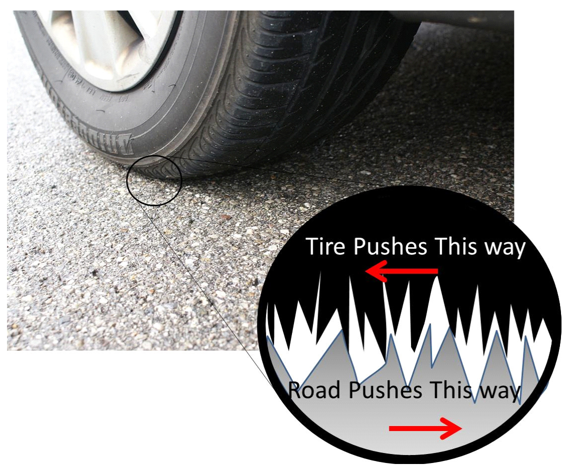


But with friction, the guy can push. However notice that the guy's foot pushes the opposite way he/she wants the box to go. The foot pushes backwards. That makes the little roughness teeth push *forwards!* This is not too much of a surprise.



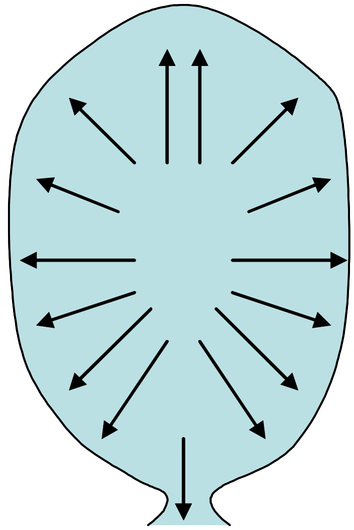
We already saw a situation like this in our homework when we wanted to keep a person standing on the back of a truck. The roughness teeth are what keeps the person moving with the truck. They must push forward.

Let’s study a car wheel. Notice that as the tire turns, it will push against the road.



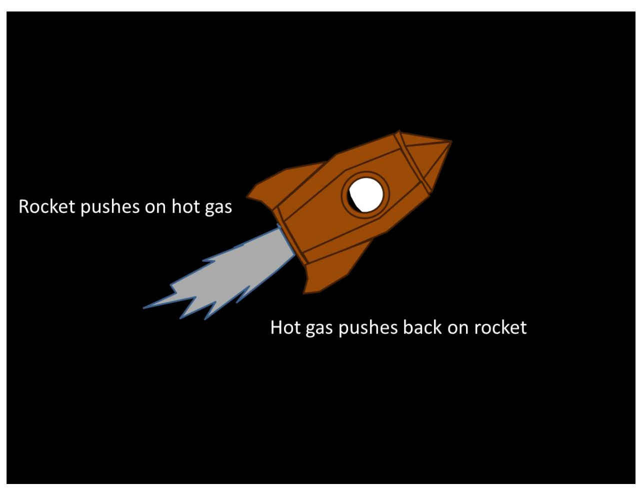
Again the roughness teeth in the road will push back on the tire. This is what pushes the car forward. Notice in propulsion problems so far there has always been a Newton’s third law pair!

Let’s take on a harder case. A balloon that has been released without being tied. The balloon moves around the room. What makes the balloon go? If you blow up the balloon, the extra air pressure in the balloon is distributed around the inside of the balloon. But if you open the stem and let if □y around, then where the stem is, there is no balloon wall to push back on the air. The air simply escapes.



Think of summing up all the forces acting on the surface of the balloon. The forces act to balance each other, except where the hole is. Thus, the force on the other side of the balloon is unbalanced. And the balloon flies around. The stretched balloon material pushes on the air, and the air pushes back.

Rockets work this way, except that instead of inflating, they use a controlled explosion using the rocket fuel. The burning rocket fuel heats gas, making the gas expand.



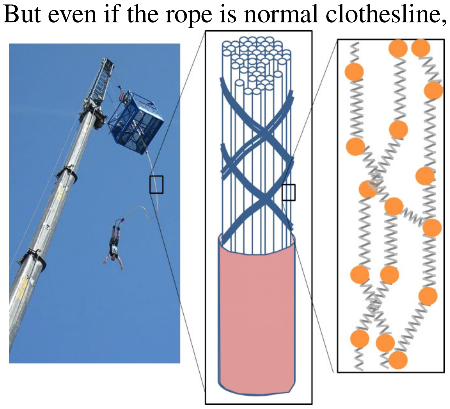
The burning of the fuel effectively pushes the gas outward, out of the rocket. The gas resists this push, pushing back on the rocket. This push from the gas propels the rocket forward. To see how gas can push on something, think of the air pressure in your tires. The air pressure is literally pushing your car upward, keeping the rims off the ground. Air pressure is a subject for PH123. But hopefully you can see that air can provide a resistive force to propel something forward. For now, notice that Newton's third law pairs are the source of propulsion.

19 Ropes and pulleys

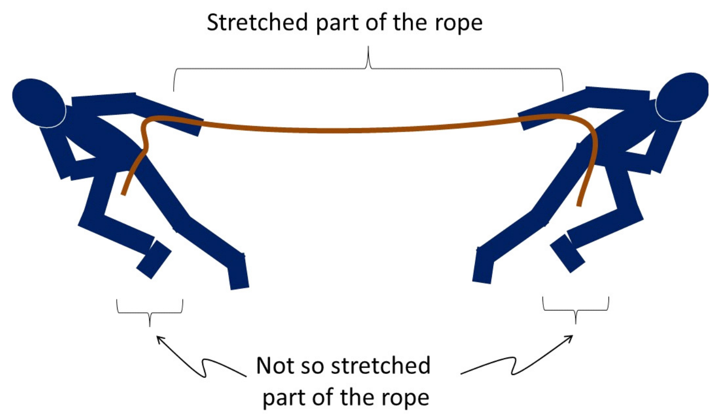
We have used ropes in our force situations already. But our ingenious ancestors invented devices that change the direction of a rope tension. We need to be able to include such details in our force calculations. So let's take a closer look at the ropes, themselves, and lets take a look at this device that changes the direction of a tension, the pulley.

Tension revisited

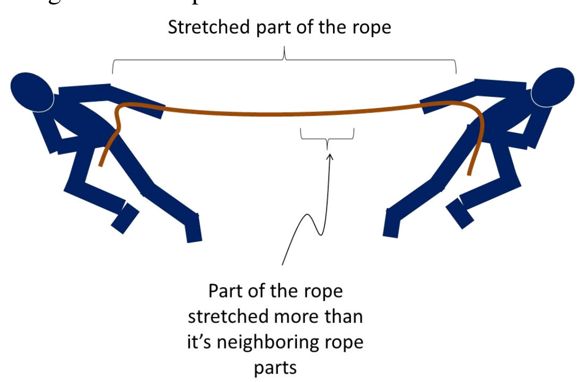
Let's review a bit. The rope is made of bundles of fibers. If the rope is a bungee rope, the fibers are very elastic. But even if the rope is normal clothesline, the fibers can stretch.



The stretch comes from spring-like bonds between the atoms of the rope material. As the rope is stretched, the bonds resist the stretch, pulling back. This is tension force. Let's consider two guys pulling on a rope in a tug-o-war.



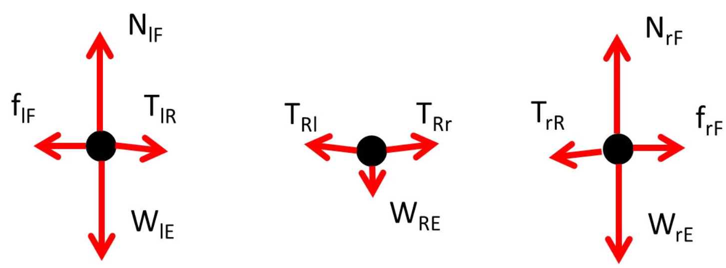
Notice that we will stretch atomic bonds for all the parts of the rope that are being used in the tug-o-war. Nearly the entire set of bonds in the whole rope are involved with creating the tension. If one region of the rope were to be stretched more than it's neighbors,



the stretchy spring-like bonds would quickly even out the stretches until the tension was mostly uniform throughout the rope.

Now Let's change our situation. Suppose that the guys are not actually pulling. They are just holding the rope.

Will there be a tension? Let's draw a free-body diagrams for each object involved. Then we can solve for the tension.



We have three objects, two guys and the rope. Notice that the rope pulls on the guys just as hard as the guys pull on the rope, well, almost. Let's write out Newton's second law for the rope

$$F_{net_{Rx}} = -T_{Rl}\cos\theta + T_{Rr}\cos\theta$$

and

$$F_{net_{Ry}} = +T_{Rl}\sin\theta + T_{Rr}\sin\theta - W_{RE}$$

Since we are in static equilibrium,

$$F_{net_x} = F_{net_y} = 0$$

so we can write

$$T_{Rl}\cos\theta = T_{Rr}\cos\theta$$
$$W_{RE} = T_{Rl}\sin\theta + T_{Rr}\sin\theta$$

Let's look at the first of the set. This tells us that the right hand and left hand tensions are the same

$$T_{Rl} = T_{Rr} = T$$

which we had already surmised. But look at the y-equation

$$W_{RE} = 2T\sin\theta$$

so the tension for the guys just holding the rope is

$$T = \frac{W_{RE}}{2\sin\theta} = \frac{m_R g}{2\sin\theta}$$

If we have a slightly heavy rope like they use in a tug-o-war, we might find the length of rope having a mass of $0.163\,4\,\mathrm{kg}$ and suppose we find the rope hands with a $5\,^\circ$ angle,

then

$$
\begin{aligned}
T &= \frac{W_{RE}}{2\sin\theta} = \frac{(0.163\,4\,\mathrm{kg})\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{2\sin(5\,^\circ)} \\
&= 9.\,186\,5\,\mathrm{N}
\end{aligned}
$$

We can see that the weight of the rope really does matter if the rope is massive enough. We get a little bit of tension because the rope has mass. So far we have been ignoring the effect of the rope's mass. We call doing this the *massless string approximation*. We can see that if the rope is heaver the approximation will fail.

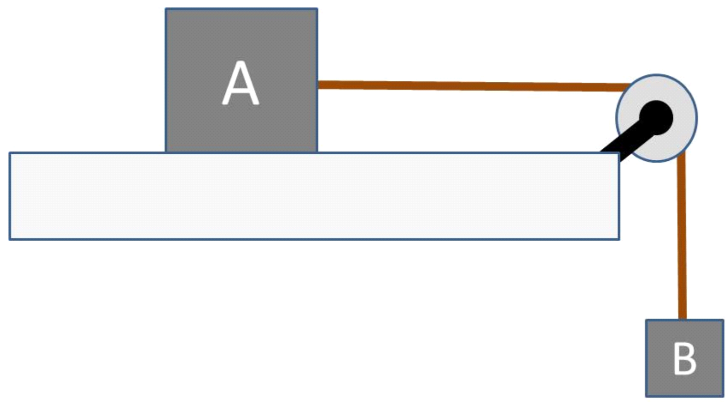
The heavy anchor chain on a large ship would really not be well approximated by a massless string!

USS John F. Kennedy Anchor Chain

But so long as the weight of the rope or string is much less than the masses of the other objects in the system, then the massless string approximation can be used. In real situations, we will have to check to see if the massless string approximation is valid before solving the problem.

Pulleys

What does a pulley do for us?



The first thing you will notice is that a pulley changes the direction of the tension! This can be quite useful. But there is a cost.

Early pulley-like devices were just wooden blocks that the rope would slide over. They would have a lot of friction. That friction would reduce the tension in the rope. So, in our example above, Block A would not be pulled with as much force due to the tension created by block B because some of the force from block B was used just to make the rope move over the pulley. The engineering work on pulleys done over the years has been to reduce friction by making part of the pulley turn. By turning like a wheel. the rope and pulley don't slide against each other. The friction is greatly reduced because it is static friction

$$f_s \leq \mu_s N$$

and so long as the wheel moves with the rope the roughness teeth don't bend and we get

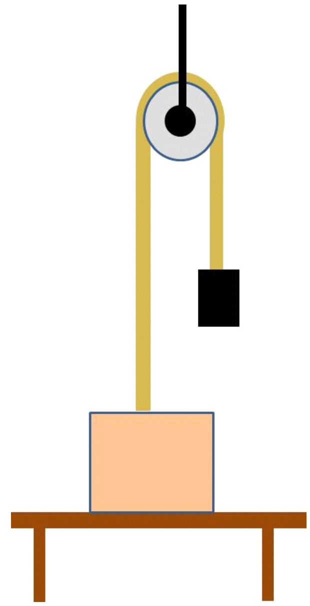
$$f_s \ll \mu_s N$$

Elevator pulley and cable.

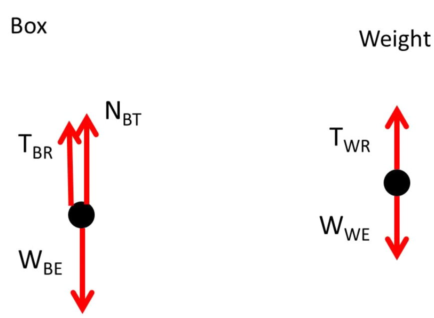
An ideal pulley would have absolutely no friction. This is an engineering feat yet to be accomplished. But many pulleys have very little friction. If the friction in the pulley axel and the static friction between the rope and the pulley wheel do not change the tension in the rope noticeably, we can ignore the friction. We call this the *frictionless pulley approximation.*

We will generally use both the massless string and the frictionless pulley approximation together in problems. Let's try one now.

A box with a weight of $100.0\,\mathrm{N}$ sits on a table. The table exerts a normal force on the box to keep it from smashing its way through the table material. But what would happen if you tied a rope to the box, and passed the rope through a pulley (that is perfectly frictionless), then attached a weight to the other end? What would the normal force be in this situation if the weight weighed $40\,\mathrm{N}$?



What type of problem is this? Well, we can see we have a rope and we have forces, so this is likely a Newton's second law problem. We will need a free body diagram for our objects.



Notice that we did not draw a diagram for the rope. If we take the massless string approximation, then the rope has no mass, so we can't exert a force on the rope! But the rope can exert a force on the box and weight.

VAR:

$$
\begin{aligned}
W_{BE} &= 100.0\,\mathrm{N} \\
W_{WE} &= 40\,\mathrm{N}
\end{aligned}
$$

BE

$$\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}_{net}}{m}$$
$$F = mg$$

We recognize that $\vec{a} = 0$ so our basic equation is

$$\vec{\mathbf{F}}_{net} = 0$$

or

$$\begin{aligned} F_{net_x} &= 0 \\ F_{net_y} &= 0 \end{aligned}$$

The pulley can't change the tension in the rope. Since we are assuming that the mass of the rope is negligible, and the pulley is frictionless. Then we can say

$$T_{BR} = T_{WR}$$

This is a constraint of the system. Now let's set up Newton's second for both objects. There are no x-direction forces, so we only need $F_{net_y} = 0$

For the box

$$0 = T_{BR} + N_{BT} - W_{BE}$$

and for the weight

$$0 = T_{WR} - W_{WE}$$

from the last equation

$$T_{WR} = W_{WE}$$

then

$$T_{BR} = W_{WE}$$
$$0 = W_{WE} + N_{BT} - W_{BE}$$

or

$$N_{BT} = W_{BE} - W_{WE}$$

and we can put in numbers

$$\begin{aligned} N_{BT} &= 100.0\,\mathrm{N} - 40\,\mathrm{N} \\ &= 60.0\,\mathrm{N} \end{aligned}$$

Of course this would have been harder (requiring some of our new math, an integral) if we did not use the massless string approximation and the frictionless pulley approximation.

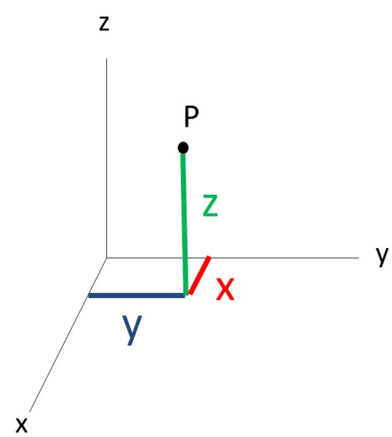
In our study of motion, we started with objects moving in straight lines, and then moved to objects moving in circles and even curved paths. In our study of forces we have studied objects moving in straight lines, you might expect that we will extend what we have learned to objects moving in circles and curved paths–and you would be right! We will start this next lecture.

20 Starting Rotational Dynamics

We are not going to learn much that is new in this lecture, but we are going to put pieces of physics that we already know together in a new way. To start with, let's consider unit vectors and coordinate systems. Then we will put Newton's second law, friction, circular motion, and radial and tangential coordinates together to solve problems where objects experience forces as they travel in circular paths.

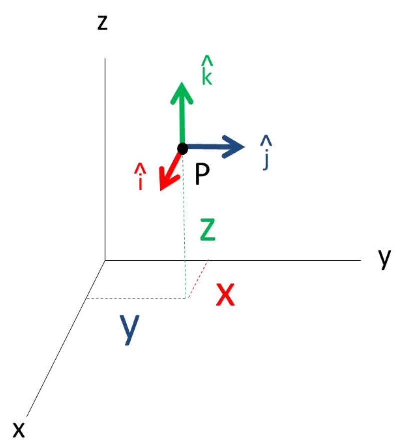
Cylindrical coordinates

So far we have mostly used an Euclidean coordinate system (xyz).



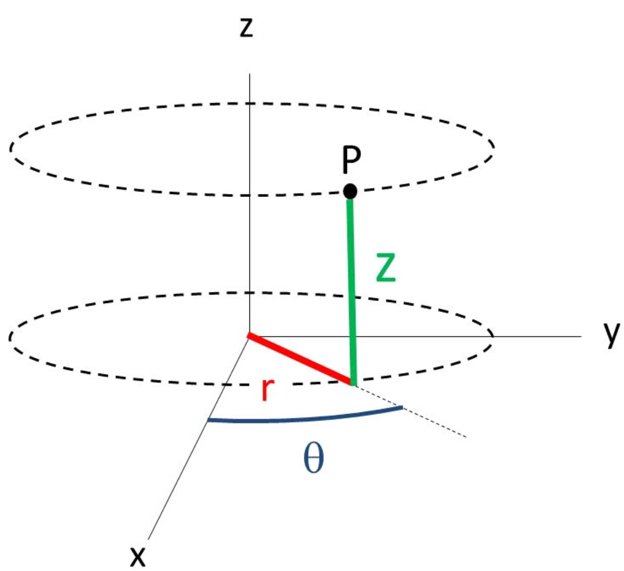
This coordinate systems has three axes, x, y, and z, and a point in space is given by how far away the point is from the xy , yz, and zx planes. A point is given by giving the three distances. We call the distance from the zy plane x. It is how far we have gone in the x-direction to get to our point. In Rexburg, it would be like saying how many blocks East we went. Likewise we call the distance from the xz plane y. It is how far we have gone in the y-direction. In Rexburg, it is like how many blocks North we went. And z is how far we are from the xy-plane. In Rexburg, it would be like how many floors up you went in a building.

We also defined three directions in this coordinate system.



These are our $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ unit vectors.

But really we have also used another coordinate system when we considered circular motion. Here it is



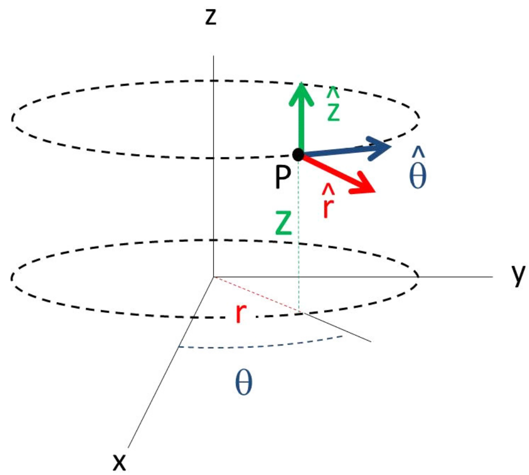
The figure shows the same point we considered in our Euclidean coordinate system. We still need three measurements in this coordinate system to describe where our point is in space. We will keep one of the measurements the same. We will call z the distance of our point from the xy plane. It is how far we went in the z-direction (how far up off the surface in Rexburg). But the other two measurements are different.

One of the new measurements is how far from the z-axis of the coordinate system we go to get to our point. But we won't follow in a block pattern like you would in a Western city like Rexburg. This is a direct "as the crow flies" measurement. We call this measurement r, because it is along a radius of the dotted circle shown. Notice that if I just tell you to go r meters, you could go in any direction. If the whole class received the same instruction, "go r meters," then you would all be lined up somewhere on the circle that has a radius, r surrounding the starting point.

The final measurement is an angle, measured from the x-axis. This will tell you which way to go from the origin. We call this angle measurement θ (or sometimes ϕ, or any Greek letter).

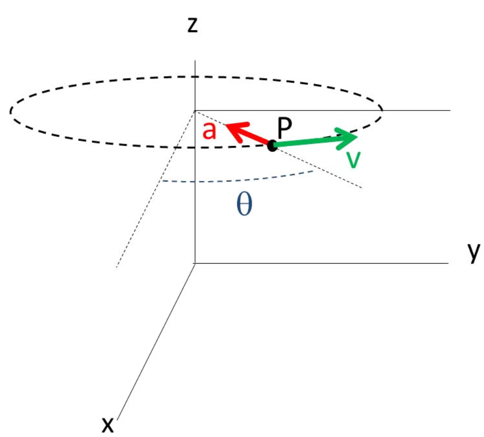
So if you were a large green monster, you could start out at the city center, choose your direction by choosing θ, then smash through the buildings a distance r, and then climb up through what was left of the building at where you stopped a distance z to get to our point P. A point can be described by giving the measurements r, θ, and z.

Of course you recognize this coordinate system as being made from polar coordinates, with a z-component added in. The proper name for this coordinate system is the cylindrical coordinate system. There are, of course, three unit vectors that describe directions in this coordinate system.



Imagine you are in an airplane circling the airport. This coordinate system would be very natural. It tells you how far you are from the airport, what direction you are from the airport, and how high above the airport you are.

We called the $\hat{r}$ direction the radial direction because it is along the radius. The $\hat{\theta}$ direction is the tangential direction (also called azimuthal direction in older books). And we will keep calling the z-direction the $\hat{k}$-direction. Because of this, sometimes this coordinate system is called the rtz-coordinate system. Think of flying in the plane. You would have a turning (centripetal) acceleration keeping you going in a circle. That would be a radial acceleration because it points in the (negative) $\hat{r}$-direction.



The plane would also have a tangential velocity in the $\hat{\theta}$ direction.

Circular motion is familiar to us, and all this is really nothing new, with the small exception that we have added in the z-axis so we can have circular motion of flying things. We have a set of equations for uniform circular motion

$$\Delta\theta = \theta_f - \theta_i$$
$$\Delta t = t_f - t_i$$
$$\omega_{ave} = \frac{\Delta\theta}{\Delta t}$$
$$\omega = \frac{d\theta}{dt}$$
$$\omega = \frac{v_t}{r}$$
$$a_r = a_c = \frac{v_t^2}{r} = \omega^2 r$$

and if the motion is uniform circular motion we know

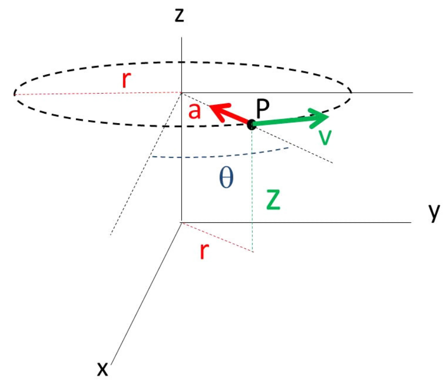
$$\begin{aligned} a_t &= 0 \\ a_z &= 0 \\ v_r &= 0 \\ v_z &= 0 \end{aligned}$$

Let's try a problem.

Suppose you decide to live in a particular apartment complex that gives obnoxious helicopter rides for signing up for an apartment (clear evidence that the rent is too high!).[10] The helicopter is flying $152.4\,\mathrm{m}$ above the ground (because any lower would be illegal) and is going around campus in a big circle (radius of $3218.7\,\mathrm{m}$) with a constant speed of $7.0\,\mathrm{m/s}$. What is the acceleration of the helicopter?

This is a uniform circular motion problem, but with our new twist that the motion is off the surface of the Earth.

[10] Yes, one semester this really happened.



We can see that the tangential and z-components of the acceleration are zero, and that the radial and z-components of the velocity are zero.

$$\begin{aligned} a_t &= 0 \\ a_z &= 0 \\ v_r &= 0 \\ v_z &= 0 \end{aligned}$$

all from our picture and knowing the helicopter is creating constant circular motion. We also know that

$$\begin{aligned} z &= 152.4\,\mathrm{m} \\ r &= 3218.7\,\mathrm{m} \\ v_t &= 7.0\,\mathrm{m/s} \end{aligned}$$

where we can see that the helicopter speed must be in the tangential direction (which direction around the circle we don't know, but if you were in the helicopter, it would be obvious).

The radial acceleration is related to the tangential speed of the helicopter. By pointing to the center of the circle, we recognize the radial acceleration as needing the title "centripital." Then we can say

$$a_r = a_c = \frac{v_t^2}{r}$$

and we know all these parts

$$a_r = \frac{\underline{v_t}^2}{\underline{r}}$$

so we can find

$$\begin{aligned} a_r &= \frac{(7.0\,\mathrm{m/\,s})^2}{3218.\,7\,\mathrm{m}} \\ &= 1.\,522\,4 \times 10^{-2}\,\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

This is not a very large acceleration, the pilot probably does not want you to feel motion sickness (and therefore question the value of that high rent payment).

Circular motion and forces

Now let's add in forces. We are force experts using Newton's second law.

$$\overrightarrow{a} = \frac{\overrightarrow{F}_{net_x}}{m}$$

But in the past we have written Newton's second law as

$$\begin{aligned} a_x &= \frac{F_{net_x}}{m} \\ a_y &= \frac{F_{net_y}}{m} \\ a_z &= \frac{F_{net_z}}{m} \end{aligned}$$

but now we need to write these as

$$\begin{aligned} a_r &= \frac{F_{net_r}}{m} \\ a_\theta &= \frac{F_{net_\theta}}{m} \\ a_z &= \frac{F_{net_z}}{m} \end{aligned}$$

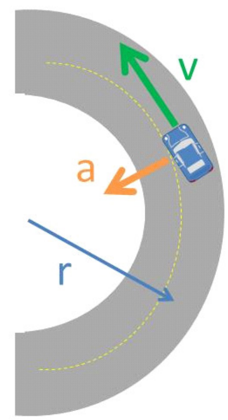
and for constant circular motion we can even write the first of our new Newton's second law set as

$$a_r = \frac{F_{net_r}}{m} = \frac{v_t^2}{r}$$

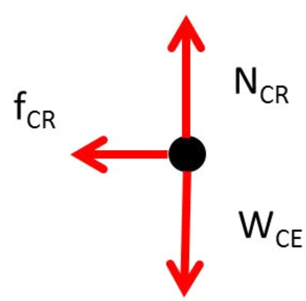
Notice that we are turning a three-dimensional problem into three one-dimensional problems. This is just what we always do with mulit-dimensional problems. But this time we have split the problem into r, θ, and z parts instead of x, y, and z parts.

Let's try another problem, one with forces involved this time:

A car travels at a constant speed of $13.\,411\,\frac{\mathrm{m}}{\mathrm{s}}$ on a level road and experiences a circular turn of radius $50.0\,\mathrm{m}$. What minimum coefficient of static friction, μ_s, between the tires and roadway will allow the car to make the circular turn without sliding? You might want to know something like this if you were designing the tire tread for race cars! A rougher tread will increase μ_s. But you don't want to make the tread too rough, or you have more friction than you need for the straight parts of the track. Here is a picture to illustrate what we want to find,



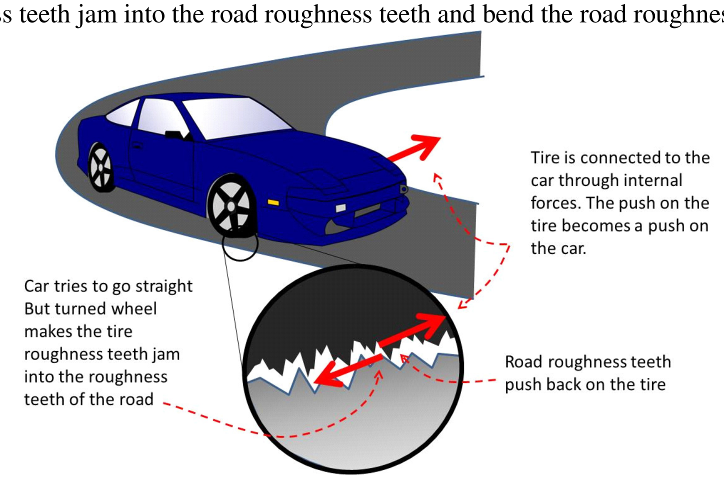
and since this is a force problem, here is the free body diagram for the car as viewed as though we were looking at the back of the car.



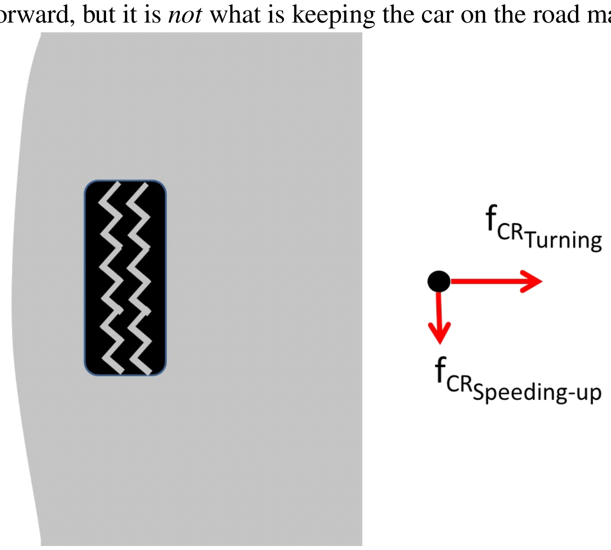
From the problem statement we know that

$$\begin{aligned}
a_t &= 0 \\
a_z &= 0 \\
v_r &= 0 \\
v_z &= 0 \\
z &= 0 \\
v_t &= 13.411\frac{\mathrm{m}}{\mathrm{s}} \\
&= 50.0\,\mathrm{m}
\end{aligned}$$

and we can see that the friction force must be in the (negative) radial direction. That means that $\overrightarrow{a}_r$ must also be in the $-\hat{r}$ direction. This makes sense because as the car tires turn, the car wants to go straight (think of Newton's first law, and also think what would happen if the road were covered with smooth, frictionless ice), but the angled tire roughness teeth jam into the road roughness teeth and bend the road roughness teeth.



Of course, the road roughness teeth push back on the tire roughness teeth. So the tire is shoved toward the center of the circle. The tire is connected to the car through a series of internal forces, so a push on the tire becomes a push on the car. The car experiences a friction force toward the center of the circle. Note that this is static friction. We really never want the tires to slide sideways. There would also be some "rolling friction" as the tires roll forward, but it is *not* what is keeping the car on the road making a turn.



The "rolling" part of the friction makes the car speed up forward, not stay in the circle. Usually you don't speed up on a turn, so while the car is turning we mostly have just $f_{CR_{Turning}}$. We can identify this static friction as a *centripetal force.* A centripetal force is the force that causes the centripetal acceleration.

Now that we have a free-body diagram, we can set up Newton's second law in our cylindrical coordinate system. Notice that the z-direciton is "up." So

$$\begin{aligned}\Sigma F_z &= 0 = N_{CR} - W_{CE} \\ \Sigma F_r &= -ma_r = -f_{CR_{Turning}}\end{aligned}$$

We can drop the "Turning" subscript, because we understand it it the turning part of the friction that is in the $-\hat{r}$ direction. We will do so in what follows. Also note that the net force in the radial direction is *negative*! We have to put a negative sign on the ma_r term because it really points in the $-\hat{r}$ direction.

From the first (z) equation we find that

$$N_{CR} = W_{CE}$$

which is not a surprise. We know that

$$f_{CR} = \mu_s N_{CR}$$

where we have an equal sign because we want the minimum μ_s. We can write

$$f_{CR} = \mu_s W_{CE}$$

and

$$-ma_r = -f_{CR} = -\mu_s W_{CE}$$

so we have for a

$$-a_r = -\frac{\mu_s W_{CE}}{m}$$

But now consider that the force, ma_r, is the force that is changing our car's direction. It is in the radial direction. This *is* our centripetal force, so we can identify $a_r = a_c$

$$-a_c = -\frac{\mu_s W_{CE}}{m}$$

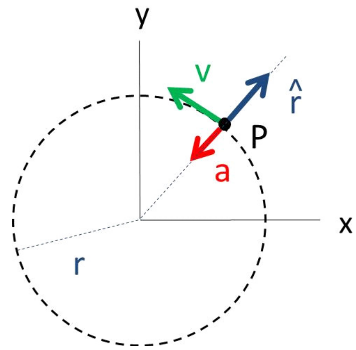
and we know that

$$a_c = \frac{v^2}{r}$$

so

$$-\frac{v^2}{r} = -\frac{\mu_s W_{CE}}{m}$$

Remember that our acceleration is radial, but it is in the $-\hat{r}$ direction toward the origin.



That is where the negative sign came from on the left hand side of the equaiton.

Now we are down to performing some algebra

$$-\frac{v^2}{r} = -\frac{\mu_s mg}{m}$$

We can cancel the minus signs

$$\frac{v^2}{r} = \mu_s g$$

so

$$\mu_s = \frac{v^2}{rg}$$

If we plug in numbers,

$$\begin{aligned} \mu_s &= \frac{\left(13.411\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{50\,\mathrm{m}\cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}} \\ &= 0.367\,05 \end{aligned}$$

Consider for a moment what this means. Will the car slip? We can look up the value of μ_s for rubber on pavement, and note that it is close to 1, so on a clear road (no ice or water) the car will not slip.

Note that we now have a centripetal force as well as a centripetal acceleration. Once again, this is not a new kind of force, but just the same old force with a new title.

More angular acceleration

Suppose that we are traveling in a circle, but the speed is not constant, then we have two acceleration components, one component towards the center, and one tangential (along a tangent line)

$$a_t \tag{20.1}$$

This is a little weird, because often in the past we have taken x and y components, but *centripetal* and *tangential* components work just as well. And we have used a_t and a_c

in circular problems before. To get back to a total acceleration vector we use,

$$a = \sqrt{a_t^2 + a_c^2} \tag{20.2}$$

and

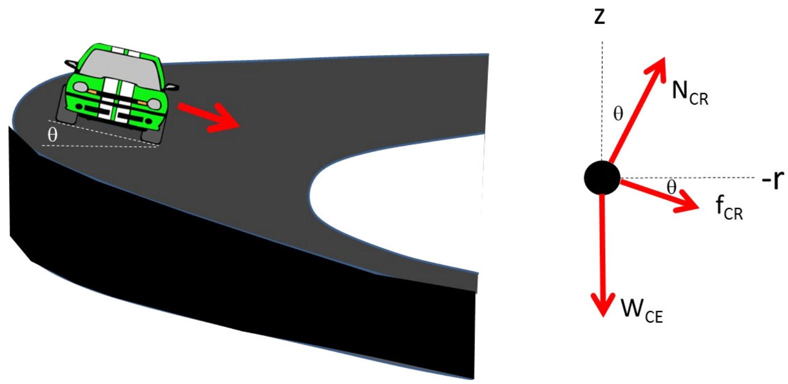
$$\theta = \tan^{-1}\left(\frac{a_t}{a_r}\right) \tag{20.3}$$

Banked roadways

Suppose we try our turning car problem again, but we play a trick to let cars travel around the curve faster. We bank the curve. This is done in real race tracks. Here is a NASCAR ® example.

Notice that the roadway is not flat. The roadway has a slope where there is a turn. So let's do our car problem again, but with a sloped roadway, say, 20.0° slope angle. Will we need as much friction to keep the car on the road?

Once again we draw a free-body diagram.



and write out Newton's second law. But notice that our coordinate system is a little strange. The z-axis is just like our usual y-axis. But the r-axis will be a little different. The minus r-axis is to the right. That is different from an x-axis!

Let's do the z-axis first.

$$F_{net_z} = ma_z = N_{CR}\sin(90^\circ - \theta) + f_{CR}\sin(360^\circ - \theta) - W_{CE}$$

Now for the r-axis. The net force is in the $-\hat{r}$ direction and all the components will be in the $-\hat{r}$ as well.

$$F_{net_r} = -ma_r = -N_{CR}\cos(90^\circ - \theta) - f_{CR}\cos(360^\circ - \theta)$$

Of course, all the negative signs cancel. But starting out with them keeps us honest in our thinking about which direction the forces go.

Using some of our beloved trig identities

$$\begin{aligned}
\sin(90^\circ - \theta) &= \cos\theta \\
\cos(90^\circ - \theta) &= \sin\theta \\
\sin(360^\circ - \theta) &= -\sin\theta \\
\cos(360^\circ - \theta) &= \cos\theta
\end{aligned}$$

we can write our Newton's second law as

$$\begin{aligned}
\Sigma F_z &= ma_z = N_{CR}\cos(\theta) - f_{CR}\sin(\theta) - W_{CE} \\
\Sigma F_r &= -ma_r = -N_{CR}\sin(\theta) - f_{CR}\cos(\theta)
\end{aligned}$$

and we can do the same thing we did in our last example, mostly...

It might be tempting to rotate our coordinate system, but we need to be careful. We know something about $a_r = a_c = v^2/r$. If we rotate our coordinates, we won't be in our cylindrical coordinates, and we will have to be careful when we calculate the centripetal acceleration. Let's not rotate coordinates this time.

We know that the car won't launch off the surface, and is not likely speeding up through the turn, so we can say that

$$a_z = 0$$

$$\begin{aligned} a_t &= 0 \\ v_r &= 0 \\ v_z &= 0 \\ z &= 0 \end{aligned}$$

$$\begin{aligned} v_t &= 13.411 \frac{\mathrm{m}}{\mathrm{s}} \\ r &= 50.0\,\mathrm{m} \\ \theta &= 20.^{\circ} \end{aligned}$$

Then our z-part of Newton's second law becomes

$$N_{CR} \cos(\theta) - f_{CR} \sin(\theta) - W_{CE} = 0$$

and we know

$$f_{CR} \le \mu_s N_{CR}$$

and again we want to take the case were we find the minimum μ_s so we will set

$$f_{CR} = \mu_s N_{CR}$$

so our Newton's second law pair now look like this:

$$N_{CR} \cos(\theta) - \mu_s N_{CR} \sin(\theta) - mg = 0$$

$$-ma_r = -N_{CR} \sin(\theta) - \mu_s N_{CR} \cos(\theta)$$

Using the first of these,

$$N_{CR} (\cos(\theta) - \mu_s \sin(\theta)) = mg$$

or

$$N_{CR} = \frac{mg}{(\cos(\theta) - \mu_s \sin(\theta))}$$

then we can take our r-equation

$$-ma_r = -N_{CR} (\sin(\theta) - \mu_s \cos(\theta))$$

and substitute in for N_{CR}

$$-ma_r = -\frac{mg}{(\cos(\theta) - \mu_s \sin(\theta))} (\sin(\theta) + \mu_s \cos(\theta))$$

and we know a_r is our centripetal acceleration so

$$a_r = a_c = \frac{v^2}{r}$$

then

$$-m\frac{v^2}{r} = -mg\frac{(\sin(\theta) + \mu_s \cos(\theta))}{(\cos(\theta) - \mu_s \sin(\theta))}$$

We know v, r, m, g, and θ, so messy as this is, we should be able to solve for μ_s

$$-m\frac{v^2}{r}\left(\cos\left(\theta\right)-\mu_s\sin\left(\theta\right)\right)=-mg\left(\sin\left(\theta\right)+\mu_s\cos\left(\theta\right)\right)$$
$$-m\frac{v^2}{r}\cos\left(\theta\right)+\mu_s m\frac{v^2}{r}\sin\left(\theta\right)=-mg\sin\left(\theta\right)-\mu_s mg\cos\left(\theta\right)$$
$$-m\frac{v^2}{r}\cos\left(\theta\right)+mg\sin\left(\theta\right)=-\mu_s mg\cos\left(\theta\right)-\mu_s m\frac{v^2}{r}\sin\left(\theta\right)$$
$$-m\frac{v^2}{r}\cos\left(\theta\right)+mg\sin\left(\theta\right)=\mu_s\left(-mg\cos\left(\theta\right)-m\frac{v^2}{r}\sin\left(\theta\right)\right)$$

$$\frac{-m\frac{v^2}{r}\cos\left(\theta\right)+mg\sin\left(\theta\right)}{-mg\cos\left(\theta\right)-m\frac{v^2}{r}\sin\left(\theta\right)}=\mu_s$$
$$\mu_s=\frac{-m\frac{v^2}{r}\cos\left(\theta\right)+mg\sin\left(\theta\right)}{-\left(m\frac{v^2}{r}\sin\left(\theta\right)+mg\cos\left(\theta\right)\right)}$$

So our answer is

$$\mu_s=\frac{-\frac{v^2}{r}\cos\left(\theta\right)+g\sin\left(\theta\right)}{-\left(\frac{v^2}{r}\sin\left(\theta\right)+g\cos\left(\theta\right)\right)}$$

We should check this. We should get the last answer if $\theta=0$

$$\mu_s=\frac{-\frac{v^2}{r}\left(1\right)+g\left(0\right)}{-\left(\frac{v^2}{r}\left(0\right)+g\left(1\right)\right)}$$

$$\frac{-\frac{v^2}{r}}{-\left(g\right)}=\mu_s$$
$$\frac{v^2}{\left(gr\right)}=\mu_s$$
$$\mu_s=\frac{v^2}{\left(gr\right)}$$
$$\begin{aligned}\mu_s &= \frac{\left(13.411\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(50.0\,\mathrm{m}\right)}\\ &= 0.367\,05\end{aligned}$$

which is the same answer we got for the un-banked road. This gives some confidence that we got our tilted problem right.

Let's put in our numbers for our tilted ramp.

$$\mu_s=\frac{-\frac{v^2}{r}\cos\left(\theta\right)+g\sin\left(\theta\right)}{-\left(\frac{v^2}{r}\sin\left(\theta\right)+g\cos\left(\theta\right)\right)}$$

so

$$\begin{aligned}\mu_s &= \frac{-\frac{\left(13.411\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{50.0\,\mathrm{m}}\cos\left(20\,^{\circ}\right)+\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)\sin\left(20\,^{\circ}\right)}{\left(-\frac{\left(13.411\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2}{50.0\,\mathrm{m}}\sin\left(20\,^{\circ}\right)-\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)\cos\left(20\,^{\circ}\right)\right)} \\ &= 2.7176\times10^{-3}\end{aligned}$$

This is *much* less than the un-titled road.

Let's see if we can tell why. For the un-tilted road, the normal force just had to support the car's weight

$$N_{CR}=W_{CE}=mg$$

so the frictional force was

$$f_{CR}=(0.36705)\,W_{CE}$$

but now look at our normal force with the banked road

$$\begin{aligned}N_{CR} &= \frac{W_{CE}}{\left(\cos\left(\theta\right)-\mu_s\sin\left(\theta\right)\right)} \\ &= \frac{W_{CE}}{\left(\cos\left(20\,^{\circ}\right)-\left(2.7176\times10^{-3}\right)\sin\left(20\,^{\circ}\right)\right)} \\ &= 1.0652W_{CE}\end{aligned}$$

The normal force has increased with the banking. That is because the car is pushing harder on the road due to the banked turn. The road is helping the car turn with the radial part of the normal force. Then the frictional force can be much less.

$$f_{CR}=\mu_s\frac{W_{CE}}{\left(\cos\left(\theta\right)-\mu_s\sin\left(\theta\right)\right)}$$

Since the minimum μ_s is now tiny, this is the case. Let's put in numbers.

$$\begin{aligned}f_{CR} &= \left(\left(2.7176\times10^{-3}\right)\right)\frac{W_{CE}}{\left(\cos\left(20\,^{\circ}\right)-\left(\left(2.7176\times10^{-3}\right)\right)\sin\left(20\,^{\circ}\right)\right)} \\ &= 2.8949\times10^{-3}W_{CE}\end{aligned}$$

This is a reduction in the required friction of

$$\frac{f_{CR}}{f_{CR}}=\frac{2.8949\times10^{-3}W_{CE}}{(0.36705)\,W_{CE}}=7.8869\times10^{-3}$$

or we now need only 0.78869% of the friction force that we needed without the banking. This allows NASCAR ® cars to travel much faster without slipping. In Rexburg, if the curves were banked might even be able to turn with snow and ice!

Although we really did not do anything radically new in this lecture, we did use physics pieces we knew in new ways. So let's review.

1. We called our rotational coordinate system a cylindrical coordinate system (or rtz coordinate system).

2. We used this coordinate system to solve constant motion problems, first with kinematics, and then using forces and Newton's second law.

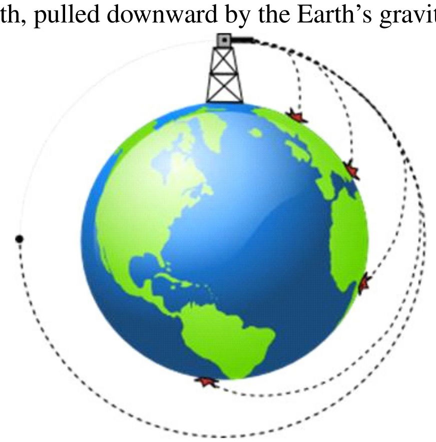
These combinations will allow us to solve more rotational problems, even with aircraft that fly off the surface. Of course, we could also let the car or airplane speed up or slow down. And that is coming. But armed with what we have done in this lecture, we are ready to take on orbital motion, loop-de-loop roller coasters, and a bunch of great problems that require both rotation and forces. And that is our next lecture.

21 Non-uniform Circular Motion Dynamics

We have studied uniform circular motion. We can extend our experience in by allowing objects to rotate or travel in circular paths *and* speed up. There are some historical tricks to this topic, so we will deal with them as well. Let's start with a special uniform circular motion problem to review, a circular orbit.

Circular Orbits

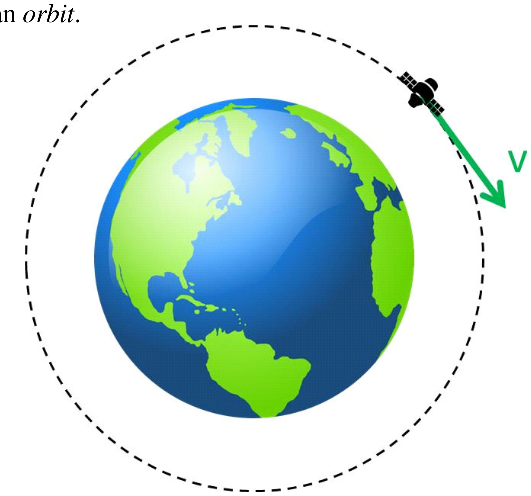
Let's take on the case of a satellite orbiting the Earth in a perfectly circular orbit. Recall that we pretended we could build a very tall tower, several hundred kilometers high. On this tower we placed a cannon. If we shot a ball out of the cannon it would travel a distance and hit the earth, pulled downward by the Earth's gravity.



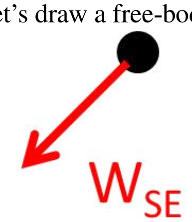
And if the muzzle velocity of the ball was small, the ball would fall to the Earth a short distance from the cannon. We could use the *flat Earth approximation* for this case. The ball does travel a little farther because the Earth is curved, but the extra distance is small compared to the total distance traveled, so we can ignore the fact that the Earth is round for this case.

But if we increase the muzzle velocity of the cannon, our ball will travel farther. Because the Earth is round, the ball will travel farther than if the cannon were on a flat surface, because the round surface of the Earth falls away from the ball's path. With the higher mussel velocity, the ball goes farther and the extra distance because of the roundness of the Earth is no longer negligible.

If we continue increasing the muzzle velocity, the ball will travel farther and farther around the globe. Eventually, if we make the muzzle velocity high enough, the ball will miss the Earth entirely. It will travel at just the right speed such that the centripetal acceleration keeps it going in the same circle. It will never get nearer the Earth. This is what we called an *orbit*.



This is what a satellite is doing. Let's draw a free-body-diagram for our satellite



Notice that there is only one force! the force due to the Earth's gravitation. Also notice that the force is radial, in the $-\hat{r}$ direction. We can give this the title of centripetal force! We can write the acceleration of our satellite using Newton's second law in cylindrical (rtz) coordinates.

$$\begin{aligned} F_{net_r} &= m_s a_r \\ F_{net_\theta} &= 0 \\ F_{net_z} &= 0 \end{aligned}$$

where the subscript s is for satellite. The acceleration is then

$$\overrightarrow{a}_r = \frac{F_{net_r}}{m_s}\left(-\hat{r}\right)$$

and we have identified this radial acceleration as our centripetal acceleration,

$$a_c = \frac{v^2}{r}$$

and the source of the force is gravitation

$$W_{SE} = G\frac{m_s M_E}{r_{sE}^2}$$

so we could write the acceleration as

$$\begin{aligned} \overrightarrow{a}_r &= \frac{F_{net_r}}{m_s}\left(-\hat{r}\right) \\ &= \frac{G\frac{m_s M_E}{r_{sE}^2}}{m_s}\left(-\hat{r}\right) \\ &= G\frac{M_E}{r_{sE}^2}\left(-\hat{r}\right) \end{aligned}$$

and we recognize this as just

$$\overrightarrow{a}_r = g\left(r\right)\left(-\hat{r}\right)$$

where $g\left(r\right)$ is the acceleration due to gravity, but we have included the $\left(r\right)$ to remind us that this is not the constant $g = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$ that we have near the Earth's surface. We can calculate what this would be for a typical satellite. A weather satellite orbits about $850\,\mathrm{km}$ above the Earth's surface. The Earth's average radius is $6370\,\mathrm{km}$. So

$$\begin{aligned} r_{sE} &= 6370\,\mathrm{km} + 850\,\mathrm{km} \\ &= 7220\,\mathrm{km} \end{aligned}$$

And the mass of the Earth is about $5.98 \times 10^{24}\,\mathrm{kg}$. So $g\left(r\right)$ for our satellite is

$$\begin{aligned} \overrightarrow{a}_r &= g\left(r\right) = G\frac{M_E}{r_{sE}^2}\left(-\hat{r}\right) \\ &= \left(6.67 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}\right)\frac{5.98 \times 10^{24}\,\mathrm{kg}}{\left(7220\,\mathrm{km}\frac{1000\,\mathrm{m}}{\mathrm{km}}\right)^2}\left(-\hat{r}\right) \\ &= 7.6516\,\frac{\mathrm{m}}{\mathrm{s}^2}\left(-\hat{r}\right) \end{aligned}$$

which is a more than 20% less than it is at the Earth's surface.

We said that muzzle velocity for our ball, or satellite velocity for our satellite is important for making a orbit happen. If we go to slow, the satellite falls to the ground. So what speed does our satellite need to go to be in orbit?

Let's use the fact that our radial acceleration is centripetal, so

$$a_r = a_c = \frac{v_t^2}{r}$$

then

$$\begin{aligned} v_t &= \sqrt{a_r r} \\ &= \sqrt{g(r)\, r} \end{aligned}$$

and numerically we have

$$\begin{aligned} v &= \sqrt{\left(7.651\,6\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(7220\,\mathrm{km}\frac{1000\,\mathrm{m}}{\mathrm{km}}\right)} \\ &= 7432.7\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

which is pretty fast ($16626\,\mathrm{mi}/\,\mathrm{h}$).

How long it takes a satellite to go around the Earth is called the satellite's *orbital period*. It is (unfortunately) given the symbol T. So we will have to be careful not to confuse it with a tension force symbol. But let's calculate what the orbital period would be for our satellite.

From our basic motion set of equations, we know

$$v = \frac{\Delta x}{\Delta t}$$

we have a arclength that is the total circumference of the orbit circle for our Δx part, so let's write Δx as Δs for arclength.

$$v = \frac{\Delta s}{\Delta t}$$

and the time it takes to go around, Δt we have called T. So

$$v = \frac{\Delta s}{T}$$

then

$$\begin{aligned} T &= \frac{\Delta s}{v} \\ &= \frac{2\pi r_{sE}}{v} \\ &= \frac{2\pi r_{sE}}{\sqrt{g(r_{sE})\, r_{sE}}} \\ &= 2\pi\sqrt{\frac{r_{sE}}{g(r_{sE})}} \end{aligned}$$

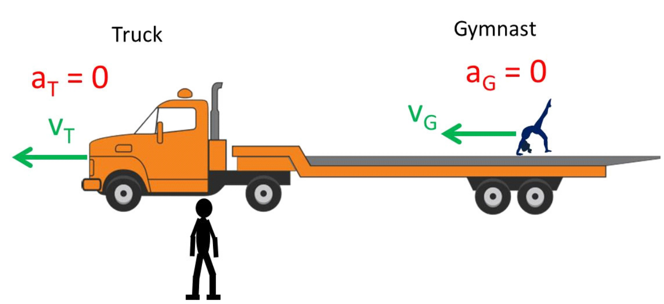
so for our satellite,

$$\begin{aligned} T &= 2\pi\sqrt{\frac{\left(7220\,\mathrm{km}\frac{1000\,\mathrm{m}}{\mathrm{km}}\right)}{7.6516\frac{\mathrm{m}}{\mathrm{s}^2}}} \\ &= 6103.4\,\mathrm{s} \\ &= 101.72\,\mathrm{min} \\ &= 1.6953\,\mathrm{h} \end{aligned}$$

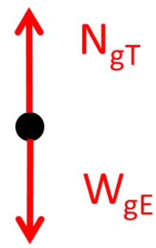
so our satellite would circle the Earth in a little more than an hour and a half!

Fictitious Forces

As we travel in circular paths, at a minimum we have a centripetal acceleration. This is a problem for us because we agreed to only use inertial reference frames for our calculations, and really we have done this so far. We do our measurements from a reference frame that is not accelerating, and study the acceleration of an object from this viewpoint. Suppose we consider, for an example, our gymnast on a flatbed truck in the Rexburg Parade.

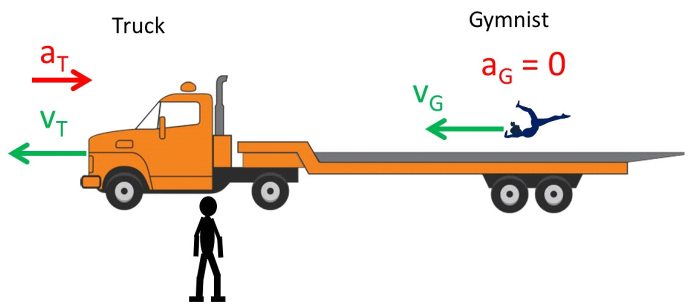


We know that as long as the truck travels with a constant speed, the truck bed is an inertial reference frame and all our physics works just fine. The gymnast is safe. The forces acting on the gymnast are all in the y-direction. She or he would be able to do his or her routine just as though they were in their "stationary" gym.

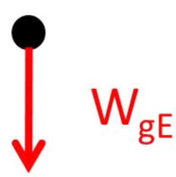


There could be static friction if the gymnast's feet are on the truck, but since there is no acceleration, the gymnast's feet won't push on the roughness teeth of the truck, so the truck roughness teeth won't push back. There would be no friction if the truck has a constant speed! We don't need a force to make the gymnast move with constant speed. Of course if there was a strong wind, then we would have some drag force. But let's assume this is negligible.

Now if the truck driver slammed on the breaks, the truck would quickly change it's motion. The gymnast would try to keep going at a constant speed (Newton's first law).



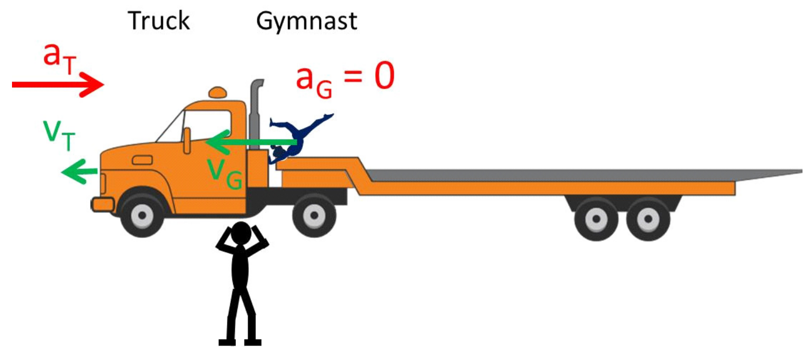
But notice that we took all our measurements of acceleration and velocity from the point of view of a person watching this happen from the street. The street is an inertial reference frame. Let's say that the gymnast has just jumped into the air as the truck puts on it's breaks. From the street viewpoint, the forces acting on the gymnast are still all in the y-direction.



and the gymnast just keeps going. She has no acceleration in the x-direction. So there is no force on her in the x-direction

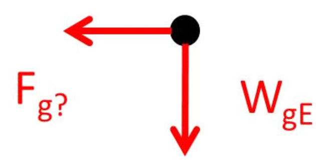
$$F_{net_x} = ma_x = m(0) = 0$$

But we can see that this is really not going to be good on the gymnast.



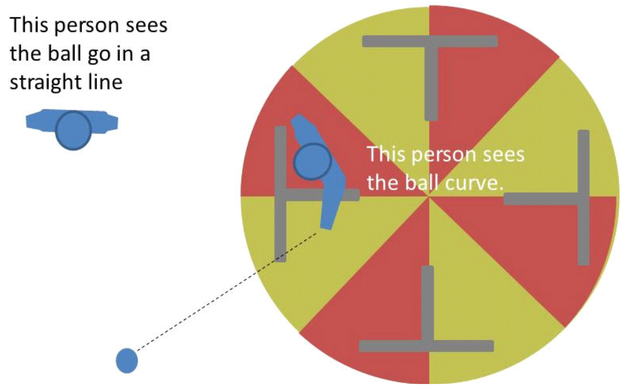
The truck has slowed, the gymnast has not, so the gymnast is going to run into the cab of the truck.

Let's consider what the gymnast would see (say, we attach a video camera on the gymnast and watch the video from her point of view). The gymnast sees herself jump, and then she sees herself thrown forward into the truck. She would see an x-direction force acting on her (in the $-\hat{\imath}$ direction).



But what would cause this force? Forces need two objects, the object that is being acted on (mover) and the environmental object creating the force. There is no environmental object creating a horizontal force on the gymnast. And really we realize that the problem is that the gymnast is using the accelerating truck as her reference system. It is an accelerated reference frame, and we won't allow that in PH121! But now we can see what an accelerated reference frame would do to our physics. It makes it look like there are forces there that are not real. We call these *fictitious forces*. They appear to be there only because we are using an accelerating point of view.

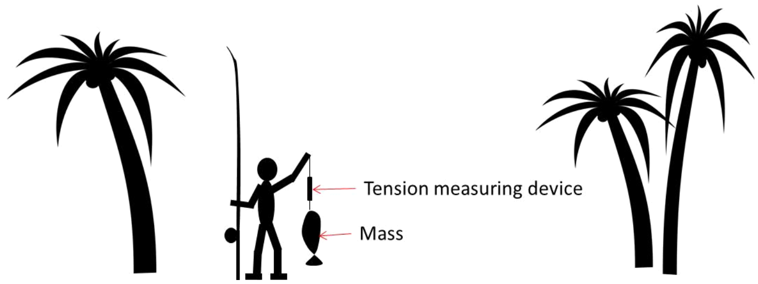
Let's look at another fictitious force case. Perhaps you played on a rotating platform as a kid. We call them merry-go-rounds. If you have a ball on a merry-go-round, and you throw the ball, to you it looks like the ball curves as it flies away from you. But for someone who is standing on the playground, not on the merry-go-round, the ball appears to go in a straight line. Which is correct? This is important, because if the ball curves, then the direction has changed. That would be acceleration. It takes a force to cause acceleration. So for you there appears to be a force, and for your partner on the playground there does not appear to be a force. Serious stuff!



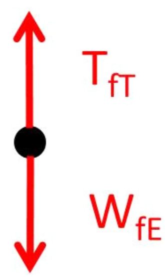
The apparent force on the ball is another fictitious force. The ball only appears to change direction because you are on a rotating platform. Really you have moved away from the ball, not the ball curved away from you. So the apparent force causing the ball to turn is not real. This particular fictitious force even has a name, it is called a *centrifugal force*. This is different than centripetal force which is a title for the force that makes things turn. To keep from confusing these two forces, remember that centri(f)ugal has an “f” for “fictitious” in it.

This centrifugal force comes up a lot because we all really live on a rotating platform, the Earth. It is common to see things appear to curve on a global scale that really don’t curve. Let’s take an example to see how important it might be to take into account the rotation of the Earth (and therefore acknowledge that we are on a rotating platform).

Suppose we are on the equator and have a mass on a rope with a tension measuring device to tell what the tension will be.



A free body diagram for the mass (a fish) might look like this



that the net force

$$F_{net_y} = m_f a$$

and the y-net force is equal to the sum of all the y-forces

$$F_{net_y} = T_{fT} - W_{fE}$$

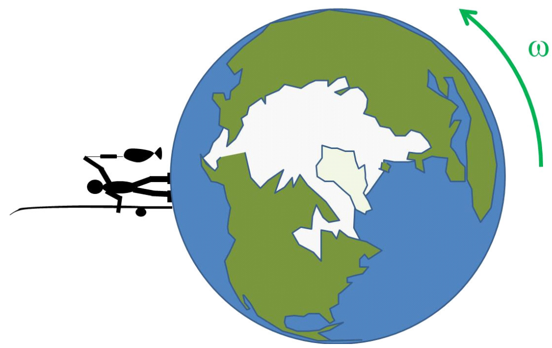
so in our flat-non-rotating Earth approximation we would say that the mass is not accelerating so

$$F_{net_y} = m_f a = 0$$
$$0 = T_{fT} - W_{fE}$$

so

$$T_{fT} = W_{fE}$$

But if we realize that the Earth is really turning,



we can see that the mass must have a radial acceleration,

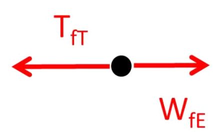
$$\overrightarrow{a}_r = a\left(-\hat{r}\right)$$

in the $-\hat{r}$ direction, and this radial acceleration could be called a centripetal acceleration

$$a_r = a_c = \frac{v^2}{r} = \omega^2 r_E$$

where r_E is the radius of the Earth. So

$$F_{net_r} = -m_f a = -m_f \left(\omega^2 r_E\right)$$



and then

$$-m_f \left(\omega^2 r_E\right) = T_{fT} - W_{fE}$$

and we can see that the tension in the rope is

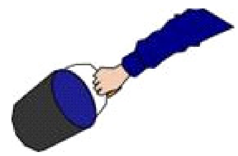
$$T_{fT} = -m_f \left(\omega^2 r_E\right) + W_{fE}$$

This is smaller than it would be if the Earth were flat and not rotating. And what is $m_f \left(\omega^2 r_E\right)$? It is a correction to our physics because we violated the no-accelerating (rotating) reference frame rule! But it *looks like* a force. This is the fictitious force called the centrifugal force. It's not real, it is a correction to our physics because we did not use a proper reference frame. But the meter in the rope really will read a little bit less because the Earth is rotating.

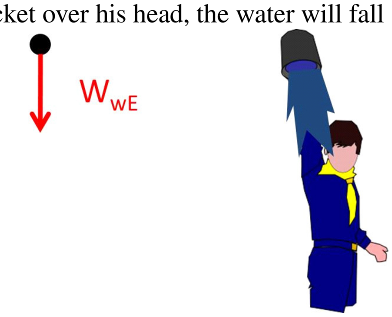
This gives us a way to test our flat earth assumption. If $m_f \left(\omega^2 r_E\right)$ is not negligible, then we need to do more work on our problem! Of course we did this at the equator. The value changes with latitude. But we get the idea. The Earth can be used as an inertial reference frame only if the correction factor is negligible. In later physics courses, we will take up this problem again!

Non-uniform Circular Dynamics

We are finally ready to deal with non-uniform circular motion. But even now, let's take a simple case. Suppose a kid has a bucket of water.



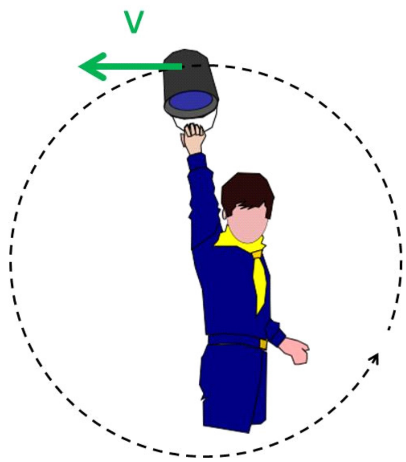
If the child holds the bucket over his head, the water will fall



This is because there is a net force on the water, so the water accelerates downward

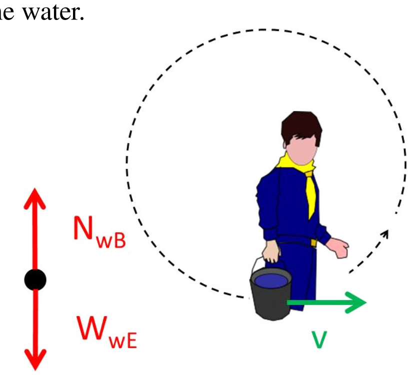
$$
\begin{aligned}
F_{net_y} &= ma_y \\
&= -W_{wE} \\
&= -mg
\end{aligned}
$$

But what if the child makes the bucket go in a circle?



Now the water has a velocity. Right at the top of the circle, the water wants to go to the left. That is because it has a velocity, so it requires a force to make the water change it's speed or direction. In this case, we want to change direction to keep the water going in a circle. So there will be a net force on the water.

We can see that this would be true if we consider what would happen if the bucket disintegrated. The water would really fly off to the left. So something about the bucket must be exerting a force on the water.



Let's look at the bucket and water on the bottom of the circle. There are two forces acting on the water. The Earth's gravitation, and a normal force due to the bottom of the bucket.

$$F_{net_y} = ma_y$$
$$F_{net_y} = N_{wB} - W_{wE}$$

and since the water is going in a circle, we know that there is an acceleration, $a_y = a_c$. This must be the centripetal acceleration that is keeping the water going in a circle.

$$ma_y = ma_c = N_{wB} - W_{wE}$$

We can see that this acceleration must be

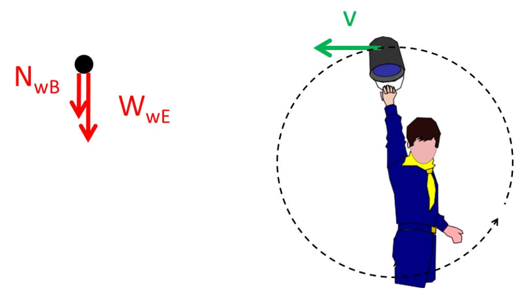
$$a_c = \frac{N_{wB} - W_{wE}}{m}$$

That is, there is a normal force from the bottom of the bucket that has a large enough magnitude to overcome the pull of gravity and also to keep the water moving in a circle.

But what about the top of the circle? The water stays in the bucket, so there must still be a centripetal acceleration.

$$F_{net_y} = ma_y = ma_c$$

but now our free-body-diagram looks like



so

$$F_{net_y} = -N_{wB} - W_{wE}$$

then, putting in the net force from Newton's second law,

$$F_{net_y} = -ma_c = -N_{wB} - W_{wE}$$
$$ma_c = N_{wB} + W_{wE}$$

and the acceleration is

$$a_c = \frac{N_{wB} + W_{wE}}{m}$$

This gave us the magnitude, but what about the direction. We will have a minus sign at the top, the acceleration is now downward, keeping the water in the circle. But notice that something has changed. If a_c is the same (the child is turning the bucket with constant circular motion). Then the normal force of the bucket must change. Let's look at the bottom of the circle again. We can solve for the normal force

$$N_{wB_{bottom}} = ma_c + W_{wE}$$

and now for the top

$$N_{wB_{top}} = ma_c - W_{wE}$$

or

The top normal force magnitude is *less* than the bottom normal force. Our forces change as the bucket goes around the circle! This is new and different!

You might have experienced this yourself if you have gone on a loo-de-loop roller coaster. At the bottom of the loop, you seem to feel "heavy," as though you had suddenly gained weight. But at the top you feel "light," as though you could fly out of the car. Indeed, at the bottom of the loop you have an extra large normal force pushing up against your nether regions because the atoms of the seat must support both your weight and the extra force that is keeping you turning in the loop. At the top, the normal force will be much less because your weight (gravitational force) is helping to provide the centripetal acceleration keeping you going along the loop. The seat does not have to push as hard to keep you in the circular motion.

We know the centripetal acceleration is

$$a_c = \frac{v^2}{r}$$

so for our coaster ride, at the bottom we have

$$N_{wB_{bottom}} = W_{wE} + m\frac{v^2}{r}$$

and at the top we have

$$N_{wB_{top}} = W_{wE} - m\frac{v^2}{r}$$

We can see from this last equation that if v is not large enough, then $W_{wE} - m\frac{v^2}{r}$ will be negative. This could only be if there is a safety system that holds the car on the track. What we are saying is that for low speeds, v, the force due to gravity W_{wE} is enough to pull the car off the track unless is is held to the track by a safety system (that provides another normal force!). But for faster speeds, the car will hold itself on the track.

Rotational Dynamics

Long ago now we learned kinematics. Then we learned Newton's laws and used Newton's laws to find the acceleration so we could use it in our kinematics equations to find how objects moved. We are going to do the same now for rotational motion. Here is our Kinematics equation set for rotation

$$\omega_f = \omega_i + \alpha \Delta t$$

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha \Delta t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\phi$$
$$\omega_{ave} = \frac{\omega_f + \omega_i}{2}$$

$$a_t = r\alpha$$

$$v_t = \omega r$$

and we can use our cylindrical (rtz) coordinate system, and what we know about centripetal acceleration to do problems that go beyond constant circular motion.

It's convenient to use cylindrical (rtz) coordinates because we know something about the radial and tangential components of the acceleration

a_r	$= a_c$ called "centripetal"	causes turning
a_t	called "tangential"	causes speeding up or slowing down

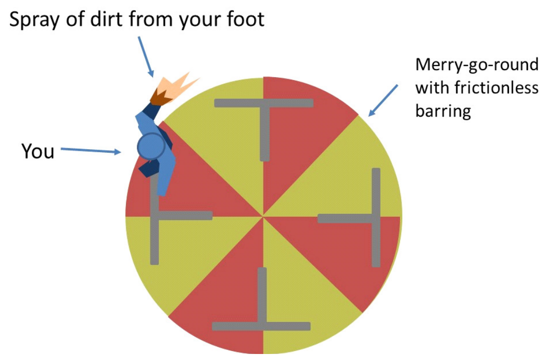
So if we have a non-zero a_r we know we will be turning and if we have a non-zero a_t we know we will change speed!

We can and have used cylindrical coordinates to write Newton's second law

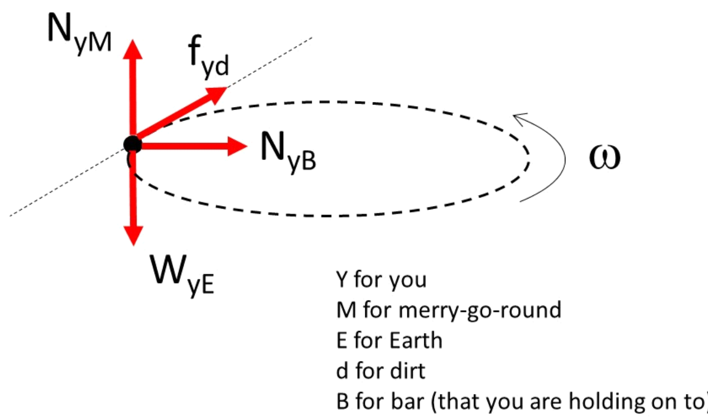
$$\begin{aligned} F_{net_r} &= ma_r = m\frac{v^2}{r} \\ F_{net_\theta} &= ma_t \\ F_{net_z} &= 0 \end{aligned}$$

The last equation may not always be true, we could accelerate upward as well. But we will save this possibility until later.

Let's try a problem. You are on the merry-go-round again. The merry-go-round has a frictionless bearing. Some big kid has spun up the merry-go-round and it is going too fast, $\omega = 18.85\,\mathrm{rad/s}$. You want to slow it down. You hold on with all your might and drag one foot to provide some friction to slow down the merry go round. If your shoe has a coefficient of kinetic friction of $\mu = 0.20$ when it is drug across dirt, how many times around do you have to go before you stop? The merry-go-round has a diameter of $2.00\,\mathrm{m}$.



The situation is shown in the next diagram.



Our free-body diagram has to be truly three dimensional.

For our basic equations we will need our Newton's second law set, and our kinetic friction equation

$$a_r = \frac{F_{net_r}}{m}$$
$$a_\theta = \frac{F_{net_\theta}}{m}$$
$$a_z = \frac{F_{net_z}}{m}$$
$$f_k = \mu_k N$$

and we will need our rotational kinematics set

$$\omega_f = \omega_i + \alpha \Delta t$$

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha \Delta t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\phi$$
$$\omega_{ave} = \frac{\omega_f + \omega_i}{2}$$
$$\Delta\phi = \phi_f - \phi_i$$
$$\Delta t = t_f - t_i$$
$$\omega_{ave} = \frac{\Delta\phi}{\Delta t}$$
$$\omega = \frac{d\phi}{dt}$$
$$\omega = \frac{v_t}{r}$$
$$\alpha = \frac{d\omega}{dt}$$
$$\alpha = \frac{a_t}{r}$$

We know that you are not accelerating in the z-direction, so

$$a_z = 0$$

and we know that the radial component of your motion is a centripetal acceleration

$$a_c = a_r = \frac{v_t^2}{r}$$

We know

$$\begin{aligned} \mu_k &= 0.20 \\ g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \\ \omega_i &= 18.85\,\mathrm{rad/\,s} \end{aligned}$$

We can write out Newton's second law in the cylindrical coordinate (rtz) system.

$$\begin{aligned} \Sigma F_{net_z} &= m_y a_z = N_{yM} - W_{yE} \\ \Sigma F_{net_r} &= -m_y a_r = N_{yB} \\ \Sigma F_{net_t} &= m_y a_t = f_{yd} \end{aligned}$$

Using what we know of acceleration we can write these as

$$\begin{aligned} 0 &= N_{yM} - W_{yE} \\ -m_y\frac{v_t^2}{r} &= N_{yB} \\ m_y a_t &= -f_{yd} \end{aligned}$$

and we see that

$$N_{yM} = W_{yE}$$

and we can use our friction formula to see that

$$\begin{aligned} f_{yd} &= \mu_k N_{yM} \\ &= \mu_k W_{yE} \end{aligned}$$

then

$$m_y a_t = -\mu_k W_{yE}$$

or

$$a_t = -\frac{\mu_k W_{yE}}{m_y} = -\frac{\mu_k m_y g}{m_y} = -\mu_k g$$

Now we know that

$$\alpha = \frac{a_t}{r}$$

so

$$\alpha = -\frac{\mu_k g}{r}$$

then, since we know ω_i and α and want $\Delta\phi$,

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\phi$$

seems the most likely of the kinematic set.

$$\frac{-\omega_i^2}{2\left(-\frac{\mu_k g}{r}\right)} = \Delta\phi$$

$$\frac{\omega_i^2 r}{2\mu_k g} = \Delta\phi$$

but this is not how many times you go around. We need to convert to revolutons, there is one revolution for every $2\pi\,\mathrm{rad}$, so

$$N_{rev} = \frac{\omega_i^2 r}{2\mu_k g}\frac{1rev}{2\pi\,\mathrm{rad}}$$

$$\begin{aligned} N_{rev} &= \frac{\left(18.850\frac{\mathrm{rad}}{\mathrm{s}}\right)^2 (1\,\mathrm{m})}{2\,(0.20)\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)}\frac{1}{2\pi\,\mathrm{rad}} \\ &= 14.426\frac{rev}{\mathrm{s}} \end{aligned}$$

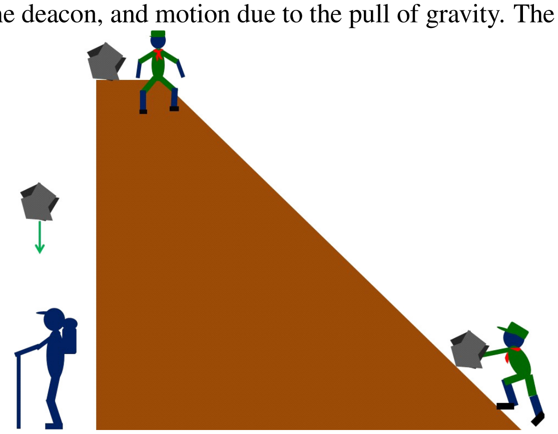
You will just have to hold on for a while.

We will return to rotation later in our lectures. In our next lecture we will take on a new way of looking at motion. One that is really more fundamental than the forces and accelerations we have been studying. But we had to work a while so we could understand it. We will study *energy* next.

22 Work

In our last lecture, we said we would study energy. We have an intuitive feel for what energy is. It is what makes something go. But what is it? You might be surprised to know that we don't have a very good answer to that question. Energy is involved in making motion happen. And we know how energy behaves. So we can define energy by describing how it acts. This is a little be like describing an airline pilot by describing what an airline pilot does (wears a uniform, flies planes[11]). And that is a perfectly good way to define something. It will take several classes to complete this definition of energy (many, many classes for the physicists among us). But we will start with how energy causes motion to happen. We will start with a form of energy that is directly related to forces moving objects.

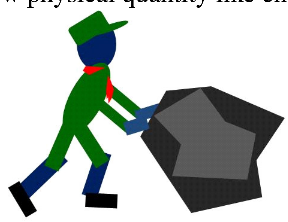
For the moving object let's take a rock, and for the environmental object let's take a deacon. The deacon is on a hike and sees the rock. You know what will happen, the rock will be moved to the edge of a cliff and pushed off the edge. We will get motion from the push due to the deacon, and motion due to the pull of gravity. The rock will travel.

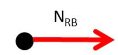


In the next figure there is a force diagram showing just the horizontal forces of such a boy pushing on a (frictionless) rock. Only the horizontal force is shown, and for some

[11] some pilots give talks in Conference

strange reason this boy is pushing a rock on a frictionless surface (which makes our math easier as we learn about a new physical quantity like energy).





Work-energy equation

Suppose we push our rock in the s-direction. The s-direction could be the x-direction, or the y-direction, or the r-direction or the θ-direction, or the z-direction or the direction along the axis of any coordinate system. We call s a generalized coordinate. So F_s would be the component of the force in the s-direction, whichever direction that actually is. We know that the force is

$$F_s = ma_s = m\frac{dv_s}{dt}$$

and we can use the chain rule to get an energy equation. This is how it works:

$$m\frac{dv_s}{dt} = m\frac{dv_s}{ds}\frac{ds}{dt} = m\frac{dv_s}{ds}v_s$$

so we can write

$$F_s = mv_s\frac{dv_s}{ds}$$

and, let's multiply both sides by ds

$$F_s ds = mv_s\frac{dv_s}{ds}ds$$

It's not obvious that this did anything useful for us, but it did! Notice that the ds terms cancel so we have

$$F_s ds = mv_s dv_s$$

and we can now integrate both sides

$$\int_{s_i}^{s_f} F_s ds = \int_{v_i}^{v_f} mv_s dv_s$$

Let's start with the right hand side. We get

$$\int_{s_i}^{s_f} F_s ds = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

But what does it mean?

The left hand side is the result of the push on the rock. It is a form of energy. We will give it a name, one that the deacon's would like. It is called *work*.

$$w = \int_{s_i}^{s_f} F_s ds$$

It might surprise you to learn that in physics "work" is a type of energy. But it's not too strange if we think about it. It takes energy to do work, so it makes some sense to think of the energy from the deacon's lunch being converted into the energy of the deacon's work in moving the rock.

The right hand side must also be a form of energy. Since it has the speed of the rock in it, it must be the energy tied up in the movement of the rock. This matches our intuition. Think of the Sunbeam class when Sunday is the day after Halloween. The kids are full of energy, and they run around and bounce off the walls. Because this amount of energy is related to moving, we will use our greek word for motion to describe this energy. We call it *kinetic energy* and we give it the symbol K.

$$K = \frac{1}{2}mv^2$$

In fact, there is a beginning energy and an ending energy in the right hand side of our equation. So the right hand side is a change in kinetic energy

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta K$$

Notice that this kinetic kind of energy can't be negative! At least as far as we know, mass can't be negative, and the speed in the equation is squared. So kinetic energy can only have positive values. Another thing to notice is that kinetic energy has no direction, it is a scalar. This is a wonderful thing! If we can solve problems using the idea of energy, we don't have to use vectors.

Work-energy theorem

Suppose just for a second that the force F_s is constant, Then our work equation would look like this

$$w = F_s \int_{s_i}^{s_f} ds$$

and we know how to take this integral

$$\begin{aligned} w &= F_s \left(s|_{s_i}^{s_f} \right. \\ &= F_s \left(s_f - s_i \right) \\ &= F_s \Delta s \end{aligned}$$

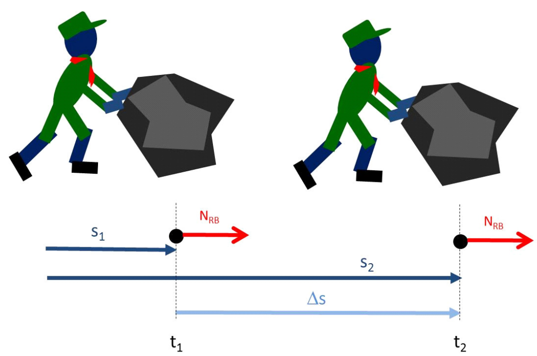
The work equation tells us that we have exerted a force and that the object has gone a displacement Δs. Work seems to be a combination of how hard we push and how much movement we accomplish.

Now let's go back to our integral form and take just the integrand and look at it carefully. We can see that it is a force times a small displacement.

$$F_s ds$$

This makes sense. The small displacement is how far the boy pushed the rock in a time dt. And we just learned that if the time is larger, Δt, and the force was constant, then we could write this as

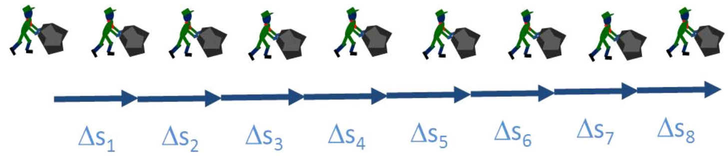
$$w = F_s \Delta s$$



The work from t_1 to t_2 would be the force F_s multiplied by how far the boy pushed the rock $\Delta s = s_2 - s_1$. We can see with our calculus experience that

$$dw = F_s ds$$

is a small amount of work, where we can assume that the distance ds is so small that the boy does not change how hard he pushes during the time dt. Then, the total work in moving the rock is the sum of lots of little ds pushes.



or

$$w = \sum F_{si} \Delta x_i$$

and of course we will let our Δs_i be infinitesimal ds's, so the sum is really an integral.

$$w = \int_{s_i}^{s_f} F_s ds$$

but the integral means, divide up the distance traveled into small segments of size ds. For each segment, the component of the force in the direction we are going can be assumed to be constant. Multiply the distance ds by the force, F_s at that ds point. This is the change in kinetic energy as the rock moves the distance ds. To get all of the change in kinetic energy, we add up the contribution for each ds segment of the path,

$$w = \int_{s_i}^{s_f} F_s ds$$

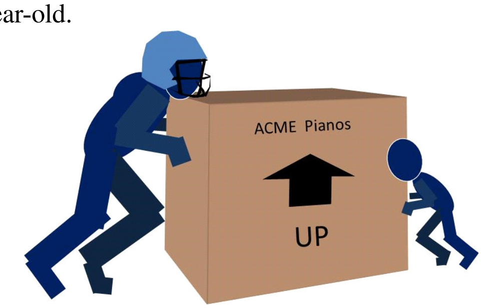
and this is the change in kinetic energy while the boy is pushing.

$$w = \Delta K$$

This equation is very important. But the idea is very simple. If you push on something hard enough that it moves, you have changed it's kinetic energy. The equation is so important that it has a name. It is called the *work energy theorem.*

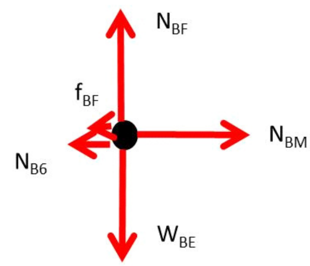
Negative work

We said that kinetic energy can't be negative, but how about work? Suppose we have two people exerting a force on a box. One is a professional American football player, The other is a six-year-old.



They push in opposite directions. How much work is being done?

To answer this, let's consider a free-body diagram.



where the subscript B is for the box and the subscript 6 is for the six-year-old and M is for the football playing (M)an. We can see that there will be a net force.

$$F_{net_s} = N_{BM} - N_{B6} - f_{BF}$$

so the total work done on the box would be

$$\begin{aligned} w &= \int_{s_i}^{s_f} F_{net_s} ds \\ &= \int_{s_i}^{s_f} \left(N_{BM} - N_{B6} - f_{BF}\right) ds \\ &= \int_{s_i}^{s_f} \left(N_{BM}\right) ds + \int_{s_i}^{s_f} \left(-N_{B6}\right) ds + \int_{s_i}^{s_f} \left(-f_{BF}\right) ds \end{aligned}$$

But which way will the box go? I suspect that the football player will push the box *and* the six-year-old to the right. So Δs will be positive. Then we can identify the first term in total work equation as the the work done on the box by the man

$$\begin{aligned} w_{BM} &= \int_{s_i}^{s_f} N_{BM} ds \\ &= N_{BM} \Delta s \end{aligned}$$

The next term is the work done by the child

$$\begin{aligned} w_{B6} &= \int_{s_i}^{s_f} -N_{B6} ds \\ &= -N_{BM} \Delta s \end{aligned}$$

The child's work is negative! What can that mean? Well, for starters, the child's force can't be the force that is making the box move. In fact, the child's force is another obstacle that the man's force must overcome to make the box move. This means that the man must do enough work (push hard enough) to make the box go, *and* to overcome the backward push of the child. The work that actually makes the box move would be

$$\begin{aligned} w_{net} &= w_{BM} + w_{B6} \\ &= N_{BM} \Delta s - N_{BM} \Delta s \\ &= \left(N_{BM} - N_{BM}\right) \Delta s \end{aligned}$$

But wait! we did not include friction. Does the frictional force do work? Well, yes, like our child is doing work. The frictional force is not moving the box forward, but it is making it harder for the man to move the box. The last term in our integral is the work due to friction

$$\begin{aligned} w_{bf} &= \int_{s_i}^{s_f} -f_{Bf} ds \\ &= -f_{Bf}\Delta s \end{aligned}$$

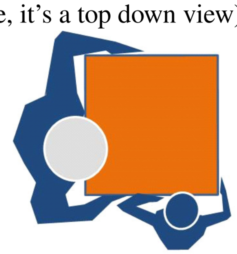
So we really should include an amount of work w_{Bf} in our net work equation

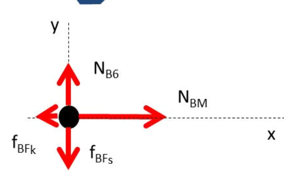
$$\begin{aligned} w_{net} &= w_{BM} + w_{B6} + w_{Bf} \\ &= N_{BM}\Delta s - N_{BM}\Delta s - f_{Bf}\Delta s \\ &= (N_{BM} - N_{BM} - f_{Bf})\,\Delta s \\ &= F_{net_s}\Delta s \end{aligned}$$

And from this we see that it is the net force that makes our box move! That is not a surprise. The net force is what makes the box's acceleration, so the net force *should be* what gives the box some kinetic energy.

Work and forces perpendicular to the direction of travel

Let's consider another case. Suppose our man and child are still pushing on the box, but the child is tired of being pushed backwards. So now the six-year-old pushes on the side of the box (see next figure, it's a top down view).





Now the child pushes at a right angle to the man. But suppose the box still goes to the right, the same direction as it did before. What can we say about the work done by the man and the child? Let's take a coordinate system where up is the z-direction and to the right is the x-direction. Then the y-direction would be to the man's left or in our top-down figure it would be toward the top of the page. Then Δs is in the x-direction

$$\Delta s = \Delta x$$

and $\Delta y = 0$ because the box is not going in the y-direction. So the man's work is

$$\begin{aligned} w_{BM} &= \int_{s_i}^{s_f} N_{BM} dx \\ &= N_{BM}\Delta x \end{aligned}$$

just as before but the work done by the child would be

$$\begin{aligned} w_{B6} &= \int_{s_i}^{s_f} N_{B6} dy \\ &= 0 \end{aligned}$$

The child is pushing, but the child has not overcome the static friction force between the box and the floor. The roughness teeth are bent, but the bonds have not broken. So the box has no y-displacement. The man has overcome the floor friction, so in the x-direction there is an amount of kinetic friction.

$$\begin{aligned} w_{bf_s} &= \int_{s_i}^{s_f} -f_{Bfs} dy \\ &= 0 \\ w_{bf} &= \int_{s_i}^{s_f} -f_{Bf} ds \\ &= -f_{Bf_k}\Delta x \end{aligned}$$

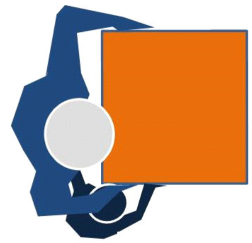
Notice that the man doesn't have to push as hard now, because he is no longer having to overcome the child's push.

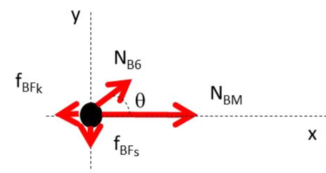
$$\begin{aligned} w_{net} &= w_{BM} + w_{Bf_k} + w_{B6} + w_{Bfs} \\ &= N_{BM}\Delta x - f_{Bf}\Delta x + 0 + 0 \\ &= (N_{BM} - f_{Bf})\,\Delta x \\ &= F_{net_x}\Delta x \end{aligned}$$

The man only has to overcome friction to keep the box moving (and avoid tripping over the child).

Work and forces at an angle

But suppose our child get's smarter. Now the six-year-old pushes mostly in the same direction as the man. The child pushes with a force that is 30° from the x-axis.





Then we can use Newton's second law to find the net force

$$\begin{aligned} F_{net_x} &= N_{BM} + N_{B6}\cos\theta - f_{BF_k} \\ F_{net_y} &= N_{B6}\sin\theta - f_{BF_s} \end{aligned}$$

and $F_{net_y} = 0$ still because the box is still going in the x-direction. because the box is not accelerating in the y-direciton. Notice that from the y-equation

$$N_{B6}\sin\theta = f_{BF_s}$$

so the static friction force in the y-direction is canceled with the y-component of the child's push. The y-components will do no work. But we can see that now part of the child's push is helping the man. The net work would be in the x-direction

$$\begin{aligned} w_{net} &= w_{BM} + w_{Bf_k} + w_{B6} + w_{Bfs} \\ &= N_{BM}\Delta x - f_{Bf}\Delta x + N_{B6}\cos\theta\Delta x + 0 \\ &= F_{net_x}\Delta x \end{aligned}$$

as expected.

And we have learned something important. If a force pushes at an angle to the direction the object is traveling, then only part of the force can be causing the motion. In our original work equation

$$w = \int_{s_i}^{s_f} F_s ds$$

we had only the s-component of the force. But now we recognize that

$$F_s = F\cos\theta_{Fs}$$

where θ_{Fs} is *the angle between* the $\overrightarrow{F}$ and $\overrightarrow{\Delta s}$ directions. This is a new concept, the angle between two vectors. Up till now we have always used the angle measured from the x-axis (or we have used our trigonometry experience to do equivalent math). But now we can't do this. We *have to* find the angle between the direction of the force is pushing and the direction the object is actually going so we can take a component of the force in the direction that the box is going.

Let's put the cosine in the basic work equation

$$w = \int_{s_i}^{s_f} F \cos\theta_{Fs} ds$$

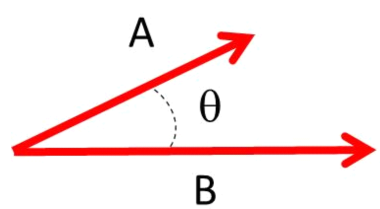
What we are really doing is finding the component of the force in the direction that the box is going and multiplying by ds!

Once again, new math

We have a quantity

$$F \cos\theta_{Fs} ds$$

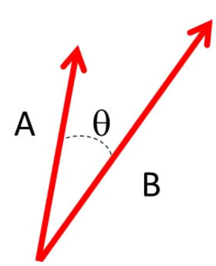
that represents something fundamentally new. We are taking a component of a vector in a direction that is not one of our axes directions. So if we have a generic vector $\overrightarrow{A}$ and we want the component in the direction of another vector $\overrightarrow{B}$ as shown in the next figure,



Then the A_B component would be.

$$A_B = A \cos\theta_{AB}$$

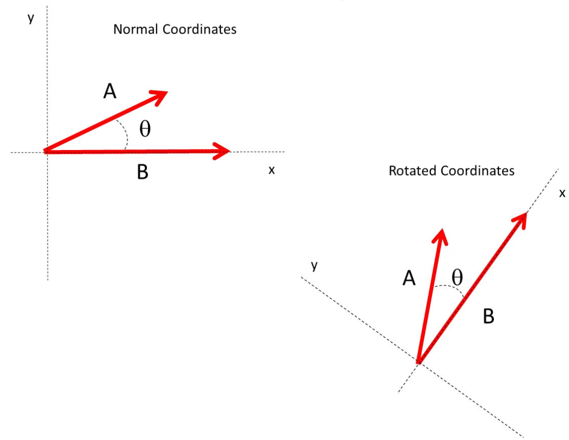
Since $\overrightarrow{B}$ is in the x-direction, this is no surprise. But suppose the vectors are as shown in the next figure:



Really the situation has not changed. The lengths are the same, the angle θ_{AB} is the same. So the result is still

$$A_B = A\cos\theta_{AB}$$

We could see that this must be true by converting to a rotated coordinate system.



These two situations look the same once you see the coordinates that support them.

Now let's consider

$$A\cos\theta B$$

This is

$$A\cos(\theta)\,B = A_B B$$

or the component of $\overrightarrow{A}$ in the $\overrightarrow{B}$ direction multiplied by B. This strange quantity has a name and a symbol. The symbol is

$$\overrightarrow{A}\cdot\overrightarrow{B} = A\cos(\theta)\,B$$

and it is called the *dot product* of $\overrightarrow{A}$ and $\overrightarrow{B}$. It turns out that (a phrase that here means that I am going to let your math professor prove all this) we can find the value of $\overrightarrow{A}\cdot\overrightarrow{B}$ if we know the components of the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$

$$\overrightarrow{A}\cdot\overrightarrow{B} = A_xB_x + A_yB_y$$

We can see that this might be true if we write $\overrightarrow{A}$ and $\overrightarrow{B}$ in terms of their components

$$\begin{aligned}\overrightarrow{A} &= A_x\hat{\imath} + A_y\hat{\jmath}\\ \overrightarrow{B} &= B_x\hat{\imath} + B_y\hat{\jmath}\end{aligned}$$

then

$$\begin{aligned}\overrightarrow{A}\cdot\overrightarrow{B} &= (A_x\hat{\imath} + A_y\hat{\jmath})\cdot(B_x\hat{\imath} + B_y\hat{\jmath})\\ &= A_x\hat{\imath}\cdot B_x\hat{\imath} + A_x\hat{\imath}\cdot B_y\hat{\jmath} + A_y\hat{\jmath}\cdot B_x\hat{\imath} + A_y\hat{\jmath}\cdot B_y\hat{\jmath}\end{aligned}$$

now consider

$$\begin{aligned}\hat{\imath}\cdot\hat{\imath} &= (1)(1)\cos(0\,^\circ)\\ &= 1\end{aligned}$$

since $\hat{\imath}$ and $\hat{\imath}$ are in the same direction, and

$$\begin{aligned}\hat{\imath}\cdot\hat{\jmath} &= (1)(1)\cos(90\,^\circ)\\ &= 0\end{aligned}$$

So, using the zeros first

$$\begin{aligned}\overrightarrow{A}\cdot\overrightarrow{B} &= A_x\hat{\imath}\cdot B_x\hat{\imath}+0+A_y\hat{\jmath}\cdot B_x\hat{\imath}+0\\ &= A_xB_x+A_yB_x\end{aligned}$$

All of this leads to

$$F\cos\theta_{Fs}ds=\overrightarrow{F}\cdot d\overrightarrow{s}$$

so we can write our work equation as

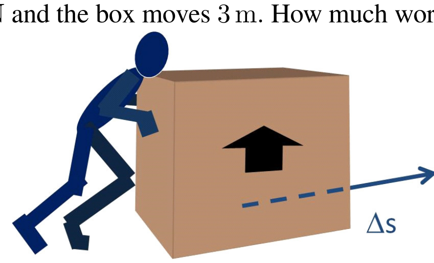
$$w=\int_{s_i}^{s_f}\overrightarrow{F}\cdot d\overrightarrow{s}$$

This just means that we take the component of the force in the s-direction and multiply by ds and integrate the result. But it is a handy, compact notation.

Work from a Constant Force

Let's try a problem.

Suppose we let our guy push his box without the "help" of a child. He pushes with a constant force of $20\,\mathrm{N}$ and the box moves $3\,\mathrm{m}$. How much work did the guy do?

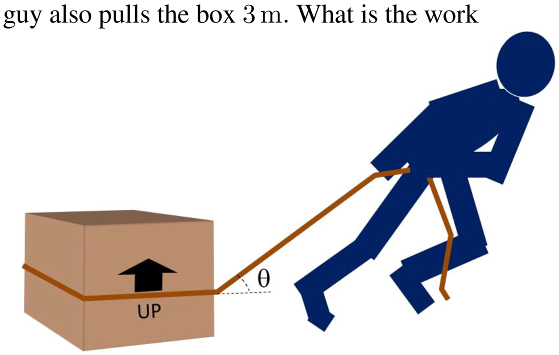


The push and the displacement are in the same direction so

$$\begin{aligned} w_{BG} &= \int_{s_i}^{s_f} \overrightarrow{F} \cdot d\overrightarrow{s} \\ &= \int_{s_i}^{s_f} N_{BM} ds \cos\theta \\ &= N_{BM}\cos\theta \int_{s_i}^{s_f} ds \\ &= N_{BM}\Delta s \cos\theta \end{aligned}$$

$$\begin{aligned} w_{BG} &= (20\,\mathrm{N})\,(3\,\mathrm{m})\cos(0\,^{\circ}) \\ &= 60.0\,\mathrm{J} \end{aligned}$$

Let's try another. Suppose another guy pulls a box with the same $20\,\mathrm{N}$ but now at an angle of $20\,^{\circ}$. This guy also pulls the box $3\,\mathrm{m}$. What is the work



$$\begin{aligned} w_{BG} &= \int_{s_i}^{s_f} \overrightarrow{F} \cdot d\overrightarrow{s} \\ &= \int_{s_i}^{s_f} N_{BM} ds \cos\theta \\ &= N_{BM}\cos\theta \int_{s_i}^{s_f} ds \\ &= N_{BM}\Delta s \cos\theta \end{aligned}$$

$$\begin{aligned} w_{BG} &= (20\,\mathrm{N})\,(3\,\mathrm{m})\cos(20\,^{\circ}) \\ &= 56.382\,\mathrm{J} \end{aligned}$$

For a second problem, let's look at the work done by gravity in making something fall.

We can say that the force done by gravity is

$$W = mg$$

downward. So if we drop a ball the work done would be

$$\begin{aligned} w_{BE} &= \int \overrightarrow{F} \cdot \overrightarrow{ds} \\ &= \int (mg)\, ds \cos\theta_{Fx} \end{aligned}$$

but our θ_{FX} will be zero because the ball is going in the negative direction so dy downward and the force due to gravity is downward. So

$$w_{BE} = \int (mg)\, dy\, (1)$$

Note that I have used dy in place of ds because I want to say explicitly that the ball is moving in the downward direction.

$$w_{BE} = \int_{yi}^{y_f} mgdy$$

the mass and the acceleration due to gravity don't change, so we have

$$\begin{aligned} w_{BE} &= mg \int_{yi}^{y_f} y^0 dy \\ &= mg \left(\frac{y^1}{1} \right) \Bigg|_{y_i}^{y_f} \\ &= mg\,(y_f - y_i) \\ &= mg\Delta y \end{aligned}$$

So the pull of Earth's gravity does work in making balls fall.

Suppose we want the ball to go back up to it's starting point. We will have to push at least a hard as the Earth's gravity is pushing down just to keep it moving upward. So we must at least have

$$N_{push} = -W$$

then the ball won't accelerate, but it will keep moving up at a constant speed. In that case we will do work

$$\begin{aligned} w_{By} &= \int_{yi}^{y_f} \overrightarrow{F} \cdot \overrightarrow{dy} \\ & \int_{yi}^{y_f} N_{push} dy \cos\theta_{Nx} \\ &= \int_{yi}^{y_f} (mg)\, dy\, (1) \\ &= mg\Delta y \end{aligned}$$

which is just as much work as the gravitational force did in making the ball fall. Of course we could have pushed harder up than gravity pushed down. In that case the ball would accelerate upward. But $mg\Delta y$ is the *minimum* work needed to put the ball back up where it started.

It is probably worth noting that while we are pushing the ball upward, gravity is trying to push the ball downward. The work done by gravity in this case is

$$\begin{aligned} w_{BE} &= \int \overrightarrow{F} \cdot \overrightarrow{ds} \\ &= \int (mg)\, ds \cos\theta_{Fx} \\ &= \int (mg)\, dy\, (-1) \end{aligned}$$

because the gravitational force is opposite the direction we are pushing. Then

$$w_{BE} = -mg\Delta y$$

as we push upward at a constant speed. The gravitational force is not causing the motion, so we expect to see negative work. This is a special situation. We could have pushed harder, and not kept the object moving at a constant rate upward. But this is a minimum work required to lift the object.

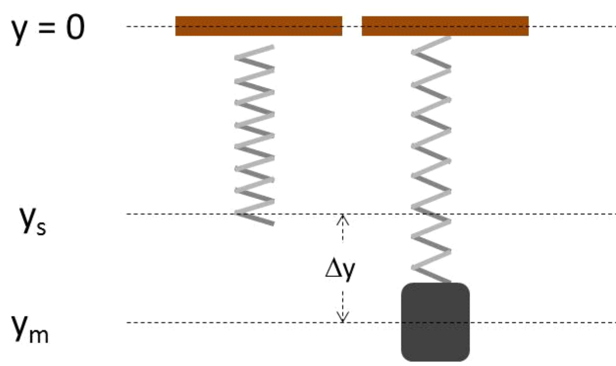
We have done some good work problems with constant forces. Of course, if the force is not constant, the problems might be a little more difficult. We will take on this more difficult situation in the next lecture.

23 Work Done By Changing Forces

In our last lecture, we had normal, tension, and gravitational forces do work. These forces didn't change as the object moved. But some forces are not at all constant. In this lecture we will study a very important example of a force that changes as an object moves. We will study an object connected to a spring.

Hook's "Law"

Think of a spring. If it is stretched, it will pull back against the stretching. If it is compressed, it pushes back against the compression. The more you pull, the harder it pulls back. Or the more you compress it the more it resists being compressed. This force depends on how stretched or compressed the spring has become. The spring likes to be at an equilibrium length and opposes any displacement from that length.



Sir Robert Hooke first came up with a mathematical expression for how the force of a spring works

$$F_s = -k\left(y - y_{\text{equilibrium}}\right) \tag{23.1}$$

where k is a constant that depends on the material and manufacturing method used in making the spring. It really tells us how stiff the spring is. Notice that the direction of the force is in the opposite the direction of the displacement. When this is true we call the force a *restoring force* because it tends to restore the object to it's equilibrium position. The quantity $y_{\text{equilibrium}}$ is where the end of the spring would be if it were not stretched. Of course, if we hang a mass on the end of the spring, the mass will pull the spring down past it's equilibrium position to a new position, y. Sometimes you will see our spring force equation written as

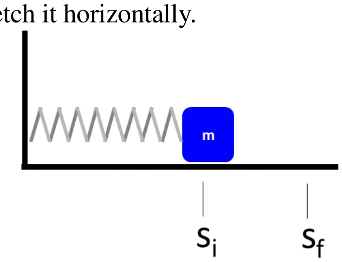
$$F_s = -k\Delta y$$

but we have to be careful to recognize that $\Delta y = y - y_{\text{equilibrium}}$ is a very special displacement. It is a displacement from a chosen starting point, the equilibrium point. We could place a small subscript on the "y" to replace the long word, "equilibrium."

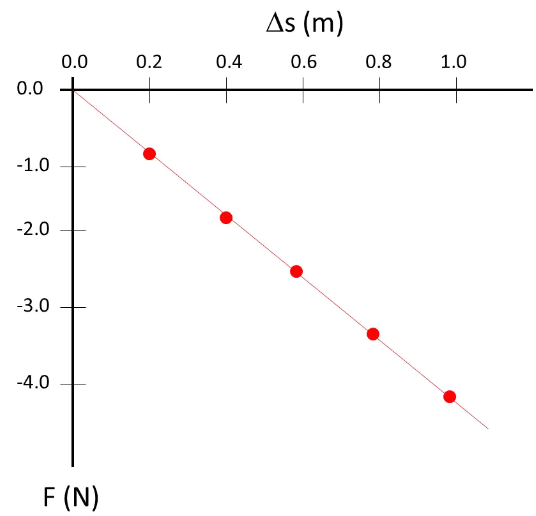
$$F_s = -k\left(y - y_o\right) \tag{23.2}$$

Anyone who has owned a slinky and has had younger siblings knows that you can overstretch a spring. Once overstretched, the spring will not regain its original shape. So with this "law" we have to be careful to use our formula in a region of x (or y) that does not damage the spring. This demonstrates that a scientific law is not something that is true all the time, but rather is a mathematical formula that describes the way we think the physical universe works in some way.

Let's take a spring and stretch it horizontally.

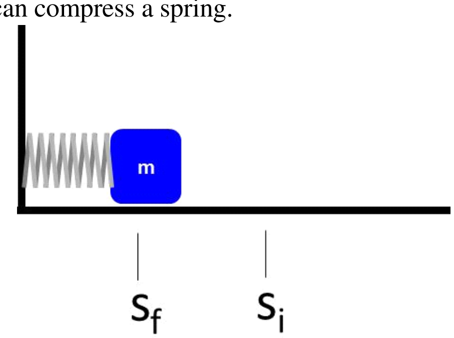


Notice that we are using a general coordinate "s." We can plot the effect of stretching the spring. The restoring force of the spring increases as we stretch the spring.



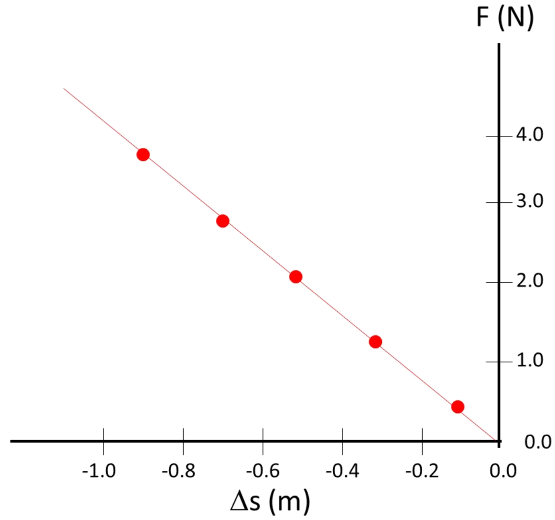
The more we stretch, the more restoring force we have.

It's also true that we can compress a spring.



If Δs is negative, then the restoring force will be positive.

$$F_s = -k\Delta s$$



Let's try a problem using springs:

Suppose we suspend a spring of length $0.50\,\mathrm{m}$ and spring constant of 0.30 from a support. And further suppose that we suspend a $0.50\,\mathrm{kg}$ mass from this spring. How much does the spring stretch?

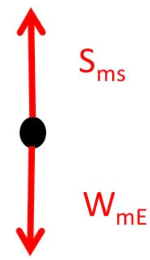
We'll call this a *spring force problem*, and the basic equation is Hooke's law plus Newton's second law

$$F_s = -k\,(y - y_o)$$

We know

$$\begin{aligned} L_s &= 0.05\,\mathrm{m} \\ k &= 28\frac{\mathrm{N}}{\mathrm{m}} \\ m &= 0.50\,\mathrm{kg} \\ g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^3} \end{aligned}$$

We will need a free body diagram



We will call the spring force $F_s = S$ for our free body diagrams. The subscript s is also for spring. When the mass stretches the spring it will come to rest.

$$F_{net_y} = S_{ms} - W_{mE} = 0$$

so

$$S_{ms} = W_{mE}$$

so

$$k\Delta s = mg$$

Notice that we did not put in the negative sign. We already took care of direction when we wrote Newton's second law. So we just put in magnitudes after we write out Newton's second law.

Then

$$\Delta s = \frac{mg}{k}$$

or

$$\begin{aligned}\Delta s &= \frac{(0.50\,\mathrm{kg})\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{\left(28\,\frac{\mathrm{N}}{\mathrm{m}}\right)} \\ &= 0.175\ \mathrm{m}\end{aligned}$$

The spring streched $17.5\,\mathrm{cm}$.

Work for a spring force

Recall our basic equation for work is

$$w = \int \overrightarrow{F} \cdot \overrightarrow{ds}$$

where we have used our new dot product notation.

But what if $\overrightarrow{F} = \overrightarrow{F}(s)$ that is, what if the force changes as we travel in the s-direction? Let's take an example, and then generalize what we find.

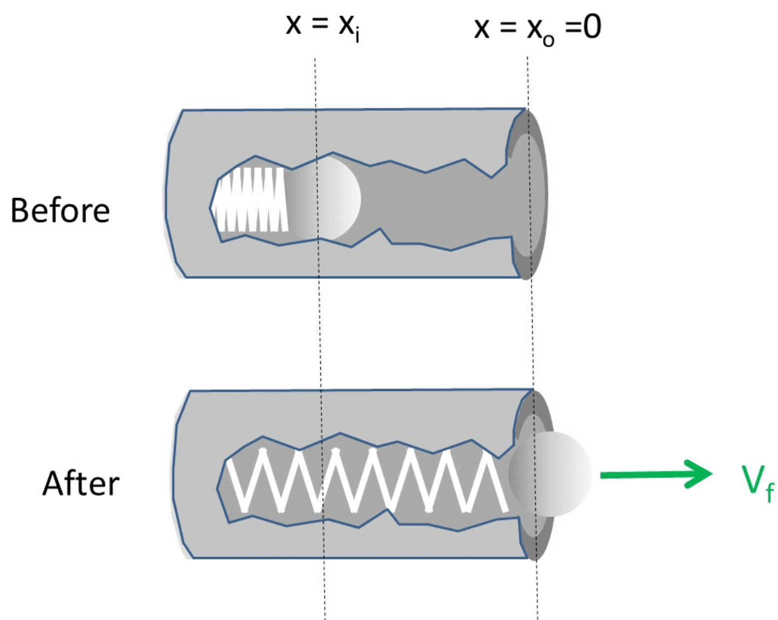
Suppose we have a spring, say, the one in our spring cannon. As the spring pushes on the ball, is the force constant? We know the answer is no, because for spring forces

$$S(s) = -k(s - s_o)$$

so as we let the spring become less compressed (we change s), the magnitude of the force changes. So in such a case, how do we find the work done by the force? We just place this force in our work equation

$$w = \int \overrightarrow{S}(s) \cdot \overrightarrow{ds}$$

or, if our spring cannon operates in the x-direction.



then the work done on the ball is

$$w_b = \int \overrightarrow{S}(x) \cdot \overrightarrow{dx}$$

or

$$w_b = \int |-k(x - x_o)| \cos \theta_{Sx} dx$$

where θ_{Sx} is the angle between the $\overrightarrow{S}$ and the $\overrightarrow{dx}$ directions. From the figure we can see that $\theta_{Sx} = 0$ so

$$w_b = \int |-k(x - x_o)| (1) dx$$

It would be convenient to set our $x = 0$ point right where the spring is at equilibrium. So

$$x_o = 0$$

We will do this, but we have to be careful and make sure $|-k(x - x_o)|$ stays a positive number. After all, it is a magnitude. The problem is that if we choose the end of the cannon as our $x = 0$ point, then x_i and all the other x values from x_i up to $x = 0$ are negative. k can't be negative, and $|-k(x - x_o)|$ must be positive. We can achieve this by using the absolute value signs, but they are clunky. We could just write

$$|-k(x - x_o)| = -k(x - x_o) \text{ for } x \leq 0$$

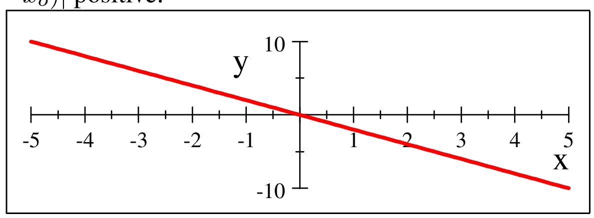
Then we can state that

$$x_o = 0$$

because we chose this as our zero point. Note that if $x > 0$ we have to make

$$|-k(x - x_o)| = +k(x - x_o) \text{ for } x > 0$$

to keep $|-k(x - x_o)|$ positive.



Plot of $-k(x - x_o)$. Note that for $x < 0$ the value of $-k(x - x_o)$ is positive. This is tricky, make sure you understand why this works!

Using $|-k(x - x_o)| = -k(x - x_o)$ for $x \leq 0$ the work would be

$$\begin{aligned} w_b &= \int -k(x - 0)\, dx \\ &= \int -k(x)\, dx \\ &= -k \int x dx \end{aligned}$$

and we need limits for our integrals, making sure $x \leq 0$ in our limits. We start with the ball at $x = x_i$ and end with the ball and the end of the spring at $x = x_o = 0$ with the spring uncompressed. So

$$w_b = -k \int_{x_i}^{0} x dx$$

Our limits fit our condition on our equation. We can do this integral with our basic integral formula that we learned earlier

$$w_b = \left(-k\left(\frac{x^2}{2}\right)\right|_{x_i}^{0}$$

$$w_b = -k\left(\frac{0}{2}\right) - \left(-k\left(\frac{x_i^2}{2}\right)\right)$$

$$w_b = -\left(-k\left(\frac{x_i^2}{2}\right)\right)$$
$$w_b = \frac{1}{2}kx_i^2$$

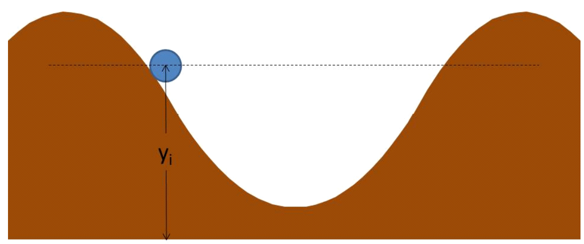
This was not too hard! We can do the same for any force that changes with position.

Work-energy missing piece

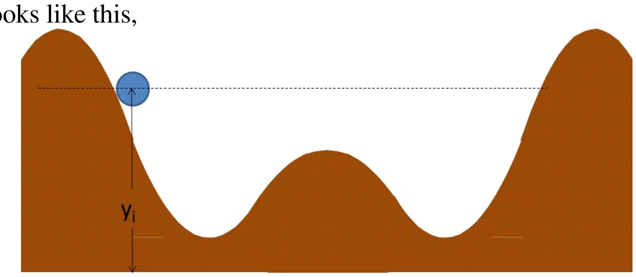
We started by saying our Deacons were pushing rocks on frictionless surfaces. But in a real situation, we know friction exists. Let's consider what happens to the energy of our motion if we do have friction.

Conservative and non-conservative forces

Let's use as our test system a ball rolling down a hill/valley system. We could have a ball roll down a hill/valley system that looks like this,



or one that looks like this,



The gravitational pull due to the Earth will do work on the ball, and that work energy will become kinetic energy. The ball will pick up speed. As the ball reaches the valley floor, the work done by gravity will be negative, so the ball will slow down as it climbs the next hill. The place where the ball stops, and turns around are called *turning points.* The turning points would be just the same height for both balls. The actual path from the beginning y_i and the ending y_f doesn't matter at all.

This is because energy is being *conserved*. The Earth's gravitational force is the force causing the ball to move, that is, causing the gravitational work to be there. It is the environment that makes the y-position matter. So we will give this gravitational force a new title. We will add the name *conservative* to gravitation. Note this is not a new force, just a new name applied to an old force.

But the words "conservative force" apply to a class of forces that all act this way. Only the beginning and ending positions matter when calculating work for these forces. For conservative forces, energy is conserved. Electrical and magnetic forces are both this type of force. And it turns out that so are spring forces (for ideal springs)!

If we have only conservative forces, then the amount of energy we have for a systems won't change. This simple idea is amazingly powerful! In the case of our ball sliding down the hill, we start with an amount of work that it took to get the ball up the hill. From a previous problem, we can guess that this will be

$$w_{gi} = mgy_i$$

As the ball slides down the hill, the amount of work energy used to get the ball up the hill will be converted into kinetic energy. But as the ball slides up the hill, the kinetic energy will be converted back into work energy. We can predict that no energy will be lost from our ball-hill system. so

$$\begin{aligned} E_i &= E_f \\ K_i + w_{gi} &= K_f + w_{gf} \end{aligned}$$

If we stop our experiment at the bottom of the hill we will see that $w_{gf} = 0$, that is, it took no work to get the ball from the bottom of the hill to the bottom of the hill–the ball's displacement $\Delta y = 0$, so no work. If we start the ball from rest then $K_i = 0$. So

$$0 + w_{gi} = K_f + 0$$

and we find that the kinetic energy at the bottom of the hill, K_f, is equal to the work it took to get the ball up the hill. We converted all our work energy into kinetic energy.

Suppose we let the ball keep going. How high will it go up the other side of valley? Assuming we have only conservative forces, then

$$\begin{aligned} E_i &= E_f \\ K_i + w_{gi} &= K_f + w_{gf} \end{aligned}$$

but now the final situation is up the hill. Just as the ball reaches it's highest point, $v_f = 0$ (the ball will momentarily stop) and so $K_f = 0$. Then

$$0 + w_{gi} = 0 + w_{gf}$$

and we see that the work to get the ball up on the other side is equal to the initial work in placing the ball up the hill.

$$mgy_i = mgy_f$$

so the ball goes up just as high on the far hill as it was originally on the starting hill. Notice we did these problems very easily just knowing that we had conservative forces. When we can say that the total amount of energy does not change in a before and after picture, we say that *energy is conserved.* This is such a powerful idea, let's write it anther way

$$\begin{aligned} E_i &= E_f \\ K_i + w_{gi} &= K_f + w_{gf} \\ w_{gi} - w_{gf} &= K_f - K_i \\ -\Delta w_g &= \Delta K \end{aligned}$$

which says as we change the amount of work in a system, we change the kinetic energy of that system. Since work comes from a force, this says that if we push something enough to make it move, we will gain kinetic energy. We convert work into kinetic energy, and notice that the amount of energy lost from our change in work becomes the same energy gained in kinetic energy. This just a restatement of the ideal of conservation of energy, but it is so important that it has it's own name. It is called the *Work-Energy Theorem.*

But there is another class of forces that are *non-conservative forces*. And the archetype of this class is friction.

Consider our two hills again. The ball's path on the two-hump hill is shorter than the ball's path on the three-hump hill. Let's see if this matters for friction forces. The work done by a friction force would be

$$w_f = \int \overrightarrow{f} \cdot \overrightarrow{ds}$$

Notice that as the ball slides down the hill, f and ds are in opposite directions. So the angle θ in

$$w_f = \int f ds \cos\theta$$

will be $180\,^{\circ}$. The cosine of $180\,^{\circ}$ is -1 so

$$\begin{aligned} w_f &= -\int f ds \\ &= -f\Delta s \end{aligned}$$

and our friction force will give us negative work. This means it would take more work to make the ball go the same speed as it would have with no friction.

Let's look at this in our energy equation. For energy to be conserved, we have to add in the energy being taken out by friction. Notice in the next equation there is a work w_{ff} which is the final amount of work taken out of the system by friction.

$$\begin{aligned}
E_i &= E_f \\
K_i + w_{gi} &= K_f + w_{gf} + w_{ff} \\
\frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff} \\
\frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff}
\end{aligned}$$

if we use our zeros, and we take final position of the ball at the bottom of the hill,

$$0 + mgy_i = \frac{1}{2}mv_f^2 + 0 + w_{ff}$$

so

$$\frac{1}{2}mv_f^2 = mgy_i - w_{ff}$$

so

$$v_f = 2\frac{mgy_i - w_{ff}}{m}$$

we can see that the ball won't go as fast if there is a friction term. Some of the original gravitational work energy is no longer available to make into the kinetic energy of the ball. And since $|w_f| = |f\Delta s|$

$$v_f = 2\frac{mgy_i - f\Delta s}{m}$$

the larger the Δs, the more original work energy becomes unavailable for making the kinetic energy of the ball.

Suppose the ball rolls all the way down the hill and up the other side. Since we had a lower speed at the bottom of the hill, the ball won't go up the other side as far. Let's show this in our math, where now the final case is up on the other side of the valley. At the turning point $v_f = 0$ we have

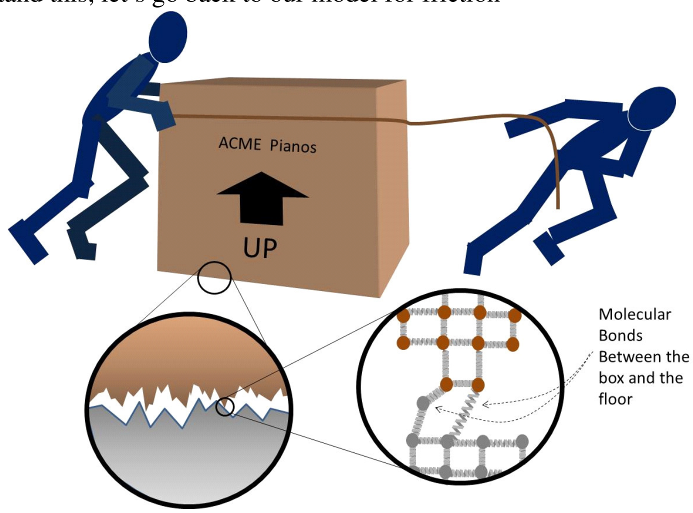
$$\begin{aligned}
E_i &= E_f \\
\frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff} \\
0 + mgy_i &= 0 + mgy_f + w_{ff}
\end{aligned}$$

and let's solve for y_f, the final height of the ball

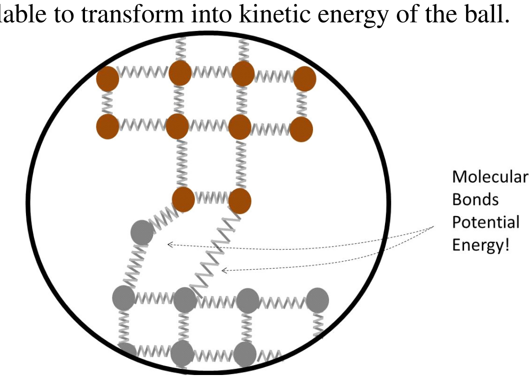
$$\begin{aligned}
mgy_f &= mgy_i - w_{ff} \\
y_f &= \frac{mgy_i - w_{ff}}{mg} \\
&= y_i - \frac{w_{ff}}{mg}
\end{aligned}$$

and as expected, the ball does not get as far up the other side. The friction force has taken energy away from both the kinetic energy and the potential energy. But where did it the energy go?

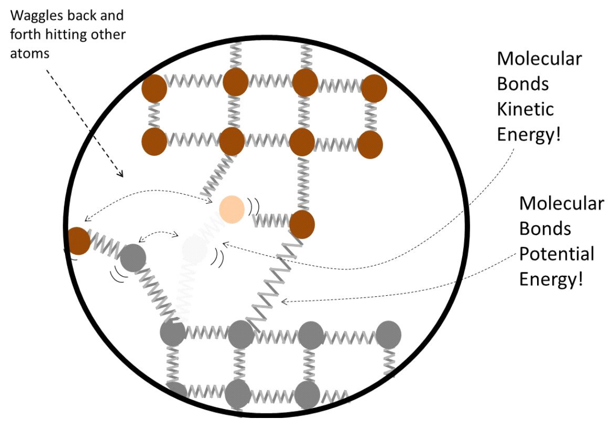
To understand this, let's go back to our model for friction



Consider our moving team again. Recall that as the guys push the box, the roughness teeth are bent, and the molecular bonds are stretched. Those stretched, spring-like bonds will build up what we might call "bond potential energy." We will deal with this type of potential energy in detail in PH220, but for now we will just model the bonds like springs. The important point for now is that some of the energy has been removed from the gravitational potential energy and turned into bond potential energy. So this energy is no longer available to transform into kinetic energy of the ball.



When the bonds break, the atoms are free to move. The spring like bond forces convert the bond potential energy into kinetic energy of the atoms. As more roughness teeth come by the atoms are struck again and again.



The atoms oscillate and strike other atoms. The whole solid is made of atoms and bonding forces. So as atoms are struck, the spring-like bonds are compressed and push back. After a short time, the whole collection of atoms in the solid are moving, compressing and uncompressing their bond forces. All this is kinetic and work energy of the atoms in the solid floor martial (and of the box material). If you put your hand on the floor or the box some of the energy would be transferred to your hand atoms by collisions between the floor atoms and your hand atoms. So soon the atoms in your hand would have some kinetic energy transferred to them and would be stretching and compressing their bonds. What you would feel is that the floor got hot. We will call the energy lost to the motion inside an object due to friction *thermal energy* because it triggers our body temperature sensors. We will study this in more detail in PH123, but for now the important point is that we have spread some of the energy all over the floor, the box, your hand, and eventually all over the room. This energy is no longer available to be transformed into kinetic energy of the box. We will say that energy that has been lost from our box system (so that it can't participate in creating motion of the box or gravitational potential energy of the box) has been *dissipated***.** The word "dissipated" means "lost from the system."

Let's be a little more specific. We call the combination of gravitational work energy, spring work energy, and kinetic energy *mechanical energy.*

$$E_{mech} = K + w_g + w_s$$

these energies are all conservative. We call the work energy due to friction *non-conservative energy* and forces like friction that dissipate energy are called *non-conservative forces.*

Let's write non-conservative work as

$$w_{nc}$$

Then from our ball sliding down the hill we can see that

$$\begin{aligned} E_i &= E_f \\ \frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_{ff} \\ K_i + w_{gi} &= K_f + w_{gf} + w_{nc} \end{aligned}$$

The left hand side is the initial mechanical energy (we don't have a spring, so $w_{si} = 0$). The right hand side is the final mechanical energy. So or we could write this as

$$E_{mech_i} = E_{mech_f} + w_{nc}$$

So the final mechanical energy must be less than the initial mechanical energy, and the difference is w_{nc}. We could write this as

$$\Delta E_{mech} = w_{nc}$$

Power

We now have a better understanding of energy and how we can use the concept of energy to solve problems. But there is more to our energy picture. The rate at which we use energy, say, the rate at which we perform work, is important. Suppose you are in the thriving metropolis next to the offices of the Daily Universe, and you see a super guy

Both you and Super Guy go to the top of the Daily Universe building. Super Guy does this in a single bound. You take the stairs. What is the difference in the work done by you and Super Guy assuming we can neglect the work done by frictional forces.

This is an energy problem so we set up our energy equation

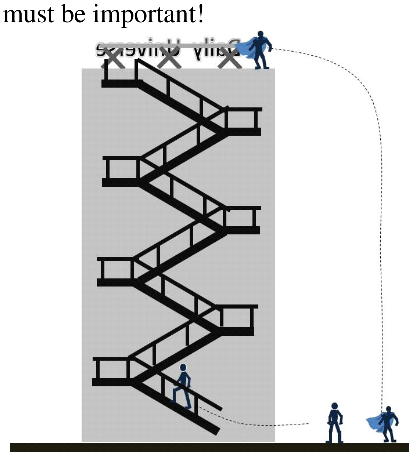
$$K_i + w_{gi} + w_{si} + w_{ei} + E_{th_i} = K_f + w_{gf} + w_{sf} + w_{ef} + E_{th_f}$$

where I have written the thermal energy due to non-concervative work as E_{th}. In this problem we don't have springs, or electric forces, so we can cancel all the spring and electric force terms

$$K_i + w_{gi} = K_f + w_{gf}$$

At the beginning you are standing on the ground stationary, and so is Super Guy. At the end you are standing stationary at the top of the Daily Universe building, and so is Super Guy. So $K_i = 0$ and $K_f = 0$. The only change in energy is that both you and Super Guy have done work against the gravitational field in order to get to the top of the building $w = mg\,(y_f - y_i)$. We can see that energy is not conserved for you or Super Guy, there must be an input of energy at the start that we missed. And that energy we missed is the chemical potential energy of the food you and Super Guy ate for breakfast that can be turned into the work you and Super Guy do to reach the top. But notice that since you and Super Guy have the same mass (just distributed differently) the two of you have done the same amount of work. But something must be different. Otherwise, we would all be Super Guy!

And you can see immediately what the difference is. Supper guy got to the top of the building in moments. It took you 15 minutes of hard running to get up the stairs. The time it takes to do work must be important!



Let's define a new quantity that is the amount of work we do divided by the time it took to do the work.

$$\frac{w}{\Delta t}$$

and we need a name and a symbol. In respect of Super Guy, let's say that this new quantity is called *power*. Super Guy is more powerful because he can do the work of going from street level to the top of a building in a very short Δt. You can do the same work, via the stairs, but it will take a much longer Δt.

$$\frac{w}{\Delta t_{small}} > \frac{w}{\Delta t_{big}}$$

The symbol for power is a capital P.

Notice that power is a rate at which work happens. If we take the Δt to be very small, so we can call it dt, then

$$P = \frac{dw}{dt}$$

and this definition works for any type of energy

$$P = \frac{dE_{any}}{dt}$$

Notice that the units for power would be J/s. This combination of units has a name, the *watt.*

$$\mathrm{W} = \frac{\mathrm{J}}{\mathrm{s}}$$

Let's go back to our definition of power, to find an alternate equation for power.

$$P = \frac{w}{\Delta t}$$

which we could write as

$$P = \frac{w_f - 0}{\Delta t}$$

where 0 is the initial work before we start pushing or pulling. Then

$$P = \frac{dw}{dt}$$

as well. And we know that

$$w = \int \overrightarrow{F} \cdot \overrightarrow{ds}$$

so the integrand, $\overrightarrow{F} \cdot \overrightarrow{ds}$ must be a little bit of work

$$dw = \overrightarrow{F} \cdot \overrightarrow{ds}$$

Let's divide both sides of this equation by dt

$$\frac{dw}{dt} = \frac{\overrightarrow{F} \cdot \overrightarrow{ds}}{dt}$$

the left hand side is power

$$P = \frac{\overrightarrow{F} \cdot \overrightarrow{ds}}{dt}$$

The right hand side could be rewriting as

$$P = \frac{F ds \cos\theta_{Fs}}{dt}$$

or even as

$$P = F\cos\theta_{Fs}\frac{ds}{dt}$$

and we recognize ds/dt as the speed of the object we are pushing or pulling with our force F. So

$$P = F\cos\theta_{Fs}v$$

and now we know that F and v are vectors, and that θ_{Fs} is the same direction as θ_{Fv} because we are going in the s direction, so v and s are in the same direction. Then

$$P = \overrightarrow{F}\cdot\overrightarrow{v}$$

So if we know the force and the velocity, we can also find the power.

Let's try a problem.

Suppose that the Daily Universe Building roof is $45\,\mathrm{m}$ above the street below. And suppose that both you and Super Guy have a mass of $90\,\mathrm{kg}$. Super Guy gets to the roof in $1.2\,\mathrm{s}$. You get to the roof in $10.1\,\mathrm{min}$. What is Super Guy's power, and what is your power?

We know

$$\begin{aligned} m &= 90\,\mathrm{kg} \\ \Delta y &= 45\,\mathrm{m} \\ \Delta t_y &= 10.1\,\mathrm{min} = 606.0\,\mathrm{s} \\ \Delta t_{SG} &= 1.2\,\mathrm{s} \\ g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Our basic equation for this would be

$$P = \frac{w}{\Delta t}$$

so

$$\begin{aligned} (w_f - w_i) &= (mgy_f - mgy_i) \\ &= mgy_f \end{aligned}$$

so

$$P = \frac{mg\Delta y}{\Delta t}$$

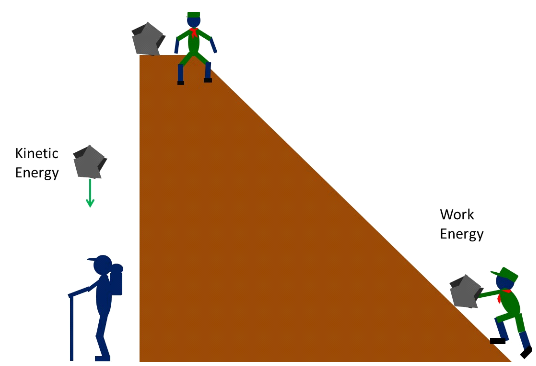
This is true for both you and Super Guy. Numerically, then

$$\begin{aligned} P_{SG} &= \frac{(90\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(45\,\mathrm{m})}{1.2\,\mathrm{s}} \\ &= 33075\,\mathrm{W} \\ P_y &= \frac{(90\,\mathrm{kg})\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(45\,\mathrm{m})}{606.0\,\mathrm{s}} \\ &= 65.495\,\mathrm{W} \end{aligned}$$

and we see that Super Guy is much more powerful than you are, just as expected.

24 Stored Energy

So now let's review. Our deacons quorum is once again hiking with rocks. To get the rocks up to the cliff, our boys do work. That work creates some kinetic energy as the rocks move Once they are on the top of the hill, the gravitational force will do work on the rocks further creating kinetic energy. But as the boys reach the top of the hill, they usually momentarily stop before pushing the rock onto the unsuspecting hikers below[12]. Where did the energy go? There was lots of work done, but we no longer have kinetic energy.



Potential Energy

So far we have learned about motion (kinematics) and forces (Newton's laws). We changed our view point to consider energy and we learned about work and kinetic energy. But we can't yet do a kinematics problem with our energy view point. In our deacons example, the deacons pushed rocks up a hill. We know we could use kinematics to find how the rocks will move, as they fall, but work and kinetic energy aren't enough.

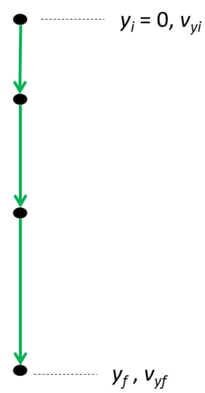
[12] Please don't really roll rocks down hills, this is dangerous. This is why I don't hike in valleys or at the bottom of cliffs!

To find the missing piece of our energy picture, let's start with the kinematic equations.

$$\begin{array}{ll}
\Delta x = v_{ix}\Delta t + \frac{1}{2}a_x \Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y \Delta t^2 \\
v_{fx} = v_{ix} + a_x \Delta t & v_{fy} = v_{iy} + a_y \Delta t \\
v_{fx}^2 = v_{ix}^2 + 2a_x \Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y \Delta y
\end{array}$$

and let's take an example problem. Let's take a falling object (a spy, a hero, a pharmacist, or whatever).

For all of these objects our motion diagram would look like this



When we were trying to find an easier way to do force problems in the last few lectures, we did the force problem and then generalized the result. Since each falling object seems to have the same motion diagram, it looks like we could do this again. Let's try to generalize our kinematic equations for falling objects. Let's use the last equation in the y-kinematics set

$$v_{fy}^2 = v_{iy}^2 + 2a_y \Delta y$$

and write out $\Delta y = y_f - y_i$

$$v_{fy}^2 = v_{iy}^2 + 2a_y \left(y_f - y_i\right)$$

In doing energy problems we learned that we can often benefit from separating our equations into a "before" picture and an "after" picture, so let's take all the initial values to the left side of the equation and all the final values to the right side. So

$$v_{fy}^2 = v_{iy}^2 + 2a_y y_f - 2a_y y_i$$

becomes

$$v_{fy}^2 - 2a_y y_f = v_{iy}^2 - 2a_y y_i$$

or

$$v_{iy}^2 - 2a_y y_i = v_{fy}^2 - 2a_y y_f$$

This looks very useful. If we know the initial speed and position, we know the combination $v_{iy}^2 - 2a_y y_i$ won't change so we can predict what the combination $v_{fy}^2 - 2a_y y_f$ will be.

I would like to make some cosmetic changes to this equation. We won't change the fact that our quantity $\left(v_y^2 - 2a_y y\right)$ is not changing. I will divide both sides of the equation by $1/2$

$$\frac{1}{2}v_{iy}^2 - a_y y_i = \frac{1}{2}v_{fy}^2 - a_y y_f$$

and multiply through by the mass of the object.

$$\frac{1}{2}mv_{iy}^2 - ma_y y_i = \frac{1}{2}mv_{fy}^2 - ma_y y_f$$

These mathematical changes did not affect the equality at all. Finally, we know that $a_y = -g$, so

$$\frac{1}{2}mv_{iy}^2 + mgy_i = \frac{1}{2}mv_{fy}^2 + mgy_f$$

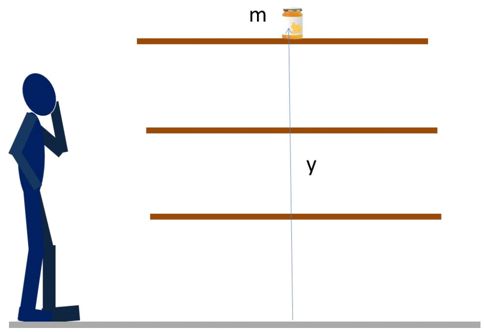
so the combined quantity $\frac{1}{2}mv^2 + mgy$ is not changing as the object moves. If a quantity doesn't change, in physics we say it is *conserved*. It is the same in our "before" picture and in our "after picture." But what is this quantity?

We recognize that one part of this of our conserved quantity

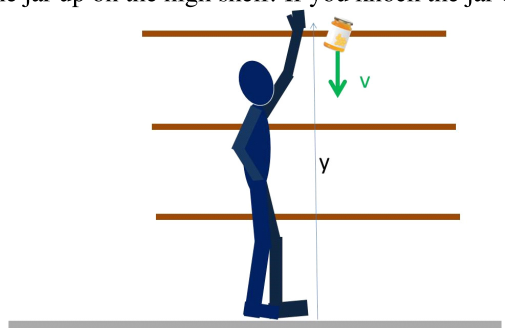
$$K = \frac{1}{2}mv^2$$

is kinetic energy. And we know all the terms in a sum must be the same kind of thing. So this combination must be *energy!* So I could use this to, say, measure the final y, and calculate the final speed without using our kinematic equations. This is wonderful enough all on it's own, but let's pause for a minute and ask, what is the mgy part?

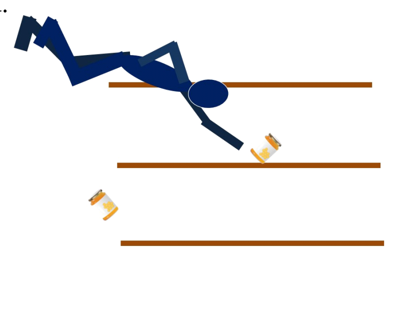
We recognize that this is the work we did in putting the object up high to begin with! This is the minimum work it took to place the object up high, like our deacons placing a rock on a hill. For another example, consider a jar on your pantry shelf.



The jar has mass, m, and is up a height y. There is no motion. But you did work at some time earlier to get the jar up on the high shelf. If you knock the jar off the shelf,



the jar will move. We could say that by doing the work of putting the jar high on the shelf we have created the potential to have motion. We have effectively stored our work for use later! We call this kind of stored energy *potential energy.* We give it the symbol U. Since it takes the Earth's gravity to create the environment where this potential energy is possible, we will say that this is *gravitational potential energy*. Think about this for a minute. If there were no gravitational pull, you, the things in your pantry, and the pantry itself would float around.



It is that our jar is up high with the Earth's gravity pulling on it that makes the jar have potential to move downward. Think that the acceleration due to gravity was in our kinematic equation that we started with. The gravitational pull is required to make gravitational potential energy. And to say we have gravitational potential energy we write a subscript, g, on our symbol for potential energy, U_g.

$$U_g = mgy$$

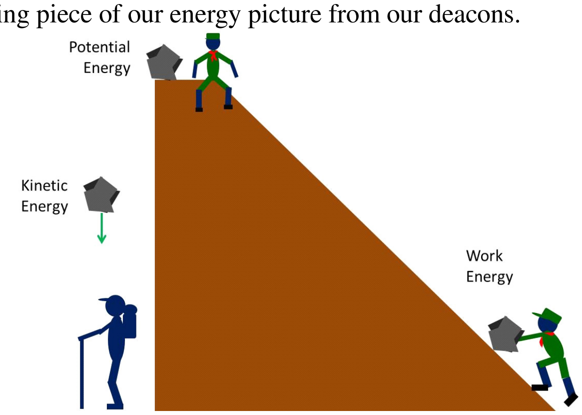
So our energy combination can be written as

$$\frac{1}{2}mv^2 + mgy = K + U$$

and our energy equation can be written as

$$\begin{aligned} \frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f \\ K_i + U_i &= K_f + U_f \end{aligned}$$

This is the missing piece of our energy picture from our deacons.



We can see that as the deacons moved the rock up the hill, some of the work they did became kinetic energy but some of the work was converted into potential energy. That is why the rock can easily move once it is pushed over the edge. The work of the boys moving the rock up the hill against the force of gravity effectively stores part of their work as potential energy. That potential energy is converted into kinetic energy as the rock falls.

Notice that we have to be very careful when we talk about work and potential energy. The work done by the gravitational force when the pushing the rock up the hill is

$$w_g = \int mg\Delta y \cos\theta$$

but the rock is going up and the gravitational force is down so $\theta = 180\,^{\circ}$. The work done by the gravitational force is negative. That makes sense because the Earth's gravitation is not making the motion, the deacon is. So we expect the losing force to make negative work.

$$w_g = -mgh$$

where

$$h = \Delta y = y_f - y_i = y_{top} - y_{bottom}$$

The gravitational potential energy change that we store when pushing the rock up the hill is given by

$$\Delta U_g = mg\Delta y = mgh$$

so we see that

$$\Delta U_g = -w_g$$

The change in potential energy is the negative of the work done by the Earth's gravitational force. Notice that the deacon's work in pushing the rock up the hill is

$$w_d = \int mg\Delta y \cos\theta$$

but this time the angle θ is 0 degrees because the rock is going in the direction of the push. So

$$w_d = +mg\Delta y = +mgh$$

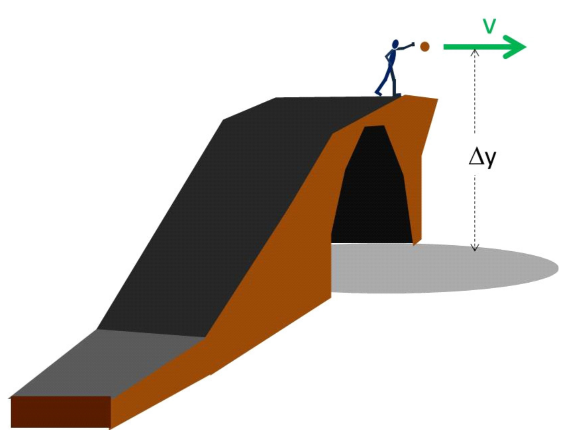
But in comparing gravitational potential energy to work, we choose to compare the gravitational potential energy to the gravitational work, not the boy's work, so we will use the minus sign.

$$\Delta U_g = -w_g$$

and this will be a general expression, so long as we always consider the same force (gravitational in this case) for both sides of the equation

$$\Delta U = -w$$

Let's try some problems using the idea of potential energy.



Suppose a boy scout throws a $0.5\,\mathrm{kg}$ rock off a bridge. The boy throws the rock with an initial velocity of $\overrightarrow{v} = 10.0\,\mathrm{m/\,s}\hat{\imath}$. The bridge is $20.\,\mathrm{m}$ high. What is the final speed of the rock just before it hits the ground below the bridge assuming air resistance is negligible?

This certainly could be a kinematics problem, but let's do this with our energy equation. We know that

$$\begin{aligned}
\overrightarrow{v}_i &= 10\frac{\mathrm{m}}{\mathrm{s}}\hat{\imath} \\
y_i &= 20\,\mathrm{m} \\
y_f &= 0\,\mathrm{m} \\
m &= 0.5\,\mathrm{kg} \\
a_y &= -g \\
g &= -9.8\frac{\mathrm{m}}{\mathrm{s}^2}
\end{aligned}$$

and our equation is

$$\begin{aligned}
K_i + U_i &= K_f - U_f \\
\frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f
\end{aligned}$$

To solve this, first let's notice that the mass is the mass of the rock in every term in our equation. So we can divide both sides by the mass of the rock.

$$\frac{1}{2}v_i^2 + gy_i = \frac{1}{2}v_f^2 + gy_f$$

and let's underline what we know

$$\frac{1}{2}\underline{v_i^2} + \underline{gy_i} = \frac{1}{2}v_f^2 + \underline{gy_f}$$

The only thing we don't know is v_{fy} which is what we want to find. We just need to do some algebra,

$$\frac{1}{2}\underline{v_i^2} + \underline{gy_i} - \underline{gy_f} = \frac{1}{2}v_f^2$$
$$\frac{1}{2}v_f^2 = \frac{1}{2}\underline{v_i^2} + \underline{gy_i} - \underline{gy_f}$$
$$\frac{1}{2}v_f^2 = \frac{1}{2}\underline{v_i^2} + \underline{g}\left(\underline{y_i} - \underline{y_f}\right)$$
$$v_f^2 = \underline{v_i^2} + 2\underline{g}\left(\underline{y_i} - \underline{y_f}\right)$$
$$v_{fy}^2 = \underline{v_{iy}^2} + 2\underline{g}\left(\underline{y_i} - \underline{y_f}\right)$$
$$v_f = \sqrt{\underline{v_i^2} + 2\underline{g}\left(\underline{y_i} - \underline{y_f}\right)}$$

or

$$\begin{aligned} v_f &= \sqrt{\left(10\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(20\,\mathrm{m} - 0\,\mathrm{m})} \\ &= 22.181\frac{\mathrm{m}}{\mathrm{s}} \\ &= 22.\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

We could compare this to doing the problem using kinematics.

We would need a full two-dimensional set of kinematic equations

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\ v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \end{array}$$

and we would need to add into our knowns

$$\begin{aligned} v_{xi} &= 10\frac{\mathrm{m}}{\mathrm{s}} \\ v_{yi} &= 0 \\ y_i &= 20\,\mathrm{m} \\ y_f &= 0\,\mathrm{m} \\ m &= 0.5\,\mathrm{kg} \\ a_y &= g = 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \\ a_x &= 0 \end{aligned}$$

then marking what we know

$$\begin{array}{ll} \Delta x = \underline{v_{ix}}\Delta t + \frac{1}{2}\underline{a_x}\Delta t^2 & \underbrace{\Delta y} = \underline{v_{iy}}\Delta t + \frac{1}{2}\underline{a_y}\Delta t^2 \\ v_{fx} = \underline{v_{ix}} + \underline{a_x}\Delta t & v_{fy} = \underline{v_{iy}} + \underline{a_y}\Delta t \\ v_{fx}^2 = \underline{v_{ix}^2} + 2\underline{a_x}\Delta x & v_{fy}^2 = \underline{v_{iy}^2} + 2\underline{a_y}\,\underbrace{\Delta y} \end{array}$$

and using our zeros

$$\begin{array}{ll} \Delta x = \underline{v_{ix}}\Delta t + 0 & \underbrace{\Delta y} = 0 + \frac{1}{2}\underline{a_y}\Delta t^2 \\ v_{fx} = \underline{v_{ix}} + 0 & v_{fy} = 0 + \underline{a_y}\Delta t \\ v_{fx}^2 = \underline{v_{ix}^2} + 0 & v_{fy}^2 = \underline{0} + 2\underline{a_y}\,\underbrace{\Delta y} \end{array}$$

we could identify that the x-component of our rock velocity does not change,

$$v_{fx} = \underline{v_{ix}}$$

and from the last of the y-set we can find the final y-component of the velocity

$$\begin{aligned} v_{fy}^2 &= \underline{0} + 2\underline{a_y}\,\underbrace{\Delta y} \\ v_{fy}^2 &= 2\underline{a_y}\,(y_f - y_i) \end{aligned}$$

$$v_{fy} = \sqrt{2\underline{a_y}\left(y_f - y_i\right)}$$

and then we need to combine the components to get the magnitude of the final velocity

$$\begin{aligned}
v_f &= \sqrt{\left(v_x\right)^2 + \left(v_y\right)^2} \\
&= \sqrt{\left(\underline{v_{ix}}\right)^2 + \left(\sqrt{2\underline{a_y}\left(y_f - y_i\right)}\right)^2} \\
&= \sqrt{\left(\underline{v_{ix}}\right)^2 + 2\underline{a_y}\left(y_f - y_i\right)} \\
&= \sqrt{\left(10\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(20\,\mathrm{m} - 0\,\mathrm{m}\right)} \\
&= 22.181\frac{\mathrm{m}}{\mathrm{s}} \\
&= 22.\,\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

with a direction of

$$\begin{aligned}
\theta &= \tan^{-1}\left(\frac{-\sqrt{2\underline{a_y}\left(y_f - y_i\right)}}{v_{ix}}\right) \\
&= \tan^{-1}\left(\frac{-\sqrt{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(20\,\mathrm{m} - 0\,\mathrm{m}\right)}}{10\frac{\mathrm{m}}{\mathrm{s}}}\right) \\
&= 1.1031\,\mathrm{rad} \\
&= -63.203\,^{\circ}
\end{aligned}$$

We got the same speed! But the energy method seemed easier. Did you notice that the easier energy method came with a cost? We did get the speed of the rock, and if that is all we wanted, there is no problem. But the energy method did not give us a direction, and it can't give direction. We have an easier way to get the speed, but at the cost of not knowing the direction.

Also notice, that like with conservation of momentum and Newton's laws, the kinematic equations are really buried down deep in the energy equation. They are still there, and for some problems (ones where we don't need to know direction) the energy approach works very well and may be much easier.

Let's do another problem. Suppose our boy scout throws another $0.5\,\mathrm{kg}$ rock, but this time he throws the rock at a $30\,^{\circ}$ angle. The initial speed is still $10\,\mathrm{m/\,s}$. If we do this with kinematics, we would have to start over and do the whole problem again. But let's try with energy.

Our initial kinetic energy is still

$$K_i = \frac{1}{2}mv_i^2$$

and our initial potential energy is still

$$U_i = mgy_i$$

and the final kinetic energy is still

$$K_f = \frac{1}{2}mv_f^2$$

and the final potential energy is still

$$U_f = mgy_f$$

nothing has changed! When we did the first problem we did all problems where the initial and final conditions are the same. We still don't know the direction, and the direction of v_f for the two problems is very different. But the final speed is the same. This is another great time savings if we only need to know the speed.

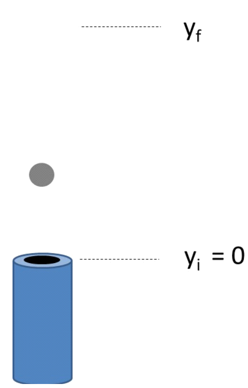
Zero point for U$_g$

You probably noticed that U_g depends on y

$$U_g = mgy$$

But we know about y. It is part of a coordinate system. And we know we can set the origin of the coordinate system anywhere we want. So does that mean that we can make U_g anything we want? To a point, this is true. Let's a problem. Let's shoot a ball out of our spring cannon. Let's try our problem shooting a ball straight up, but let's do it twice, once with the $y = 0$ point at the muzzle of the cannon, and once with the $y = 0$ point at the top of the ball's ight.

For the first case, we have the situation as shown in the next figure.



We can see that

$$\begin{aligned} y_i &= 0 \\ v_f &= 0 \end{aligned}$$

if we let v_f be right at the top of the ball's flight, and let's say that the top of the ball's flight is

$$y_f = 1.34\,\mathrm{m}$$

above the muzzle of the cannon.

We believe from what we have done in this lecture, that the quantity $K + U_g$ won't change, so

$$K_i + U_{gi} = K_f + U_{gf}$$

and we can write this out using our equations for kinetic energy and gravitational potential energy

$$\begin{aligned} K_i + U_{gi} &= K_f + U_{gf} \\ \frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f \end{aligned}$$

The masses cancel

$$\frac{1}{2}v_i^2 + gy_i = \frac{1}{2}v_f^2 + gy_f$$

Now let's use our zeros, so

$$\frac{1}{2}v_i^2 + 0 = 0 + gy_f$$

then

$$v_i^2 = 2gy_f$$

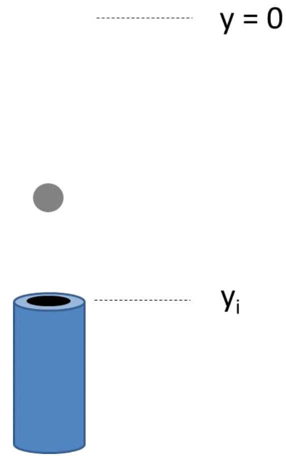
and finally

$$v_i = \sqrt{2gy_f}$$

or

$$\begin{aligned} v_i &= \sqrt{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(1.34\,\mathrm{m})} \\ &= 5.12\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Now let's do the problem again but with $y = 0$ at the top of the ball's flight. The situation is as shown in the next figure.



Now we have

$$\begin{aligned} y_f &= 0 \\ v_f &= 0 \end{aligned}$$

and we realize that it must be true that

$$y_i = -1.34\,\mathrm{m}$$

Note the minus sign!

We still believe that the quantity $K + U_g$ won't change, so

$$\begin{aligned} K_i + U_{gi} &= K_f + U_{gf} \\ \frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f \end{aligned}$$

The masses cancel

$$\frac{1}{2}v_i^2 + gy_i = \frac{1}{2}v_f^2 + gy_f$$

Now let's use our zeros, but they are different in the second case

$$\frac{1}{2}v_i^2 + y_i = 0 + 0$$

then

$$v_i^2 = -2gy_i$$

and finally

$$v_i = \sqrt{-2gy_f}$$

and the initial velocity is then

$$\begin{aligned} v_i &= \sqrt{-2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(-1.335\,\mathrm{m})} \\ &= 5.1153\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

we got the same result.

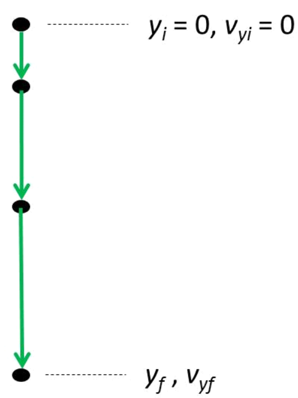
This is amazing, we can pick our $y = 0$ point anywhere that is convenient, and the math still works. But WARNING, once you have picked a $y = 0$ point for a problem, you have to keep the same $y = 0$ point for the whole problem. You can't switch your origin of your coordinate system half way through the problem!

25 Conservation of Energy

Derivation of our energy equation

You may be content. We have an energy equation. But I divided by 2 and multiplied by m with no comment on why I chose to do that. We can get our energy equation in a more elegant way, one that matches demonstrates a technique we will use in the future.

Start with Newton's second law for a falling object.



$$F_{net_y} = ma_y$$

and

$$a_y = \frac{dv_y}{dt}$$

so

$$F_{net_y} = m\frac{dv_y}{dt}$$

we know that for our falling object that our force is

$$F_{net_y} = m\left(-g\right)$$

And let's use some of the calculus you are learning. We would like to find dv_y/dt in terms of dv_y/dy. We could use the chain rule

$$\frac{dv_y}{dt} = \frac{dv_y}{dy}\frac{dy}{dt} = \frac{dv_y}{dy}v_y$$

so we can write our net force

$$F_{net_y} = m\frac{dv_y}{dt} = m\frac{dv_y}{dy}v_y$$

and this is equal to $-mg$

$$m\frac{dv_y}{dy}v_y = -mg$$

and lets multiply both sides by dy

$$mv_y dv_y = -mgdy$$

We can integrate both sides of this

$$\int_{v_i}^{v_f} mv_y dv_y = -\int_{y_i}^{y_f} mgdy$$

We use our integration math tool, remember if

$$u = at^n$$

then the anti-derivative of u is

$$\int_{t_1}^{t_2} udt = \int_{t_1}^{t_2} at^n dt = \left.\frac{at^{n+1}}{n+1}\right|_{t_1}^{t_2} = \frac{at_2^{n+1}}{n+1} - \frac{at_1^{n+1}}{n+1}$$

We can use this for each sides of our equation

$$\begin{aligned} \int_{v_i}^{v_f} mv_y^1 dv_y &= -\int_{y_i}^{y_f} mgy^0 dy \\ \left.\frac{mv^{1+1}}{1+1}\right|_{v_i}^{v_f} &= \left.\frac{-mgy^{0+1}}{0+1}\right|_{y_i}^{y_f} \\ m\frac{v_f^2}{2} - m\frac{v_i^2}{2} &= -mgy_f + mgy_i \end{aligned}$$

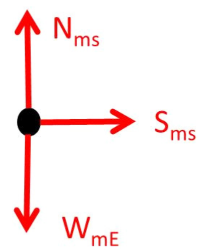
Let's separate the initial and final parts.

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

and we can see that this is really just our energy equation.

Spring potential energy

Since a spring can exert a force, we can store up energy in the spring by stretching or compressing it. This energy will be a potential energy. We will want an equation for this spring storred energy. To find the equation, let's do what we did in our last lecture to find the gravitational potential energy. We started with the net force. Let's take a horizontal spring resting on a (frictionless) surface. The free body diagram would be



We can write out Newton's second law

$$\begin{aligned} F_{net_y} &= ma_y = 0 \\ &= N_{ms} - W_{mE} \end{aligned}$$

which tells us that

$$N_{ms} = W_{mE}$$

and

$$\begin{aligned} F_{net_x} &= ma_x \\ &= S_{ms} \end{aligned}$$

so

$$ma_x = S_{ms}$$

We know

$$S_{ms} = -k\Delta x$$

so

$$ma_x = -k\Delta x$$

Again we write

$$a_x = \frac{dv_x}{dt}$$

so

$$m\frac{dv_x}{dt} = -k\Delta x$$

Let's write dv_x/dt in terms of dv_x/dx. We use the chain rule

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{dv_x}{dx}\frac{dx}{dt} \\ &= \frac{dv_x}{dx}v_x \end{aligned}$$

so

$$m\frac{dv_x}{dx}v_x = -k\left(x - x_0\right)$$

where x_o is the equilibrium position of the spring. Then

$$mv_x dv_x = -k\left(x - x_0\right)dx$$

and we integrate both sides

$$\int_{v_i}^{v_f} mv_x dv_x = \int_{x_i}^{x_f} -k\left(x - x_0\right)dx$$

The right hand side is

$$\int_{v_i}^{v_f} mv_x dv_x = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

The left hand side is

$$\begin{aligned}
\int_{x_i}^{x_f} -k\left(x - x_0\right)dx &= \int_{x_i}^{x_f} -kxdx + \int_{x_i}^{x_f} -k\left(-x_0\right)dx \\
&= \left(-k\left(-\frac{1}{2}x^2\right)\right|_{x_i}^{x_f} + \left(-kx_0x\right|_{x_i}^{x_f} \\
&= +\frac{1}{2}kx_f^2 - \left(+\frac{1}{2}kx_i^2\right) + \left(-kx_0x_f - \left(-kx_0x_i\right)\right) \\
&= +\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 - kx_0x_f + kx_0x_i \\
&= +\frac{1}{2}kx_f^2 - kx_0x_f - \left(\frac{1}{2}kx_i^2 - kx_0x_i\right)
\end{aligned}$$

At this point let's play a trick we learned in high school. I want to add zero to this

$$\int_{x_i}^{x_f} -k\left(x - x_0\right)dx = +\frac{1}{2}kx_f^2 - kx_0x_f - \left(\frac{1}{2}kx_i^2 - kx_0x_i\right) + 0$$

this does not affect the value of the right hand side at all. But let's write zero as

$$0 = \frac{1}{2}kx_0^2 - \frac{1}{2}kx_0^2$$

then

$$\int_{x_i}^{x_f} -k\left(x - x_0\right)dx = \frac{1}{2}kx_f^2 - kx_0x_f - \left(\frac{1}{2}kx_i^2 - kx_0x_i\right) + \frac{1}{2}kx_0^2 - \frac{1}{2}kx_0^2$$

which we can rearrange as

$$\int_{x_i}^{x_f} -k\left(x - x_0\right)dx = \left(\frac{1}{2}kx_f^2 - kx_0x_f - \frac{1}{2}kx_0^2\right) - \left(\frac{1}{2}kx_i^2 - kx_0x_i - \frac{1}{2}kx_0^2\right)$$

and we can factor the quadratic terms

$$\begin{aligned}
\int_{x_i}^{x_f} -k\left(x - x_0\right)dx &= \left(\frac{1}{2}k\left(x_f - x_o\right)\left(x_f - x_o\right)\right) - \left(\frac{1}{2}k\left(x_i - x_o\right)\left(x_i - x_o\right)\right) \\
&= \frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_i - x_o\right)^2
\end{aligned}$$

then substituting in our results for the right and left hand sides our energy equation is

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_i - x_o\right)^2$$

The right hand side we recognize as

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = K_f - K_i$$

so the left hand side must be our spring energy change as we move the spring from an initial position x_i to a final position x_f

$$\frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_i - x_o\right)^2 = U_{sf} - U_{si}$$

Then grouping initial and final terms gives

$$\frac{1}{2}mv_i^2 - \frac{1}{2}k\left(x_i - x_o\right)^2 = \frac{1}{2}mv_f^2 - \frac{1}{2}k\left(x_f - x_o\right)^2$$

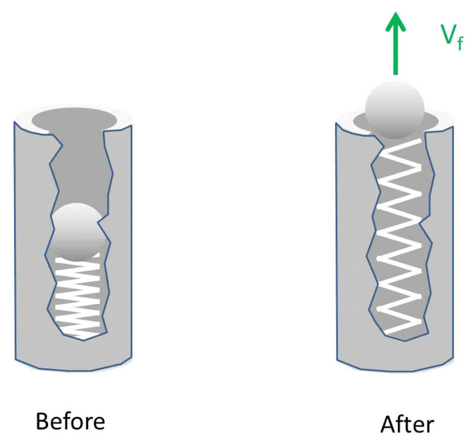
which is

$$K_i + U_{si} = K_f + U_{sf}$$

We have identified that the potential energy of a spring is given by

$$U_s = \frac{1}{2}k\left(x - x_o\right)^2$$

Let's try a problem with this.



Our spring cannon gives a muzzle velocity of about $5.1\,\mathrm{m/s}$. We found this last time. Let's calculate the spring constant for our spring. But this time, instead of using our work equations, let's use the spring potential energy. The spring is compressed $0.05\,\mathrm{m}$ when the cannon is loaded. The initial speed of the ball is $0\,\mathrm{m/s}$. The final position of the spring is at it's equilibrium point. Recall that our ball has a mass of $67.1\,\mathrm{g}$.

We know

$$\begin{aligned} v_i &= 0 \\ v_f &= 5.1\frac{\mathrm{m}}{\mathrm{s}} \\ y_o &= 0 \\ y_i &= -0.05\,\mathrm{m} \\ y_f &= y_o \\ m &= 67.1\,\mathrm{g} = 0.067\,1\,\mathrm{kg} \\ g &= 9.8\frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Our basic equations are

$$K_i + U_{gi} + U_{si} = K_f + U_{gf} + U_{sf}$$
$$K = \frac{1}{2}mv^2$$
$$U_s = \frac{1}{2}k\left(y - y_o\right)^2$$

We can write out our conservation of energy equation as

$$\frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}k\left(y_i - y_o\right)^2 = \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}k\left(y_f - y_o\right)^2$$

and use our zeros

$$0 + mgy_i + \frac{1}{2}k\left(y_i - 0\right)^2 = \frac{1}{2}mv_f^2 + 0 + \frac{1}{2}k\left(y_f - y_o\right)^2$$
$$mgy_i + \frac{1}{2}k\left(y_i\right)^2 = \frac{1}{2}mv_f^2 - 0$$

so

$$mv_f^2 = 2mgy_i + k\left(y_i\right)^2$$
$$\frac{mv_f^2 - 2mgy_i}{\left(y_i\right)^2} = +k$$

so

$$k = \frac{m\left(v_f^2 - 2gy_i\right)}{\left(y_i\right)^2}$$

$$\begin{aligned} k &= \frac{\left(0.067\,1\,\mathrm{kg}\right)\left(\left(5.1\frac{\mathrm{m}}{\mathrm{s}}\right)^2 - 2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(-0.05\,\mathrm{m}\right)\right)}{\left(-0.05\,\mathrm{m}\right)^2} \\ &= 724.41\frac{\mathrm{N}}{\mathrm{m}} \end{aligned}$$

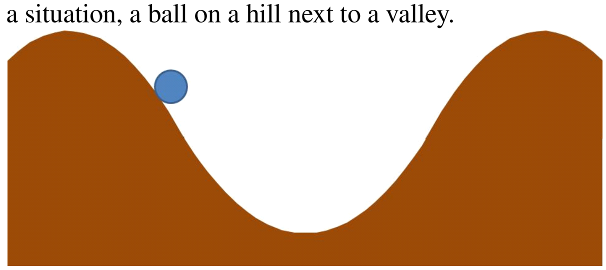
Adding in springs and spring forces and spring potential energy allows us to design wonderful things from dart guns to car shocks to optical isolation systems we use on big laser systems.

We will continue to look at springs and energy in our next lecture.

Conservation of Energy and Energy Graphs

We spent a lot of time learning to interpret motion diagrams and position, velocity and acceleration vs. time graphs. Using these graphs and diagrams we could understand how something moved. We also drew diagrams for forces. And for energy, we need to learn to draw *before* and *after* pictures to find the initial and final speeds and positions. But energy is such a useful way to look at motion that there are some standard ways to show energy for a system and we will need to be able to use these ways to depict energy.

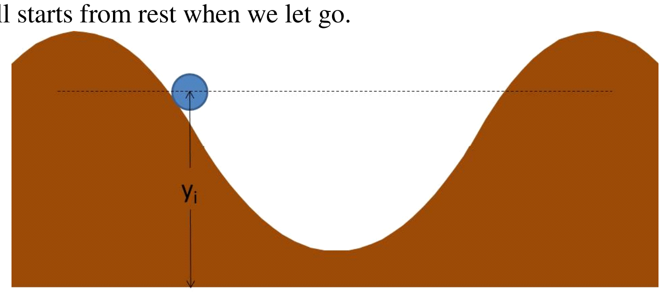
Let's start with a situation, a ball on a hill next to a valley.



We know a lot about this situation. The ball will have an initial energy

$$\begin{aligned} E_i &= K_i + U_i \\ &= \frac{1}{2}mv_i^2 + mgy_i \\ &= mgy_i \end{aligned}$$

since the ball starts from rest when we let go.



So the initial energy is all potential energy and that potential energy depends on the initial height of the ball. We know that if there is nothing to take energy away from the system, then energy is conserved

$$E_f = E_i$$

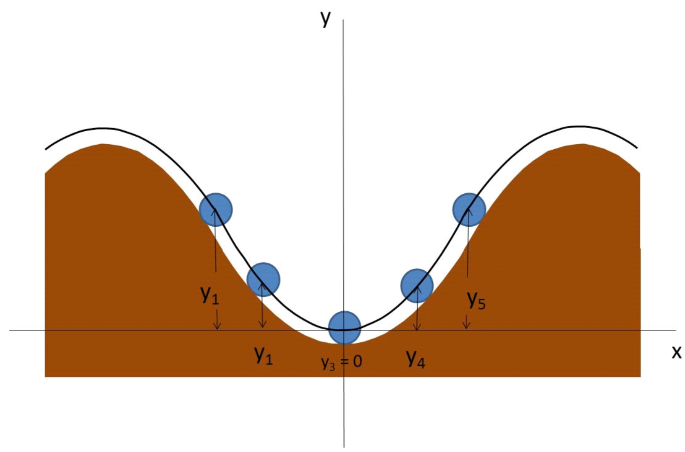
So at some time later we will have a final energy

$$E_f = \frac{1}{2}mv_f^2 + mgy_f$$

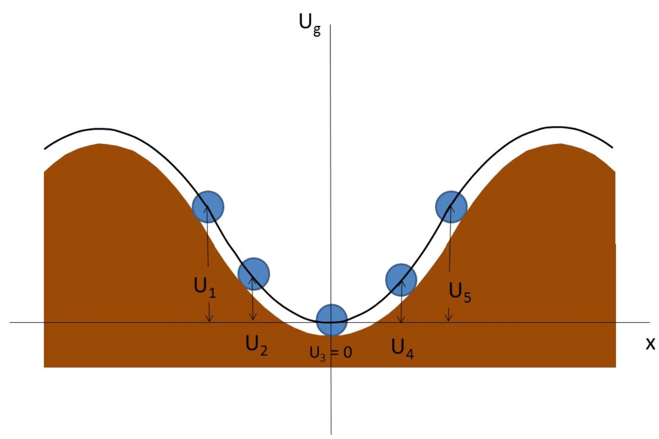
and this final energy must be equal to the initial energy

$$mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

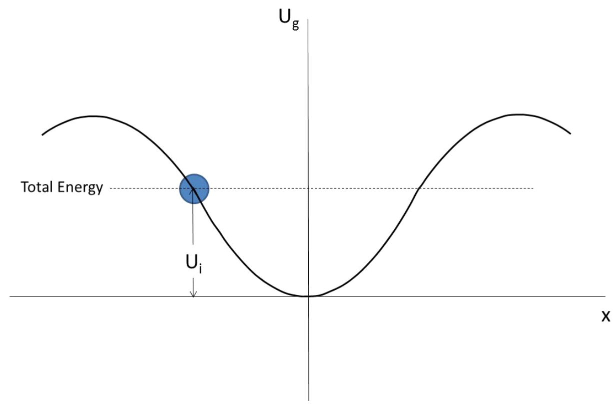
Let's think about how high up the ball will be as the ball travels down the hill and through the valley to the other side. The potential energy, U_g depends on the height of the ball. We can plot the ball height for each x-position along the ball path.



Since the gravitational potential depends on the y-position. a graph of the gravitational potential vs. x should look like the position (y vs. x) graph along the ball's path.



Of course, we could draw the graph without the hill



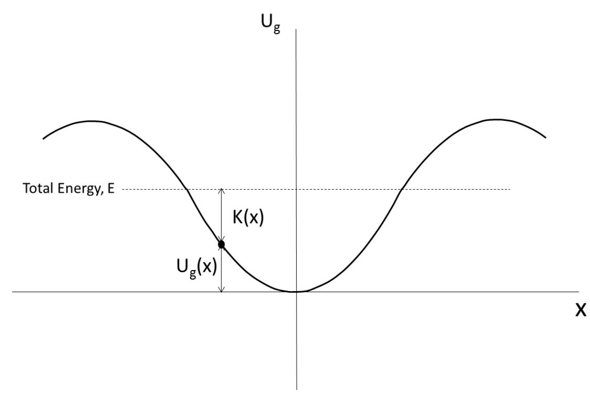
This type of graph is very commonly used to describe the energy of a system. Notice that if we plot $U_g(x)$ and we also plot the total energy, $E(x)$ then from the graph we can always find the kinetic energy too because

$$E(x) = K(x) + U_g(x)$$

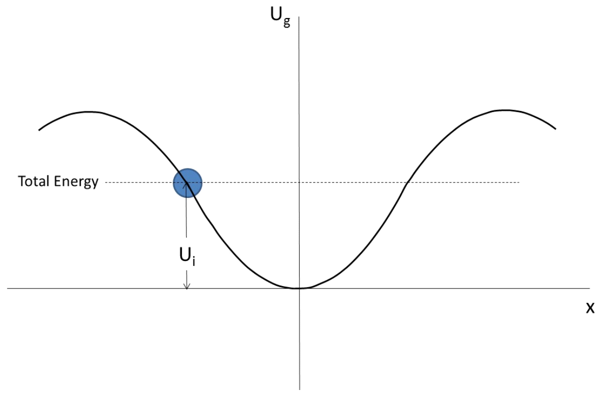
so

$$K(x) = E(x) - U_g(x)$$

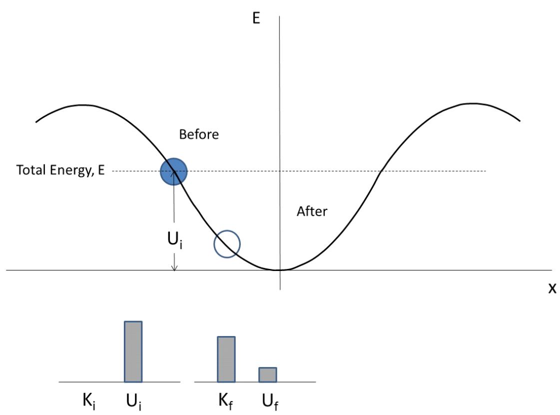
so one graph shows the entire energy situation for the system!



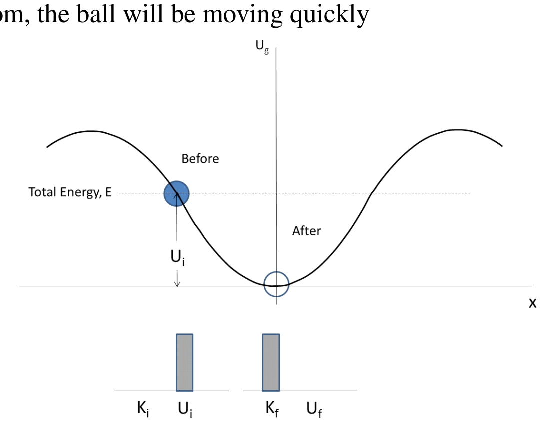
Since the kinetic and potential energies must add up to the total energy $E(x) = K(x) + U_g(x)$ we could plot the energy at a point in another way



We could use a bar chart to tell us how much of the energy is potential energy at a given x-position and how much is kinetic energy at that x-position. Notice in the figure, when $U_g(x) = E(x)$ then $K(x)$ is zero. And the bar chart shows that. After we let go of the ball, the amount of potential energy decreases and the amount of kinetic energy increases.



Right at the bottom, the ball will be moving quickly



We can see from the bar graph that all of the potential energy has been changed to kinetic energy. Remember that

$$K(x) = \frac{1}{2}mv(x)^2$$

so

$$v(x) = \sqrt{\frac{2K(x)}{m}}$$

and the speed will be fastest right at the bottom.

If the ball is allowed to keep going it will roll up the other side of the valley. We could predict that it will stop at the same height on the other side of the valley.

$$U_g(x_f) = E(x_f) - K(x_f)$$

so when $v(x_f) = 0$ then $K(x_f) = 0$ so

$$U_g(x_f) = E(x_f)$$

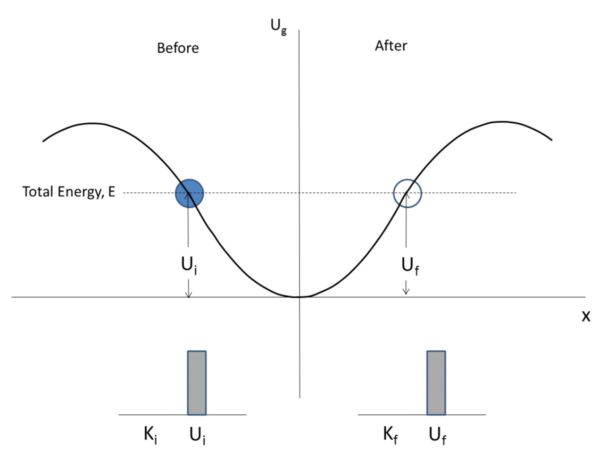
and we started with

$$E(x_f) = U_i = mgy_i$$

so

$$U_g(x_f) = mgy_i$$

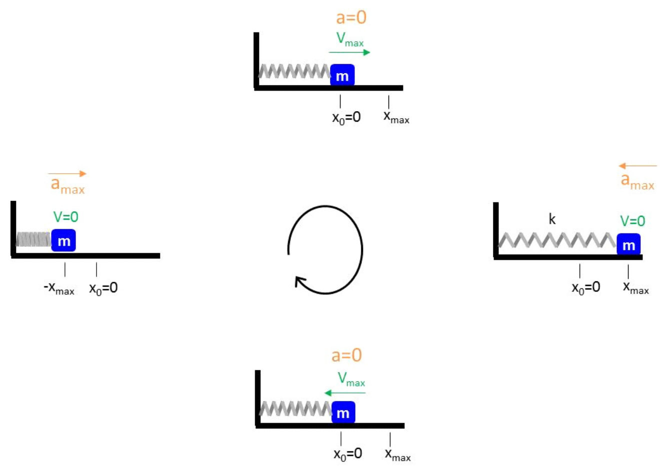
so the ball will stop (brie y) when it reaches the same height on the other side.



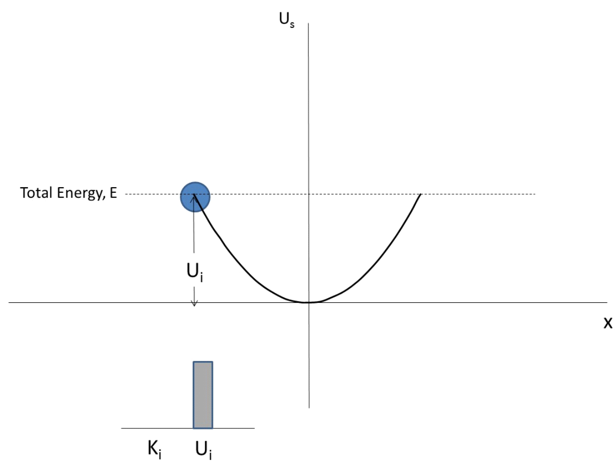
If we let the ball keep going, it will roll back down into the valley and go back and forth over and over again.

Spring potential and energy graphs

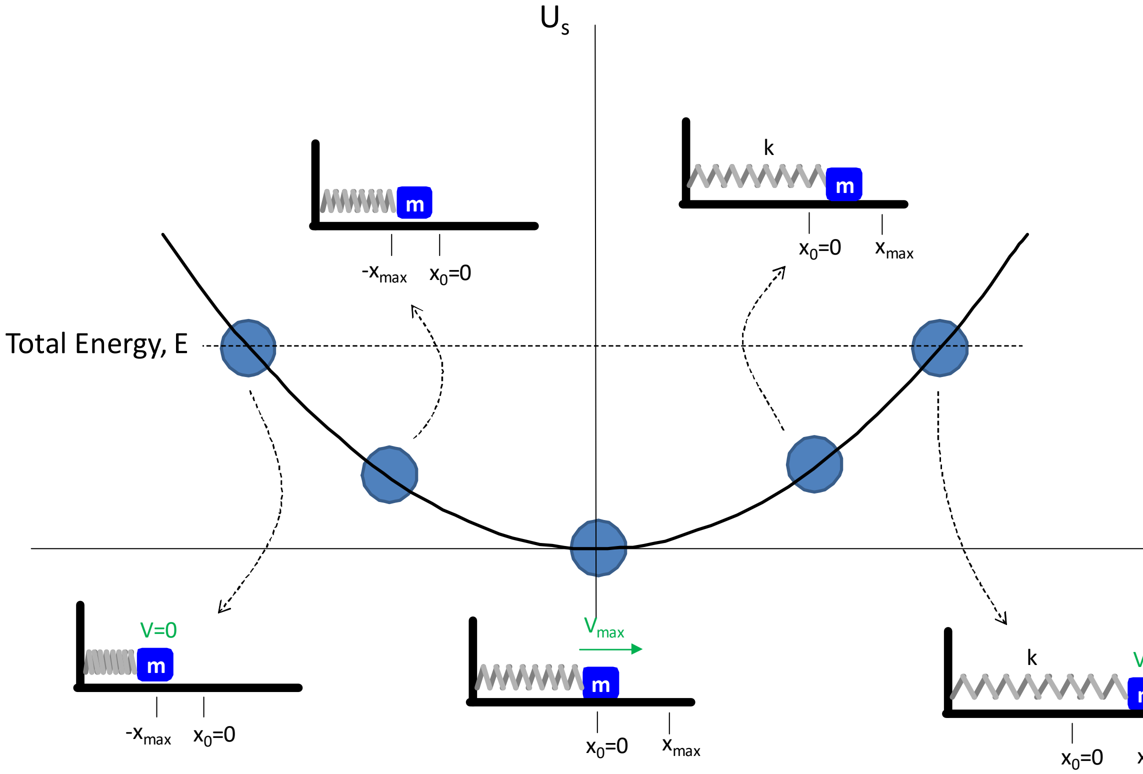
Let's try using our new graphs for spring potential energy. Consider what would happen if we compress a spring and let go.

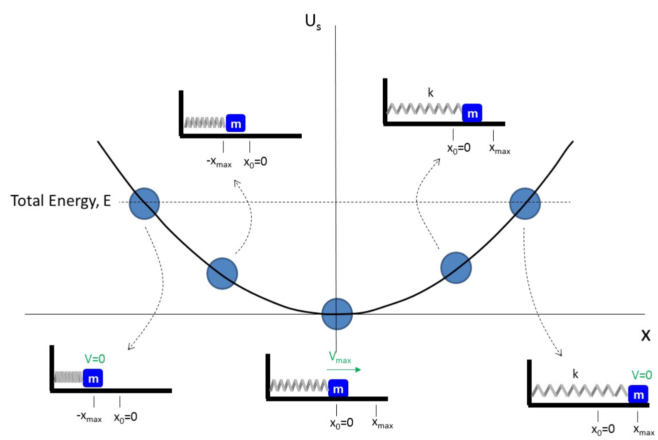


This is a little bit like letting the ball go on the hill by the valley

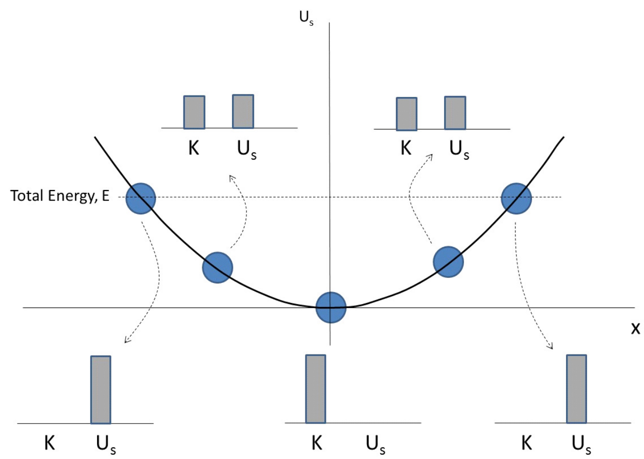


At first, we have all spring potential energy. But quickly this spring potential energy starts to change into kinetic energy.





The mass picks up speed, so it gains kinetic energy and loses potential energy. At the midpoint the mass has all kinetic energy and no potential energy. But it has momentum so it is hard to stop! So it will keep going and it will start stretching the spring, building up potential energy and slowing down so there will be less kinetic energy.

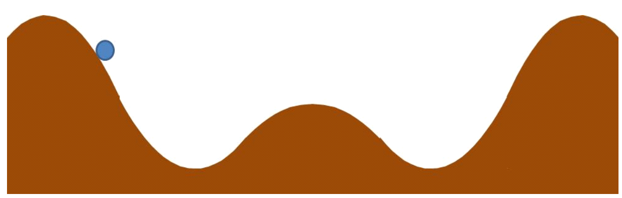


Notice how similar this set of graphs is to the gravitational potential energy set. Since the graphs look so much the same, we can use gravitational potential energy to help us gain intuition into other types of potential energy and how the objects that experience the potential energy will act.

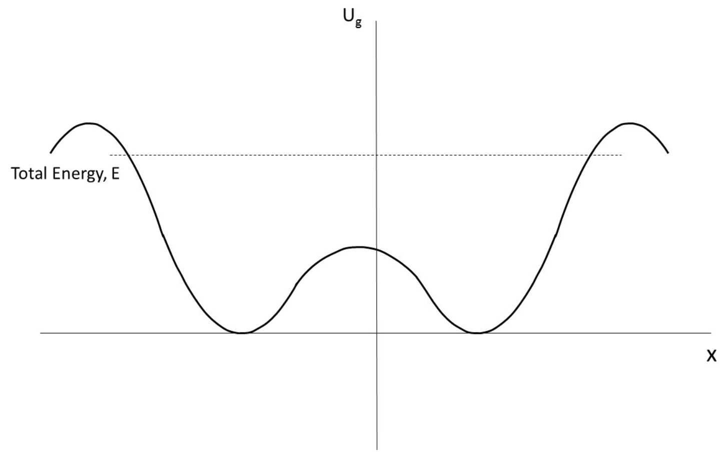
In the case of the spring, we can see that the mass will go back and forth, a little like the ball went back and forth between the two hills.

Equilibrium points

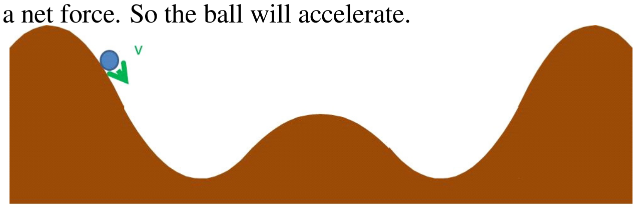
Suppose we have a more complicated hill and valley system.

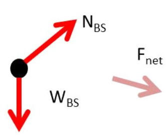


We recognize that our gravitational potential energy graph would have the same shape as the hill/valley system.

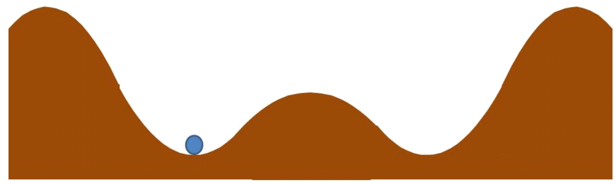


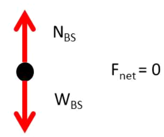
Let's relate our potential energy graphs to Newton's laws. At the point where the ball starts there is a net force. So the ball will accelerate.



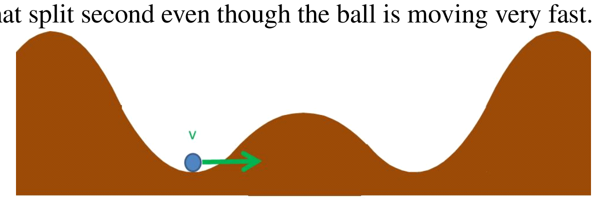


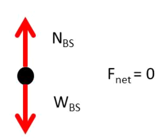
But suppose we start the ball at the bottom of the valley.



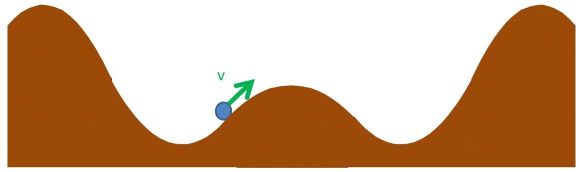


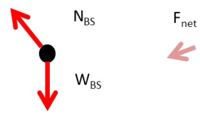
The condition $\overrightarrow{F}_{net} = 0$ is what we have called equilibrium. Notice that our force diagram would be the same if we started the ball on the hill and let it go down to the valley. The forces would be the same at the bottom of the valley, but now with the ball moving. Still the bottom of the valley is an equilibrium. The equation $\overrightarrow{F}_{net} = 0$ tells us that we are not accelerating, but it does not tell us if we are moving. Right at the bottom of the hill the ball is not speeding up or slowing down for a split second. So, indeed, $\overrightarrow{F}_{net} = 0$ or that split second even though the ball is moving very fast.



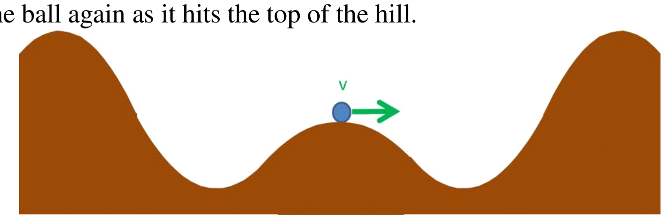


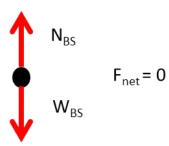
As the ball goes up the little hill, it will slow down, and a free body diagram can shown us why. There is a net force, so there is an acceleration in the opposite direction the ball is going.





Let's join the ball again as it hits the top of the hill.





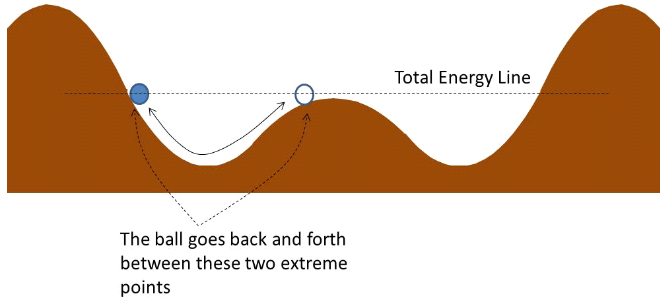
Once again the net force is zero. Of course the ball is moving. We can tell that the middle hill is lower than the side hills, so

$$K_i + U_i = K_{lh} + U_{lh}$$

where the subscripts "lh" are for "little hill." Then

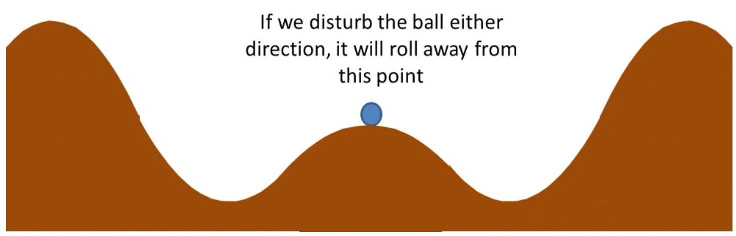
$$\begin{aligned} K_{lh} &= K_i + U_i - U_{lh} \\ &= 0 + mgy_i - mgy_f \\ \frac{1}{2}mv_{lh}^2 &= mg\left(y_i - y_f\right) \end{aligned}$$

so since the hills have different heights the speed of the ball won't be zero. We can see that this is another equilibrium point. But what would happen if we started the ball lower on the hill?



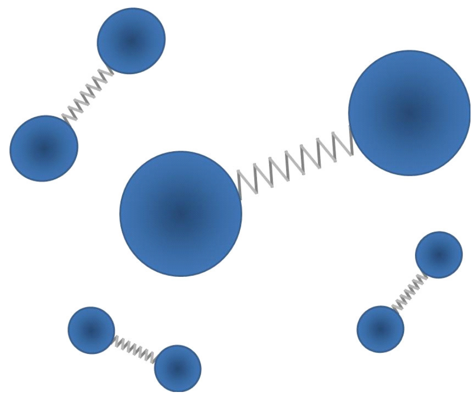
We already know the ball will only go up as high as it started. So if we start it lower than the top of the middle hill, the ball will roll back and forth between the left hill and the middle hill. If we allowed friction in our problem, eventually the ball would stop right at the bottom of the valley. If we gave the ball a small kick, again it would return to the bottom of the valley. This equilibrium position is an important one, we call it a *stable equilibrium* because an object with this particular potential energy won't move away from that point or will return to that point if disturbed.

Let's consider our other equilibrium. If we place the ball on the top of the middle hill, it is in equilibrium, but any disturbance either direction will result in a net force away from the top-of-the-hill equilibrium position.

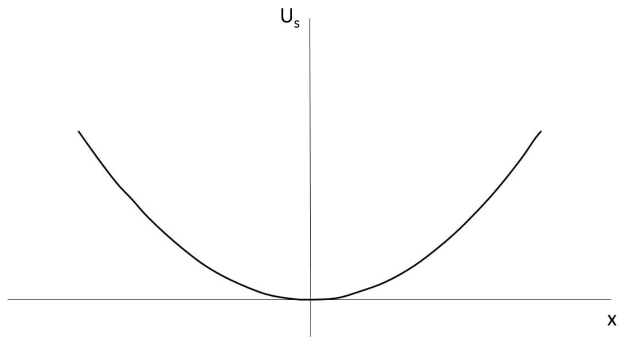


The ball would have to have someone go get it and put it back on the top of the hill. It won't return there of it's own accord. We call an equilibrium point where the object would leave the point permanently if disturbed an *unstable equilibrium.*

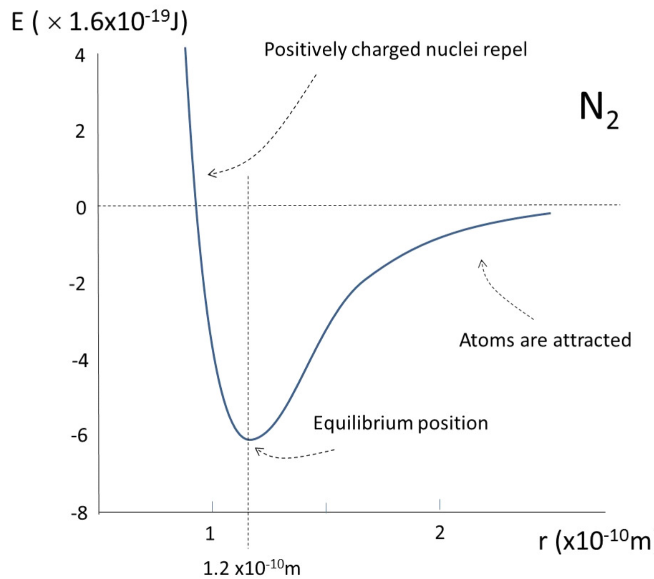
Let's use what we have learned to take on a real problem. Our atmosphere is mostly made of diatomic nitrogen or N_2. We have used spring forces as a model for molecular bonds. So let's picture the two nitrogen atoms as tied together by a spring.



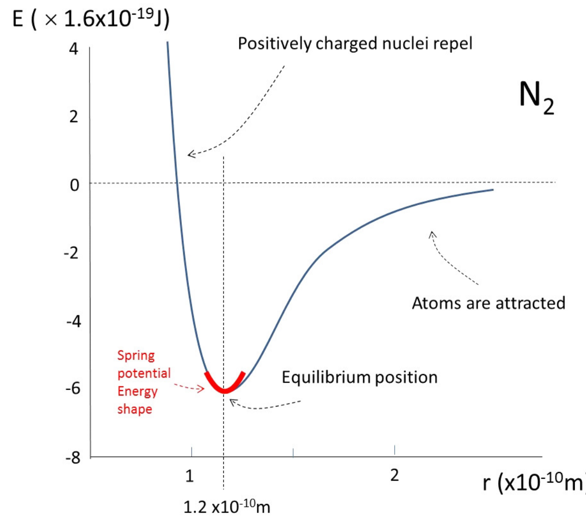
As the spring is compressed, the spring pushes back. As the spring is stretched it pulls back. So we expect a potential energy that looks like our spring potential energy.



But there is a little more to our molecule. We know that if we pull the two atoms hard enough, we can pull the molecule apart. So as the distance between the atoms gets larger, the potential energy must show that we can get the atoms apart. If we push the atoms closer together the strong electrical repulsive force between the two nuclei will make it harder and harder to put get the atoms closer together. So the potential energy must show that it is hard to force the atoms together. Here is what the potential energy graph looks like for one possible diatomic nitrogen molecule.



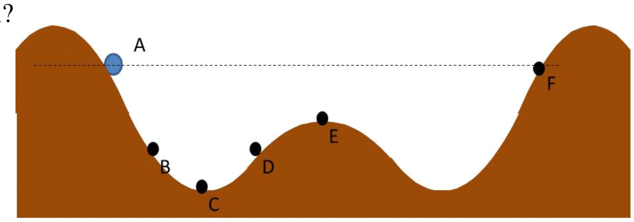
Notice that the potential energy graph is not symmetric. The left hand side shows that the potential energy gets very large as we push the atoms together. We could mentally think of this as leaving one atom in place and pushing the other atom toward the first. The moving atom is like our ball and the potential energy from the electrical force is like the hill. It makes a potential energy. So moving our mover atom toward the environmental atom is like moving the ball up a very steep hill. On the right hand side, we can see that we have a much less steep hill. This is part of the graph that tells us that we could break the molecular bond by pulling our mover atom farther away. It is going "up hill," but much less slowly. Now observe that in between the two sides there is a part of the graph that looks a lot like the spring potential energy. Here is our N_2 graph again but with a spring potential energy graph superimposed over the equilibrium point of our N_2 grapy.



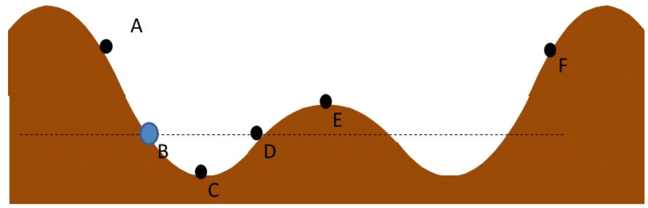
Notice that it fits pretty well. Then we could predict that if we pull on one of the atoms just a little, the bond would pull it back. It is like our ball near the stable equilibrium point. The atom would oscillate back and forth but never get very far from the equilibrium position. If we allow a friction-like force to remove the extra energy, then the atom would settle right back to the equilibrium position.

Turning points

Let's go back to our hill. If we start the ball from rest as shown, where will the ball stop on the far hill?



We now know that this will be at point F, when the energy is all potential energy again. But at point F, the force due to gravity and the normal force don't point the same direction, so we will have a net force. The ball will roll back down the hill. Since the ball stopped, and then went the other direction, we can say that the ball turned around at this point. So we call point F a *turning point*. When the ball reaches it's original position, it will also turn around, so the initial position and point F are both turning points for this situation. Bus suppose we start the ball with less initial energy.



Now points B and D will be turning points.

26 Force and Potential Energy

Let's start this lecture with a review. Let's take our equation for spring potential energy

$$U_s = \frac{1}{2}k\left(s - s_o\right)^2$$

and consider a before and after case. Let's review our spring cannon case where we compress the spring and then let it go back to it's equilibrium position.

The initial potential energy of the spring in our case is

$$U_{si} = \frac{1}{2}k\left(x_i - x_o\right)^2$$

and

$$U_{sf} = \frac{1}{2}k\left(x_f - x_o\right)^2$$

but $x_f = x_o$ for this case, so

$$U_{sf} = \frac{1}{2}k\left(x_o - x_o\right)^2 = 0$$

so the change in spring potential energy is

$$\begin{aligned}
\Delta U_s &= U_{sf} - U_{si} \\
&= 0 - \frac{1}{2}k\left(x_i - x_o\right)^2 \\
&= -\frac{1}{2}k\left(x_i - x_o\right)^2
\end{aligned}$$

The spring has lost energy, so ΔU_s is negative. This is just the spring potential energy lost by the spring as it transfers it's stored energy into kinetic energy of the ball. But this is the same amount of energy as was done on the ball as work!

And this is a general result

$$\Delta U = U_f - U_i = -w$$

Another way to look at it is that to get ΔU stored in the spring, we had to do work to compress it. How much work would we have to do? again we would have

$$w_b = \int \overrightarrow{S}\left(x\right) \cdot \overrightarrow{dx}$$

where both S and dx are both negative, making the angle between them $\theta_{Sx} = 0$ again. We have the same integral. So we would have the same result, but with the limits changed.

$$w_b = \int_{x_o}^{x_f} -k\left(x - x_o\right)(1)\,dx$$

$$w_b = \left(-k\left(\frac{x^2}{2}\right)\right|_{x_o}^{x_f} + \left(kx_o x dx\right|_{x_o}^{x_f}$$

$$w_b = -k\left(\frac{x_f^2}{2}\right) - \left(-k\left(\frac{x_0^2}{2}\right)\right) + kx_o x_f - kx_o x_0$$

$$w_b = -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_o^2 + kx_o x_f - kx_o^2$$

$$w_b = -\frac{1}{2}kx_f^2 - \frac{1}{2}kx_o^2 + kx_o x_i$$

$$\begin{aligned} w_b &= -\frac{1}{2}k\left(x_f^2 - 2x_o x_f + x_o^2\right) \\ &= -\frac{1}{2}k\left(x_f - x_o\right)^2 \end{aligned}$$

But the change in potential energy as we compress the spring is

$$\begin{aligned} U_f - U_i &= \frac{1}{2}k\left(x_f - x_o\right)^2 - \frac{1}{2}k\left(x_o - x_o\right)^2 \\ &= \frac{1}{2}k\left(x_f - x_o\right)^2 \end{aligned}$$

and since x_f for the compression is equal to x_i for the launch,

$$U_f - U_i = -w_b$$

To gain an amount of spring potential energy ΔU_s by doing work on the spring we need an amount of work

$$\Delta U_s = -w$$

the work done is the inverse of the change in potential energy.

Force from potential energy

We know how to find potential energy now if we know the force causing that potential energy to exist. Because we know that $w = -\Delta U$, so we can write

$$\begin{aligned} \Delta U &= -w \\ &= -\int \overrightarrow{F}(s) \cdot \overrightarrow{ds} \end{aligned}$$

So if I know the force on a mover object, and I know how far I move the mover object I can find the change in potential energy. Let's see how this works.

Let's take the case of a falling ball. We know the potential energy for a falling ball is due to the gravitational force, and that the gravitational potential is mgy. Then we know the answer for this case, so we can study the procedure. Here it goes!

$$W_{bE} = mg$$

and the ball falls in the $-y$ direction, so

$$ds = -dy$$

then

$$\begin{aligned}
\Delta U &= -\int \overrightarrow{F}(s) \cdot \overrightarrow{ds} \\
&= -\int mg\,(-\hat{\jmath}) \cdot dy\,(-\hat{\jmath}) \\
&= -\int mgdy \cos\theta_{Wy}
\end{aligned}$$

where θ_{Wy} is the angle between the $-\hat{\jmath}$ and $-\hat{\jmath}$ directions. Since both F_g and ds are in the same direction the angle must be $\theta_{Wy} = 0$, and $\cos(0\,^{\circ}) = 1$, so

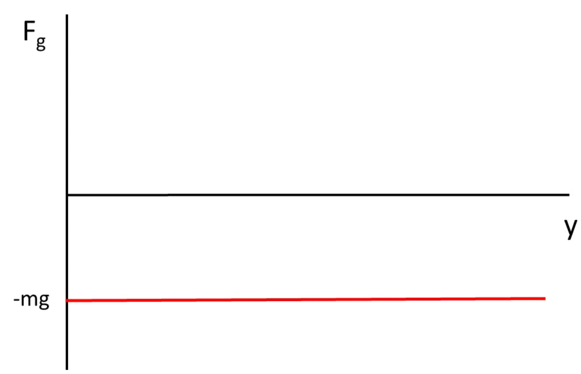
$$\Delta U = -\int mgdy$$

The mass and the acceleration are not changing, so we can take them out of the integral

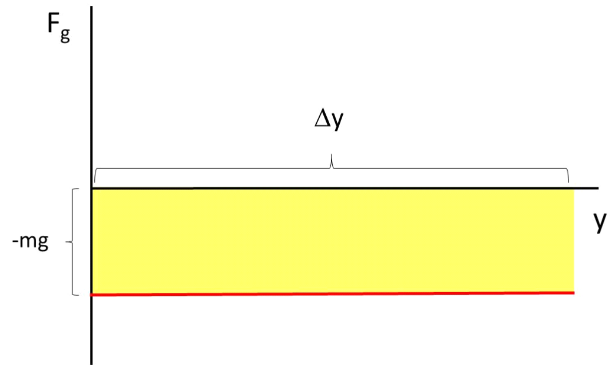
$$\begin{aligned}
\Delta U &= -mg \int dy \\
&= -\left(mgy\right|_{y_i}^{y_f} \\
&= -\left(mgy_f - mgy_i\right) \\
&= U_f - U_i
\end{aligned}$$

And we can see that we did, indeed, find the potential energy change for our mass.

Let's look at a graph of the situation. We learned earlier that an integral is a way to find the "area" under a curve. So let's graph force vs. y



The "area" would be



$$\text{“A”} = -mg \times \Delta y$$

which is just what we expect for our potential energy difference.

$$\Delta U = -mg\Delta y$$

It would also be good to find the force if we know the potential energy change. Let's go back to our relationship between force and the change in potential energy.

$$\Delta U = -\int \overrightarrow{F}(s) \cdot \overrightarrow{ds}$$

The integrand is a small amount of potential energy change. The integral adds up many small changes to get the entire potential energy change. So we could say that

$$dU = -\overrightarrow{F}(s) \cdot \overrightarrow{ds}$$

where dU is a small change in potential energy.

$$dU = -F ds \cos\theta_{Fs}$$
$$dU = -F \cos\theta_{Fs} ds$$

We can again write

$$F \cos(\theta_{Fs}) = F_s$$

so that

$$dU = -F_s ds$$

Then

$$F_s = -\frac{dU}{ds}$$

where all along we have used our generic component direction s. We could have used x, or y

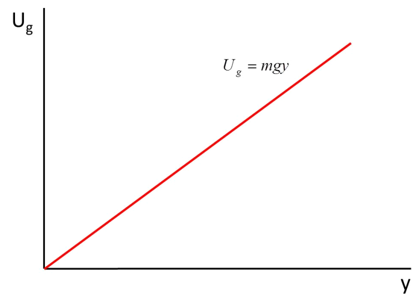
$$F_x = -\frac{dU}{dx}$$
$$F_y = -\frac{dU}{dy}$$

We can try this again for our falling ball. If we know $U_g = mgy$ then

$$\begin{aligned} F_y &= -\frac{d}{dy}(mgy) \\ &= -mg \end{aligned}$$

which is just right!

We should study this graphically to understand it better. If we plot U_g as a function of y, we get a graph that looks like this



This is not too surprising. Consider the equation for a straight line

$$y = mx + b$$

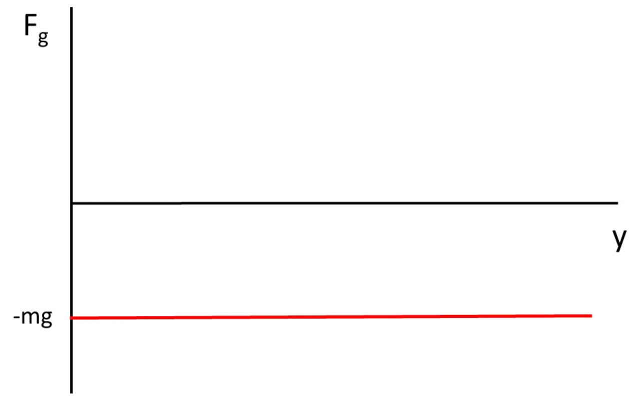
and we see that with the y-axis now the U_g-axis and with the x-axis now the y-axis. then

$$\begin{aligned} y &= mx + b \\ Ug &= mg(y) + 0 \end{aligned}$$

then we can see that if the slope of the curve is mg, we have a straight line that goes through the origin.

Now that we are more familiar with calculus, we easily recognize that dU/dt is the slope of the U vs. y graph.

$$\begin{aligned} F_y &= -\frac{dU}{dy} \\ &= -mg \end{aligned}$$



This was pretty easy for a constant force like the force due to gravity. But it would be more complicated for a spring force. Let's try it!

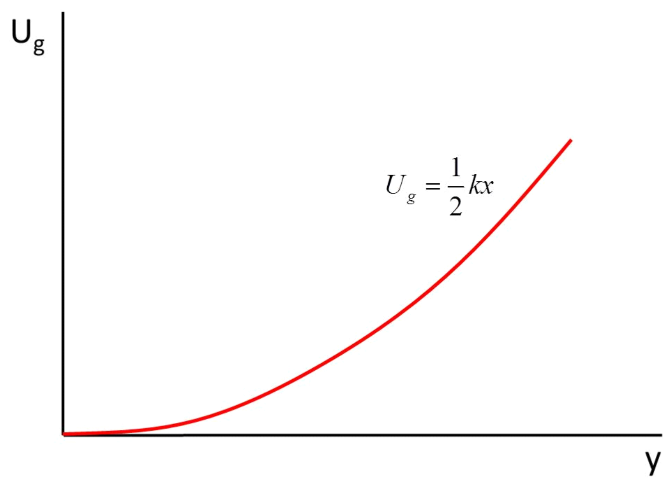
We know that

$$U_s = \frac{1}{2}k\left(x - x_o\right)^2$$

let's choose our origin so that $x_o = 0$ to make the math easy, then

$$U_s = \frac{1}{2}k\left(x\right)^2$$

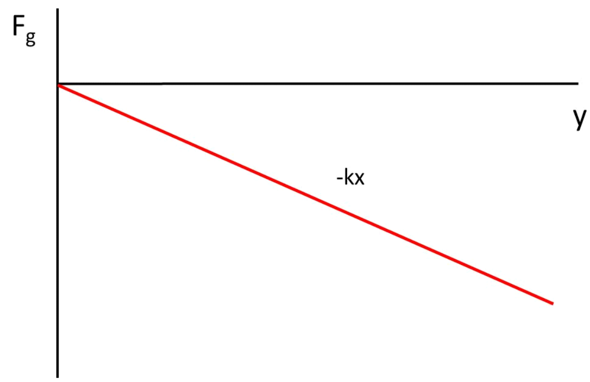
The graph looks like this



then the force would be

$$\begin{aligned} F_x &= -\frac{dU_s}{dx} \\ &= -\frac{d}{dx}\left(\frac{1}{2}k\,(x)^2\right) \\ &= -\frac{1}{2}\,(2kx) \\ &= -kx \end{aligned}$$

The plot looks like this.



And we are have found the spring force! We will use these techniques again in PH123 and PH220. We have solved our problem of non-constant forces. Next lecture we will take on frictional forces.

Work-Energy Theorem,

We already know the work energy theorem

$$w = \Delta K$$

but we want to include our idea of potential energy into this basic equation, and we have a missing piece to fill in.

Work-energy and potential energy

Now that we have the concept of thermal energy, we can begin to give some detail to our work-energy equation.

$$w = \Delta K$$

First let's split our work term into two terms, a term for conservative work and a term for non-conservative work.

$$w_c + w_{nc} = \Delta K$$

The non-conservative work takes energy out of the mechanical system and turns it into thermal energy. We observe this by noting that friction makes temperature go up. So w_{nc} will cause a change in thermal energy. Let's give thermal energy a symbol E_{th}, so the change in thermal energy is

$$\Delta E_{th} = -w_{nc}$$

The minus sign may seem mysterious, but remember that to get the same ΔK the conservative work would have to be larger to overcome the non-conservative work due to friction, so really the minus sign is real. Another way to look at this is that the energy due to friction work is lost to the mechanical system. If we have friction, we lose some energy so, like in our ball-hill/valley example, ΔK will be smaller. Since it is a loss, w_{nc} must be negative. Then

$$w_c - \Delta E_{th} = \Delta K$$

Let's also think about what forces are conservative. We have the gravitational force, and spring forces. And for these forces, we have learned that

$$\begin{aligned} \Delta U_g &= -w_g \\ \Delta U_s &= -w_s \end{aligned}$$

that is, the energy stored as potential energy is minus the work done to store the energy. We could split our w_c into specific types of work

$$w_c = w_g + w_s$$

then the work energy theorem would be

$$w_g + w_s - \Delta E_{th} = \Delta K$$

or, using potential energy

$$-\Delta U_g - \Delta U_s - \Delta E_{th} = \Delta K$$

Usually we prefer to not have negative signs, so let's take all the negative terms to the other side of the equation

$$0 = \Delta K + \Delta U_g + \Delta U_s + \Delta E_{th}$$

This is a very exciting equation (really it is!). It tells us that if we find the change in all these energy terms, these changes sum to zero. Another way to write this is

$$0 = K_f - K_i + U_{gf} - U_{gi} + U_{sf} - U_{si} + E_{th_f} - E_{th_i}$$

or

$$K_i + U_{gi} + U_{si} + E_{th_i} = K_f + U_{gf} + U_{sf} + E_{th_f}$$

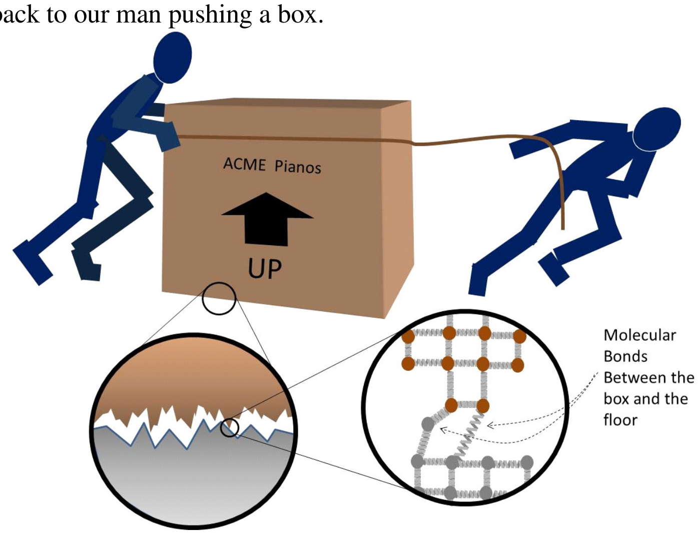
which is conservation of energy again! But this time we have included friction and similar dissipative terms. Note that we have included a term for all the possible types of energy we have studied. In PH220 we will find that there are more conservative forces (e.g. ones for electrical forces) which will also create potential energies. And we can just add these into our conservation of energy equation.

$$K_i + U_{gi} + U_{si} + U_{ei} + E_{th_i} = K_f + U_{gf} + U_{sf} + U_{ef} + E_{th_f}$$

This will become a general procedure, if you have a new kind of energy in your problem, just add it into our conservation of energy equation until all forms of energy for the problem are accounted for.

But is energy always conserved? Is this equation always true?

Let's go back to our man pushing a box.



Some of the thermal energy generated by friction increases the temperature of the box, but some of the energy increases the temperature of the floor. So if we include the floor in our system, energy is conserved, but if we only consider the box, the energy is not conserved. Energy is leaving the box.

In fact, if we just consider the box, the push and pull from the guy and rope are adding energy to the box system. So we have energy coming in and some energy coming out and we would have to account for both of these before we could say that energy is conserved for the box or not. Generally if we allow external forces to act on our system we would have to guess that energy will not be conserved for the system. After all, we are adding or subtracting energy from our system (the box). But if we consider a system that includes more of the environment (say, the box, the guys, the rope, and the floor), it would be more likely that for the larger system energy would be conserved. If a system has no external forces acting on it, we call it an *isolated system*. For isolated systems our conservation of energy equation always works.

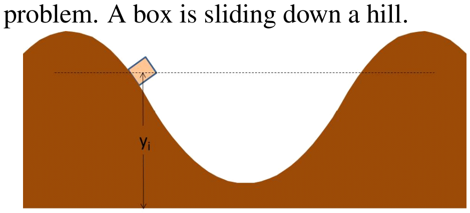
But as long as we account for every external energy input and every external energy dissipation in our energy equation, we can use it.

$$K_i + U_{gi} + U_{si} + U_{ei} + U_{push} + E_{th_i} = K_f + U_{gf} + U_{sf} + U_{ef} + E_{th_f}$$

You might think this distinction of external forces and systems is somewhat artificial, after all we are choosing our system and not nature, so, you might ask, if we take the ultimate system, the universe, is energy conserved? We believe that it is!

In doing energy problems, then, we have to be sure to identify all the types of energy that our system has. We can use bar energy graphs and energy vs. position graphs to aid in this process. Then we write our energy equation to include each type. Then we use algebra to solve for the item we are looking for to finish the problem.

Let's try an example problem. A box is sliding down a hill.



The box starts from rest. The hill is $20.\,\mathrm{m}$ high. The box has a mass of $20.\,\mathrm{kg}$. The box is going $10.\,\mathrm{m/\,s}$ at the bottom of the valley. How much energy was lost due to friction?

This is a conservation of energy problem but with friction.

We know

$$\begin{aligned} y_i &= 20\,\mathrm{m} \\ y_f &= 2\,\mathrm{m} \\ v_i &= 0 \\ v_f &= 10.0\frac{\mathrm{m}}{\mathrm{s}} \\ m &= 20.\,\mathrm{kg} \end{aligned}$$

Our basic equation is

$$K_i + U_{gi} + U_{si} + E_{th_i} = K_f + U_{gf} + U_{sf} + E_{th_f}$$

but we don't have any springs, so we can cancel all the spring potential terms (and any other terms for forces we don't have).

$$K_i + U_{gi} + E_{th_i} = K_f + U_{gf} + E_{th_f}$$

with

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ U_g &= mgy \end{aligned}$$

and we know so

$$K_i + U_{gi} = K_f + U_{gf} + E_{th_f} - E_{th_i}$$
$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f + \Delta E_{th}$$

Let's use our zero

$$0 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f + \Delta E_{th}$$

then

$$mgy_i - mgy_f - \frac{1}{2}mv_f^2 = \Delta E_{th}$$
$$mg\left(y_i - y_f\right) - \frac{1}{2}mv_f^2 = \Delta E_{th}$$
$$\Delta E_{th} = m\left(g\left(y_i - y_f\right) - \frac{1}{2}v_f^2\right)$$

$$\begin{aligned} \Delta E_{th} &= (20.\,\mathrm{kg})\left(\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(20\,\mathrm{m} - 2\,\mathrm{m}) - \frac{1}{2}\left(10.0\frac{\mathrm{m}}{\mathrm{s}}\right)^2\right) \\ &= 2528.0\,\mathrm{J} \end{aligned}$$

27 Momentum

Let's consider a situation. You are moving a water barrel on a hand truck. You exert a force, and the massive barrel of water accelerates.

But as you push the barrel seems to be easier to accelerate. As you look behind you, you realize why. The barrel is leaking. We know

$$\overrightarrow{F}_{net} = m\overrightarrow{a}$$

but now we realize that m could change for an object. We have not dealt with changing mass yet.

To see how to account for the possibility that mass could change, let's review our basic motion equations. Remember that

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$$

that is, our acceleration is a change in velocity in a change in time. We have dealt with forces leading to a change in velocity. Let's put the change of velocity with respect to time in our Newton's second law equation.

$$\overrightarrow{F}_{net} = m\frac{d\overrightarrow{v}}{dt}$$

This version of Newton's second law still does not allow the mass to change. But suppose we take the mass inside the derivative.

$$\overrightarrow{F}_{net} = \frac{d\left(m\overrightarrow{v}\right)}{dt}$$

Now the two things that affect the result of a force are both inside the derivative. If you have taken FDMAT112, you will remember that we could write this as a total derivative

$$\overrightarrow{F}_{net} = m\frac{d\overrightarrow{v}}{dt} + \overrightarrow{v}\frac{dm}{dt}$$

and we can clearly see both effects. The first term says if we keep the mass the same, a stronger force gives a larger acceleration. The second says that if we change the mass, say, by leaking water out of our barrel, that it takes less force to have the same velocity. And of course we could change both mass and velocity at once, and both would effect the result of our force.

Newton wanted to combine mass and change in velocity. After all, they are the two things that together relate a force to the motion of an object. Let's follow Newton's lead and define a new quantity

$$\overrightarrow{p} = m\overrightarrow{v}$$

It has all the things that make motion difficult to change. Consider a bull elephant. The elephant might not really move too fast, but because it is big, it is hard to stop. The mass makes it hard to change the motion of the elephant. But also consider a speeding bullet. The bullet's mass is small, but it's velocity is large. The large velocity makes it hard to change the motion of the bullet. In both cases it takes a large force to change the motion.

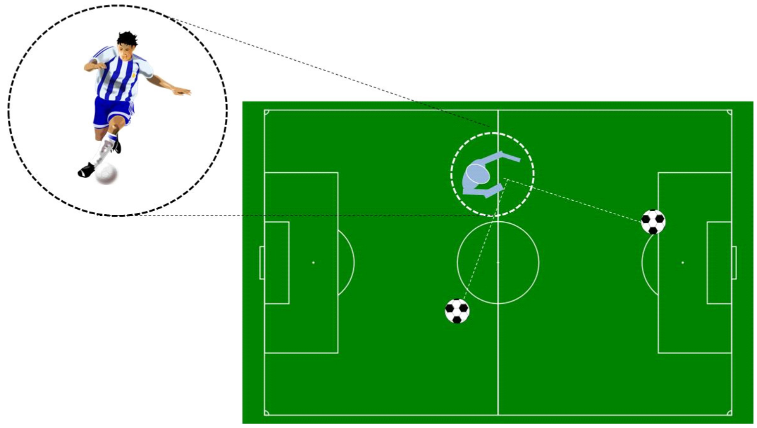
This combined quantity has a name. It is *momentum*.

We could write Newton's second law as

$$\overrightarrow{F}_{net} = \frac{d\overrightarrow{p}}{dt}$$

and indeed, this is how Newton wrote Newton's second law! We should add this form of Newton's second law to our Newton's second law set of equations.

Let's consider what happens when a soccer (futbol) player kicks a ball. The ball comes toward the player. The player exerts a force, a kick, and the motion of the ball changes.



We will deal with the details of the kick in a minute. But let's consider the change in motion. Before the kick we have one momentum, $\overrightarrow{p}_i$ and after the kick we have another momentum, $\overrightarrow{p}_f$ We could define a change in momentum

$$\Delta\overrightarrow{p} = \overrightarrow{p}_f - \overrightarrow{p}_i$$

Then

$$\overrightarrow{F}_{ave} = \frac{\Delta\overrightarrow{p}}{\Delta t}$$

This is only the average force, because all we know is a "before" and "after" case. Recall that the average acceleration is given by

$$a_{ave} = \frac{\Delta v}{\Delta t}$$

Our average force is like this. Let's rewrite $\Delta\overrightarrow{p}$

$$\Delta\overrightarrow{p} = \overrightarrow{p}_f - \overrightarrow{p}_i$$

then with our average force this must be equal to

$$\Delta\overrightarrow{p} = \overrightarrow{p}_f - \overrightarrow{p}_i = \overrightarrow{F}_{ave}\Delta t$$

The term $\overrightarrow{F}_{ave}\Delta t$ combines the magnitude and direction of the force with the duration of the force. This combination could be useful. How long a force acts matters, so a strong force acting over a short time could have the same effect as a weaker force acting over a longer time. We have a name for the combination $\overrightarrow{F}_{ave}\Delta t$. It is called the *average impulse.* And it is given the symbol

$$\bar{J} = \overrightarrow{F}_{ave}\Delta t$$

So we can write a new equation for momentum and the average impulse

$$\overrightarrow{p}_f - \overrightarrow{p}_i = \bar{J}$$

This is our first version of what is called the *impulse-momentum theorem.* The name is not really that important. But the theorem says that if a force acts on an object for a duration Δt, the momentum of the object will change.

Let's try a quick problem.

Our soccer player has a ball kicked to him with an initial velocity of $v_{ix} = -10.0\,\mathrm{m/s}$. The soccer player kicked the ball back the opposite direction with a final speed of $v_{fx} = 20.0\,\mathrm{m/s}$. If the force of the kick lasts for $\Delta t = 0.005\,\mathrm{s}$, what is the average force during the kick? The soccer ball has a mass of $0.45\,\mathrm{kg}$.

To solve this we can use our basic equation for impuse-momentum

$$\overrightarrow{p}_f - \overrightarrow{p}_i = \bar{J}$$

or

$$\overrightarrow{p}_f - \overrightarrow{p}_i = \overrightarrow{F}_{ave}\Delta t$$

Of course we will turn this two-dimensional problem into two one-dimensional problems by taking components of the momentum vectors. We end up with two one-dimensional problem parts, one for the x-direction and one for the y-direction

$$p_{fx} - p_{ix} = F_{ave_x}\Delta t$$
$$p_{fy} - p_{iy} = F_{ave_y}\Delta t$$

But in our problem we only have x-values. So

$$\begin{aligned} p_{fx} - p_{ix} &= F_{ave_x}\Delta t \\ 0 &= 0 \end{aligned}$$

we know $\overrightarrow{p} = m\overrightarrow{v}$, so for our remaining x-equation we have

$$mv_{fx} - mv_{ix} = F_{ave_x}\Delta t$$

and finally we can find the average force

$$F_{ave_x} = \frac{mv_{fx} - mv_{ix}}{\Delta t}$$
$$F_{ave_x} = \frac{m\left(v_{fx} - v_{ix}\right)}{\Delta t}$$
$$\begin{aligned} F_{ave_x} &= \frac{(0.45\,\mathrm{kg})\,(20.0\,\mathrm{m/s} - (-10.0\,\mathrm{m/s}))}{0.005\,\mathrm{s}} \\ &= 2700.0\,\mathrm{N} \end{aligned}$$

Notice that the average force is very large! Usually interaction forces are so large that for the very small time that the interaction force operates, we can ignore other forces. The other forces operate, but they are small enough to ignore during our Δt. Before and after the interaction these other forces (like gravity, and friction) dominate. So we can't ignore them outside of the interaction time. But during Δt we usually can ignore other forces. This is an approximation, of course. And the approximation has a name. We call this leaving out of other forces during the interaction time Δt the *impulse approximation*.

Of course we did all this only with an average change. We could consider that the force changes in time

$$\overrightarrow{F} = \overrightarrow{F}(t)$$

then our Newton's second law would be

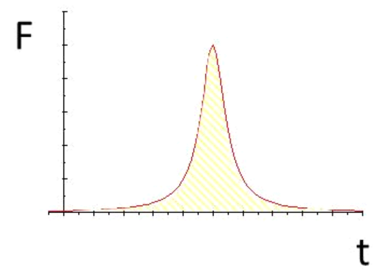
$$\overrightarrow{F}_{net}(t) = \frac{d\overrightarrow{p}}{dt}$$

which we could write as

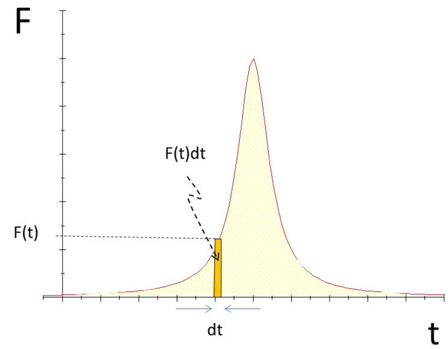
$$d\overrightarrow{p} = \overrightarrow{F}_{net}(t)\,dt$$

This says that for each moment, dt, of the small amount of impulse $\overrightarrow{F}_{net}(t)\,dt$ creates a small change in the momentum, $d\overrightarrow{p}$.

Let's look at a typical interaction, like our soccer ball kick.



We can see that the force does not stay constant. Our little bit of momentum change $d\overrightarrow{p}$ represents a little bit of impulse If we plot F vs. t, we can see that $\overrightarrow{F}_{net}(t)\,dt$ is the "area" of a little box. $F(t)$ is the height, and dt is the width. So an impulse is an area on a F vs. t graph.



To find all the impulse for an interaction, we would need to add up all the little bits of impulse.

Averages are funny things, it is perfectly legal to redefine the time of our average force to be the travel time Δt. What does this mean?

In the first graph we have the full definition of impulse

$$\int d\overrightarrow{p} = \int \overrightarrow{F}_{net}(t)\,dt$$

The left hand side gives us

$$\int d\overrightarrow{p} = \overrightarrow{p}_f - \overrightarrow{p}_i$$

so

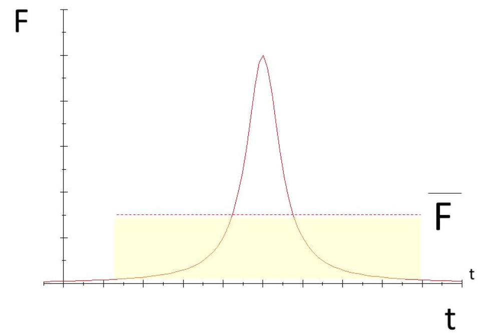
$$\overrightarrow{p}_f - \overrightarrow{p}_i = \int \overrightarrow{F}_{net}(t)\,dt$$

which says the impulse is the integral of the curve F vs. t graph, or, since an integral finds and "area," the impulse is the "area" under the curve of a F vs. t graph.

Let's go back and compare this to our average impulse equation.

$$\overrightarrow{p}_f - \overrightarrow{p}_i = \overrightarrow{F}_{ave}\Delta t$$

With what we know now, we can say that the average impulse is a rectangular region on a F vs. t graph that has the same area as $\int \overrightarrow{F}_{net}(t)\,dt$



Notice that the average force is at times very different from the actual inteaction force! But if we don't need the details of the interaction, the average often will do.

Total momentum of a system

So far we have just discussed the momentum of one object, like a soccer ball. But suppose we have a system, like a car, with lots of internal parts. The total momentum of the whole system is the vector sum of the momentum of each part.

$$\begin{aligned}\overrightarrow{P} &= \overrightarrow{p}_{part\,1} + \overrightarrow{p}_{part\,2} + \overrightarrow{p}_{part\,3} + \cdots + \overrightarrow{p}_{part\,\mathrm{N}} \\ &= \sum_{i=1}^{N} \overrightarrow{p}_{part\,\mathrm{i}}\end{aligned}$$

This is a little cumbersome to write, so let's abbreviate

$$\begin{aligned}\overrightarrow{p}_{part\,1} &= \overrightarrow{p}_1 \\ \overrightarrow{p}_{part\,2} &= \overrightarrow{p}_2 \\ &\vdots \\ \overrightarrow{p}_{part\,\mathrm{N}} &= \overrightarrow{p}_N\end{aligned}$$

so our total momentum would be

$$\begin{aligned}\overrightarrow{P} &= \overrightarrow{p}_1 + \overrightarrow{p}_2 + \overrightarrow{p}_3 + \cdots + \overrightarrow{p}_N \\ &= \sum_{i=1}^{N} \overrightarrow{p}_i\end{aligned}$$

Notice that these are vector sums! so we have to take components

$$P_x = \sum_{i=1}^{N} p_{x_i}$$

$$P_y = \sum_{i=1}^{N} p_{y_i}$$

$$P_z = \sum_{i=1}^{N} p_{z_i}$$

where as usual, we turn our three-dimensional problem into three one-dimensional problems by taking components of the vectors.

It turns out that there is a terrific time savings in using momentum. But to see the time savings in solving problems, we need to show that the internal forces causing internal momentum, say, of the car engine parts, do not matter in calculating the motion of the car system as a whole. Let's do that now.

Let's take the total momentum of our car,

$$\overrightarrow{P} = \sum_{i=1}^{N} \overrightarrow{p}_i$$

and let's take the time derivative of this momentum, that would give the net force acting on our car, since $\overrightarrow{F}_{net} = d\overrightarrow{p}/dt$.

$$\frac{d\overrightarrow{P}}{dt} = \sum_{i=1}^{N} \frac{d\overrightarrow{p}_i}{dt} = \sum_{i=1}^{N} \overrightarrow{F}_{net_i}$$

Now let's look at an individual engine part, let's give it the subscript k because it is the k^{th} part in our sum from 1 to N. We labeled each part with a number, and k is somewhere in the middle.

$$1, 2, 3, \cdots k, \cdots N$$

It would have forces acting on it. Some would be internal forces, and some external forces. Since we add up the forces to get the net force, we could separate the addition into two groups, the internal and external groups, add up all the internal forces separately, and add up all the external forces separately, and then add the two results together to get the net force on our part.

$$\overrightarrow{F}_{internal_k} = \sum_{j \neq k} \overrightarrow{F}_{kj}$$

where we are using the usual subscripts on our forces. Each of these forces is an internal force acting on k, so k is the first subscript. The other forces are labeled j, and j goes from 1 to N, but has to skip k because our engine part k can't exert a force on itself.

For the external forces, let's say we have a bunch of objects, $e_1, e_2, e_3, \cdots e_N$ all exerting a force on part k. Then we could write all the external forces summed up as

$$\overrightarrow{F}_{external_k} = \sum_{M=1}^{N} \overrightarrow{F}_{ke_M}$$

and the net force acting on part k would be

$$\overrightarrow{F}_{net_k} = \sum_{j \neq k} \overrightarrow{F}_{kj} + \sum_{M=1}^{N} \overrightarrow{F}_{ke_M}$$

But this is just for part k. Let's find the net force acting on the whole system, like our car system.

We would just add up all the forces from all the parts

$$\begin{aligned} \overrightarrow{F} &= \sum_k \left(\sum_{j \neq k} \overrightarrow{F}_{kj} + \sum_{M=1}^{N} \overrightarrow{F}_{ke_M} \right) \\ &= \sum_k \sum_{j \neq k} \overrightarrow{F}_{kj} + \sum_k \sum_{M=1}^{N} \overrightarrow{F}_{ke_M} \end{aligned}$$

The last term is just the sum of all the external forces. We could call it $\overrightarrow{F}_{external_{net}}$ The first term is the sum of all the internal forces. But we know something about internal forces. They come in pairs, $\overrightarrow{F}_{ij}$ and $\overrightarrow{F}_{ji}$ and we know that for each pair

$$\overrightarrow{F}_{ij} = -\overrightarrow{F}_{ji}$$

so if we add them all up, the sum must be zero!

$$\overrightarrow{F} = 0 + \overrightarrow{F}_{external_{net}}$$

This means that when we look at the total momentum for a system,

$$\frac{d\overrightarrow{P}}{dt} = \overrightarrow{F}_{external_{net}}$$

and we can ignore all the internal forces when we find the motion of the system. You might say that this is obvious! We have been doing this for years. You never even think about the motion of your pistons as you travel in your car. But now we can express this thought mathematically.

Conservation of momentum

Let's take on a special case. Suppose for a system there are no external forces, or that the external forces sum to zero so there is no system net force.

Then

$$\frac{d\overrightarrow{P}}{dt} = 0$$

$$\overrightarrow{P}_f - \overrightarrow{P}_i = 0$$

or

$$\overrightarrow{P}_f = \overrightarrow{P}_i$$

For such a system, the total momentum won't change. This gives us a powerful way to calculate what motion our objects will experience!

If two objects smash into each other, then the atoms of object one will experience a force from the impact of object 2, and we might expect a reciprocal force as well on the atoms of object two from the impact of object 1. Let's write the impulse momentum theorem for both objects:

$$\overrightarrow{\mathbf{F}}_{21}\Delta t = m_1\overrightarrow{\mathbf{v}}_{1f} - m_1\overrightarrow{\mathbf{v}}_{1i}$$
$$\overrightarrow{\mathbf{F}}_{12}\Delta t = m_2\overrightarrow{\mathbf{v}}_{2f} - m_2\overrightarrow{\mathbf{v}}_{2i}$$

Since

$$\overrightarrow{\mathbf{F}}_{21} = -\overrightarrow{\mathbf{F}}_{12}$$

then

$$\overrightarrow{\mathbf{F}}_{21}\Delta t = -\overrightarrow{\mathbf{F}}_{12}\Delta t$$

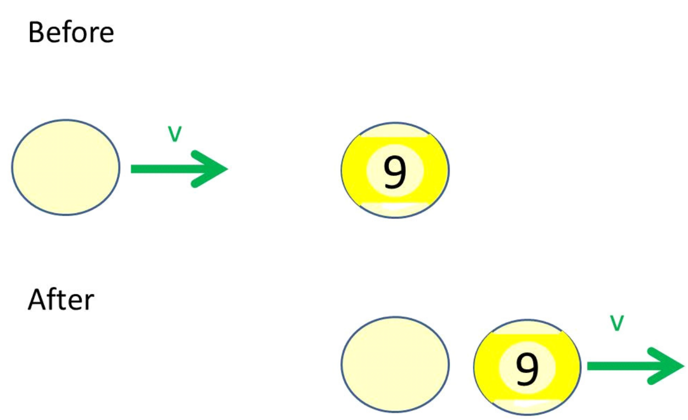
and

$$m_1\overrightarrow{\mathbf{v}}_{1f} - m_1\overrightarrow{\mathbf{v}}_{1i} = -\left(m_2\overrightarrow{\mathbf{v}}_{2f} - m_2\overrightarrow{\mathbf{v}}_{2i}\right)$$

or

$$m_1\overrightarrow{\mathbf{v}}_{1f} + m_2\overrightarrow{\mathbf{v}}_{2f} = m_1\overrightarrow{\mathbf{v}}_{1i} + m_2\overrightarrow{\mathbf{v}}_{2i}$$

Notice how like an energy equation this is. The initial momentum is equal to the final momentum. The total momentum did not change for the system. Just like energy, when the total momentum does not change we will say that *momentum is conserved.*



We could add in more objects, and we would find that the complicated collision would still show momentum begin conserved so long as there is no external force acting.

When no external forces act on a system, the total momentum of the system is conserved

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Note that we must be careful to define our isolated system. A rocket ship launching does not seem to have it's momentum conserved, but we must take into account the reaction of the Earth to have the complete system. The Earth does move a little, but not very much because it's mass is so much bigger.

Two identical pool balls travel toward each other and collide. The cue ball has an initial velocity $v_{Ci} = v$, and the 9 ball is stationary. We notice that after the collision, the 9 ball moves off and the cure ball has stopped. So $v_{Cf} = 0$. what is the velocity of the 9 ball after the collision. velocities are v_{1i} and v_{2i}. What is the velocity of each ball after the collision? Let's assume we can neglect friction.

Start with the idea of conservation of momentum

$$P_i = P_f$$

The collision will be short, and we expect that during the collision, all other horizontal forces can be ignored. So we will use the impulse approximation.

The initial total momentum of the two ball system would be

$$P_i = p_{ci} + p_{9i}$$

and the final total momentum would be

$$P_f = p_{cf} + p_{9f}$$

so we can write

$$p_{ci} + p_{9i} = p_{cf} + p_{9f}$$

then

$$m_c v_{ci} + m_9 v_{9i} = m_c v_{cf} + m_9 v_{9f}$$

We recognize that both balls have the same mass, so mass cancels out. Let's use our zeros

$$v_{ci} + 0 = 0 + v_{9f}$$

$$v_{ci} = v_{9f}$$

We see that the two balls have switched velocities.

Notice that the clue that a problem is a conservation of momentum problem is that we are given a "before" and an "after" look at the motion of an object. The procedure for a conservation of momentum problem has about three steps

1. First we write out that the total momentum does not change $P_i = P_f$.
2. Then fill in the momentum for each part of the system.
3. The resulting equation may allow us to solve for the thing we are looking for.

It's probably appropriate to note how very difficult this simple problem would be if we just used forces. We would have had to study the interaction, the collision, itself. And the forces would change throughout the entire interaction. We would have to perform the impulse integral. But using conservation of momentum, we avoided all of that difficulty. But there was a cost. In what we have done we don't know *anything* about the actual interaction. All we know is the velocity of the balls before the interaction and the velocity of the balls after the interaction. But if you are a pool shark, that's all you need to know!

In our next lecture, we will explore conservation of momentum and see just how useful it can be.

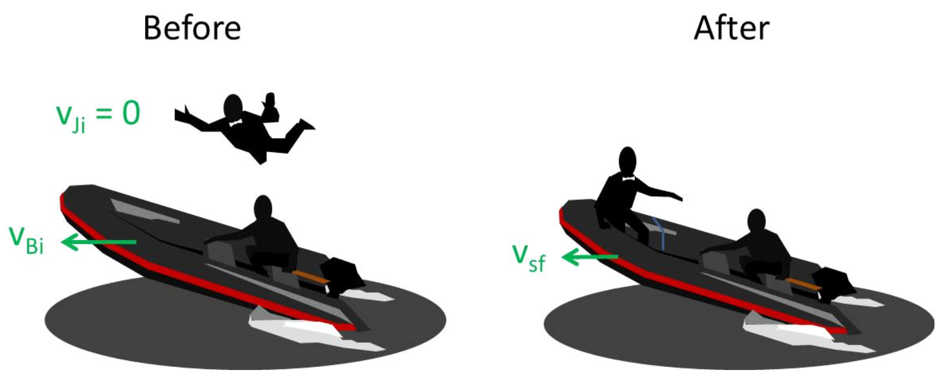
28 Momentum Problems

We just got to the point last time where we could do a simple problem using conservation of momentum. We will need to practice this type of problem in order to get good at it. So let's take on a few interesting problems in this lecture.

Inelastic collisions

Suppose our two pool balls hit, but instead of bouncing off each other, suppose that the balls stuck together. It's a strange name, but we call a collision where the objects stick together an *inelastic collision*. Really this sort of collision happens more than you might think. Let's take a specific example.

James Bland (000) is pursuing a nefarious spy. The spy $(75\,\mathrm{kg})$ is leaving in a small boat $(90\,\mathrm{kg})$. To stop the spy, Bland $(70\,\mathrm{kg})$ jumps into the boat. He does not bounce off (that would not be very Bland-like), he stays in the boat. This is a totally inelastic collision! If the initial speed of the boat is $-15\,\mathrm{m/\,s}$. What is the speed of the boat plus Bland after the collision?



Remember that using conservation of momentum does not allow us to study the forces involved in the collision. We can only look at what we had before the collision and what we had after the collision. So we should draw a before and after picture. We, as usual, we just draw dots for our objects.

Let's list what we know

$$\begin{aligned} m_{bg} &= 75\,\mathrm{kg} \\ m_b &= 90\,\mathrm{kg} \\ m_J &= 70\,\mathrm{kg} \\ v_{Bi} &= -15\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

where the subscripts bg is for the "bad guy" and b is for the boat. Since Bland and boat both begin with "B," we will use J for "James." In this case, all the forces except the collision forces are smaller during the collision forces during the collision, so we can invoke the impulse approximation. we can expect momentum to be conserved.

Our basic equation is

$$P_i = P_f$$

The actual impact will be short. We expect that the boat will go slower in the water after Bland hits. If the collision lasts for a short time Δt, then we will take our before picture an infinitesimal time before Δt starts, and we will take our after look at the momentum an infantilism time after Δt ends. Clearly if we wait a long time after the collision, the water drag and engine push, etc., will have to be accounted for. But we will look just barely before and just barely after the collision. Then

$$P_i = P_f$$

The initial total momentum of the system would be

$$P_i = p_{Bi} + p_{Ji}$$

where B is for "boat plus bad guy" and J is for "James." Then the final total momentum would be

$$P_f = p_{Bf} + p_{Jf}$$

and we can write our total momentum equation as

$$\begin{aligned} P_i &= P_f \\ p_{Bi} + p_{Ji} &= p_{Bf} + p_{Jf} \end{aligned}$$

We know

$$\overrightarrow{p} = m\overrightarrow{v}$$

and let's consider drawing our coordinate system so the boat is moving in the x-direction only. The initial x-direction speed of James is $v_{Ji} = 0$. He is just falling into the boat. The initial speed of the boat is $v_{Bi} = -15\,\mathrm{m/\,s}$. But once Bland is in the boat, the boat and Bland stay together. So the final speed will be v_{sf} where s is for the boat-Bland-bad guy (s)ystem. And the mass of the system will be the combined masses of the boat, bad-guy, and Bland. Then we can write our momentum equation as

$$(m_b + m_{bg})\, v_{Bxi} + m_J v_{Jxi} = (m_b + m_{bg} + m_J)\, v_{sxf}$$

we should use our zeros

$$(m_b + m_{bg})\, v_{Bxi} + 0 = (m_b + m_{bg} + m_J)\, v_{sxf}$$

then

$$v_{sxf} = \frac{(m_b + m_{bg})\, v_{Bxi}}{(m_b + m_{bg} + m_J)}$$

or

$$\begin{aligned} v_{sxf} &= \frac{(90\,\mathrm{kg} + 75\,\mathrm{kg})\left(-15\frac{\mathrm{m}}{\mathrm{s}}\right)}{(90\,\mathrm{kg} + 75\,\mathrm{kg} + 70\,\mathrm{kg})} \\ &= -10.532\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

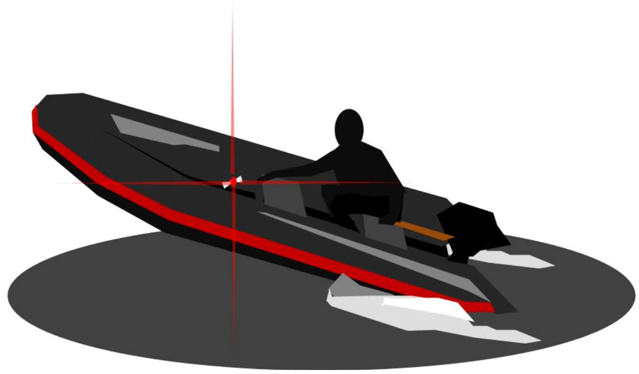
By jumping into the boat, the boat and bad guy have slowed. I suppose this gives James more time to fight the bad guy.

Of course, we could have had a much less exciting problem. Any time we have a collision where the two colliding objects stick together, we have a *totally inelastic collision*. So throwing a bale of hay on a moving truck would be a totally inelastic collision. A

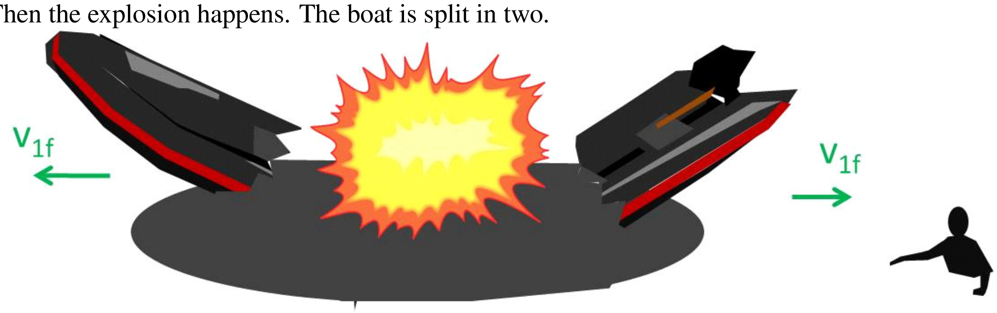
Little brother jumping onto his big brother and holding on would also be such a case. Or an obnoxious kid throwing a lump of clay at another kid where the clay sticks would be another. We could have a *partially* inelastic collision if one of the objects collapses or dents or is in some what deformed during the collision.

Explosions

But let's continue with our James Bland scenario. Suppose while we watch Mr. Bland there is a fight, and Mr. Bland's tie is knocked off. The scuffle continues, and Mr. Bond allows the bad guy to knock him out of the boat. The bad guy smiles, but we see that in the front of the boat there is a small blinking red light.



The tie is a bomb! just in time, the bad guy sees the blinking light and jumps out of the boat letting the engine die (it's early in the movie, you can't get rid if the bad guy yet!). The boat's velocity fades to $v_{bi} = 0\,\mathrm{m/\,s}$ due to drag forces caused by the air and water. Then the explosion happens. The boat is split in two.

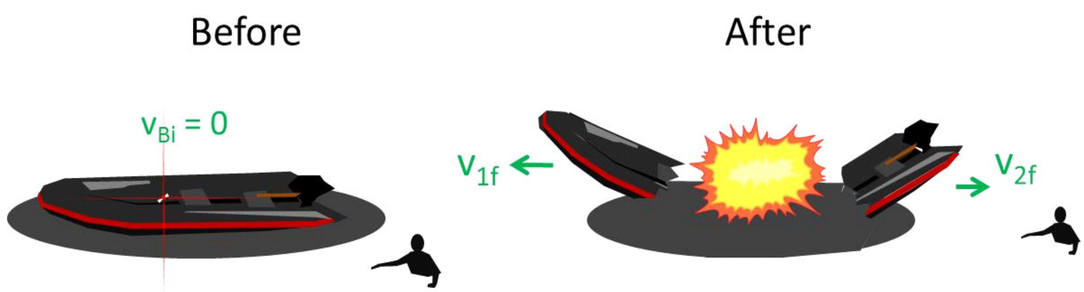


What can we say about the velocities of the parts of the boat just after the explosion? Let's call the two parts of the boat part 1 and part 2. Suppose the mass of part 1 is

$m_1 = 20\,\text{kg}$ and the mass of part 2 is $m_2 = 70\,\text{kg}$ (part two has the motor). Long after the explosion, the boat parts will come to rest on the water (or under the water). But we want the final velocities of the boat parts just after the explosion.

This would also be a difficult problem to solve with forces. Notice that the boat is resting before the explosion, so there is no net external force. And all the explosion forces are internal forces.

We can treat this as a conservation of momentum problem. The explosion is short lasting, and so we can use the impulse approximation. We can draw a before and after picture.



or better, more like this.

We know

$$\begin{aligned} m_B &= 90\,\text{kg} \\ m_1 &= 20\,\text{kg} \\ m_2 &= 70\,\text{kg} \\ v_{Bi} &= 0\,\frac{\text{m}}{\text{s}} \end{aligned}$$

and our basic equation is

$$\vec{P}_i = \vec{P}_f$$

and momentum is

$$\overrightarrow{p} = m\overrightarrow{v}$$

To approach this problem, let's first realize that all of our motion is in the x-direction. Then

$$P_{ix} = P_{fx}$$

and before the explosion the momentum is

$$\begin{aligned} P_{ix} &= p_{Bi}x \\ &= m_B v_{Bix} \\ &= 0 \end{aligned}$$

and the final momentum is

$$P_{fx} = m_1 v_{1fx} + m_2 v_{2fx}$$

so

$$0 = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1 v_{1f} = -m_2 v_{2f}$$
$$\frac{v_{1fx}}{v_{2fx}} = -\frac{m_2}{m_1}$$

Since m_2 is larger than m_1, we can see that v_{1fx} is larger than v_{2fx}. In fact,

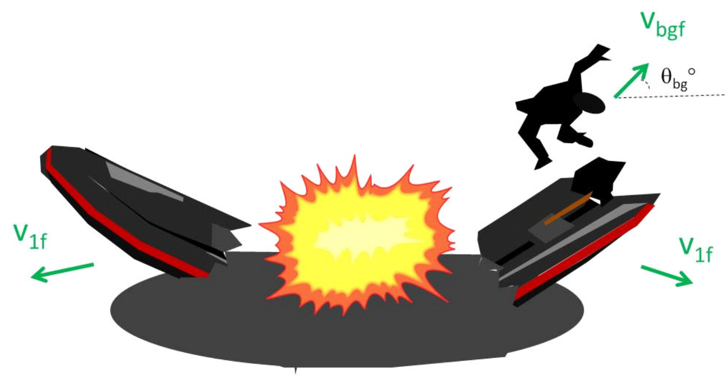
$$v_{1fx} = -\frac{m_2}{m_1} v_{2fx}$$

$$\begin{aligned} v_{1fx} &= -\frac{70}{20} v_{2fx} \\ &= -3.5 v_{2fx} \end{aligned}$$

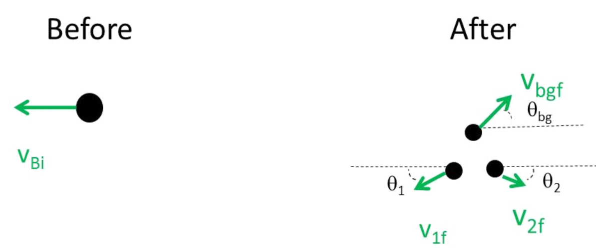
We now know that part 1 of the boat leaves the explosion 3.5 times as fast at part 2. Although we don't know the actual speeds, our experience with boats tells us that at $1/3.5$ the speed of part 1, boat part 2 is not likely to take out the bad guy. James has more work to do.

Momentum in two dimensions

Suppose that the bad guy did not see the tie-bomb. The explosion would throw the bad guy off the boat as well as splitting the boat into pieces. The bad guy will be fine (it's still early in the movie). But he will □y through the air due to the explosion. James is watching all this with his wrist-watch velocity meter. He sees that the initial velocity of the boat is $v_{si} = -15\,\mathrm{m/\,s}$ and that part one of the boat has a final speed of $v_{1f} = 85\,\mathrm{m/\,s}$ at an angle of $190\,°$. He also measures $v_{2f} = 26.0\,\mathrm{m/\,s}$. at $0\,°$. But he is unable to measure the final speed and direction of the bad guy in time. That final velocity is critical to figuring out if the bad guy will get away unharmed. So we need to know that speed and direction.



Here is our picture



and what we know

$$
\begin{aligned}
v_{1f} &= 85\frac{\mathrm{m}}{\mathrm{s}} \\
\theta_1 &= 190\,° \\
v_{2f} &= 26.0\frac{\mathrm{m}}{\mathrm{s}} \\
\theta_2 &= -15\,° \\
v_{bg} &= ? \\
\theta_{bg} &= ? \\
v_{si} &= -15\,\mathrm{m/\,s}
\end{aligned}
$$

and from the previous problems we know

$$\begin{aligned} m_{bg} &= 75\,\mathrm{kg} \\ m_1 &= 20\,\mathrm{kg} \\ m_2 &= 70\,\mathrm{kg} \end{aligned}$$

and let's define the total mass of the system to be

$$M_s = m_{bg} + m_1 + m_2 = 75\,\mathrm{kg} + 20\,\mathrm{kg} + 70\,\mathrm{kg} = 165\,\mathrm{kg}$$

Our basic equations are the same as before

$$\overrightarrow{P}_i = \overrightarrow{P}_f$$
$$\overrightarrow{p} = m\overrightarrow{v}$$

But we realize that this is a two-dimensional problem, so we will need to turn it into two one-dimensional problems by taking components.

$$\begin{aligned} p_{1x} &= m_1 v_{1f} \cos\theta_1 \\ p_{2x} &= m_2 v_{2f} \cos\theta_2 \\ p_{bgx} &= m_{bg} v_{bgf} \cos\theta_{bg} \end{aligned}$$

and

$$\begin{aligned} p_{1y} &= m_1 v_{1f} \sin\theta_1 \\ p_{2y} &= m_2 v_{2f} \sin\theta_2 \\ p_{bgy} &= m_{bg} v_{bgf} \sin\theta_{bg} \end{aligned}$$

then

$$\overrightarrow{P}_i = \overrightarrow{P}_f$$

becomes

$$\begin{aligned} P_{ix} &= P_{fx} \\ P_{iy} &= P_{fy} \end{aligned}$$

or

$$\begin{aligned} P_{ix} &= m_w v_{si} \\ P_{iy} &= 0 \end{aligned}$$

and

$$\begin{aligned} P_{fx} &= p_{1x} + p_{2x} + p_{bgx} \\ P_{fy} &= p_{1y} + p_{2y} + p_{bgy} \end{aligned}$$

then

$$P_{ix} = P_{fx}$$

becomes

$$m_s v_{si_x} = p_{1x} + p_{2x} + p_{bgx}$$

and

$$P_{iy} = P_{fy}$$

becomes

$$0 = p_{1y} + p_{2y} + p_{bgy}$$

Now let's substitute in our momenta.

$$M_s v_{si_x} = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 + m_{bg} v_{bgf} \cos\theta_{bg}$$
$$M_s v_{si_y} = m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2 + m_{bg} v_{bgf} \sin\theta_{bg}$$

We have two equations and two unknowns, v_{bgf} and θ_{bg}. There are several ways to wade through the algebra to find a solutions. Let's first solve for the terms with θ_{bg}

$$M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2) = +m_{bg} v_{bg} \cos\theta_{bg}$$
$$M_s v_{si_y} - (m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2) = +m_{bg} v_{bg} \sin\theta_{bg}$$

There is no initial speed of the system in the y direction. So we can write the set as

$$M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2) = +m_{bg} v_{bg} \cos\theta_{bg}$$
$$0 - (m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2) = +m_{bg} v_{bg} \sin\theta_{bg}$$

Now let's divide the y equation by the x-equation

$$\frac{m_{bg} v_{bg} \sin\theta_{bg}}{m_{bg} v_{bg} \cos\theta_{bg}} = \frac{-(m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2)}{M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2)}$$

the m_{bg} and v_{bg} factors cancel

$$\frac{\sin\theta_{bg}}{\cos\theta_{bg}} = \frac{-(m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2)}{M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2)}$$

using some trigonometry

$$\tan\theta_{bg} = \frac{-(m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2)}{M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2)}$$

an inverse tangent get's us the angle

$$\theta_{bg} = \tan^{-1}\left(\frac{-(m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2)}{M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2)}\right)$$

$$\begin{aligned}\theta_{bg} &= \tan^{-1}\left(\frac{-\left((20\,\mathrm{kg})\left(85\,\frac{\mathrm{m}}{\mathrm{s}}\right)\sin(190\,^\circ) + (70\,\mathrm{kg})\left(24.0\,\frac{\mathrm{m}}{\mathrm{s}}\right)\sin(-15\,^\circ)\right)}{(165\,\mathrm{kg})\,(15\,\mathrm{m/\,s}) - \left((20\,\mathrm{kg})\left(85\,\frac{\mathrm{m}}{\mathrm{s}}\right)\cos(190\,^\circ) + (70\,\mathrm{kg})\left(24.0\,\frac{\mathrm{m}}{\mathrm{s}}\right)\cos(-15\,^\circ)\right)}\right) \\ &= 16.117\,^\circ\end{aligned}$$

Now that we know the angle, we can find v_{bg}. Let's use the x-equation

$$M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2) = +m_{bg} v_{bg} \cos\theta_{bg}$$

then

$$v_{bg} = \frac{M_s v_{si_x} - (m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2)}{m_{bg} \cos\theta_{bg}}$$

or

$$\begin{aligned} v_3 &= \frac{(165\,\mathrm{kg})(-15\,\mathrm{m/\,s}) - \left((20\,\mathrm{kg})\left(85\,\frac{\mathrm{m}}{\mathrm{s}}\right)\cos(190\,^\circ) + (70\,\mathrm{kg})\left(24.0\,\frac{\mathrm{m}}{\mathrm{s}}\right)\cos(-15\,^\circ)\right)}{(75\,\mathrm{kg})\cos(16.117\,^\circ)} \\ &= -33.636\frac{\mathrm{m}}{\mathrm{s}} \\ &= -75.242\frac{\mathrm{mi}}{\mathrm{h}} \end{aligned}$$

This is pretty fast. But bad guys in movies seem to be very resilient.

Once again, consider how hard this problem would be if we just used Newton's second law and kinematics. Of course, in fact we did use Newton's second law and Newton's third law to find conservation of momentum. So our Force laws are sitting there behind all that we are doing. And we would need kinematics to find where the bad guy would eventually fall even with what we have done. But we will leave this for another problem.

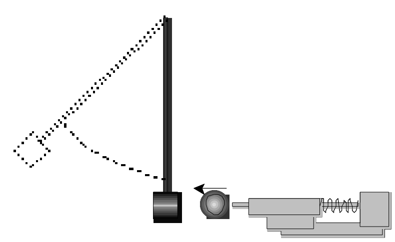
29 Energy and Momentum

We have studied conservation of energy and conservation of momentum separately. But conservation of energy and conservation of momentum are even more powerful if we use them together. Let's take on some really tough problems in this lecture, and see how our conservation laws work together to make them easier.

Energy Example: Ballistic Pendulum

You might wonder, can we combine our conservation of energy and conservation of momentum to do even more complicated dynamic problems. And of course, we can!

In class I will show you a strange device. It is a combination of a spring cannon, and a pendulum. The spring cannon shoots a ball into a cup at the bottom of the pendulum. Then the pendulum swings upward. You probably can guess that we could safely determine how fast the ball is shot (the muzzle velocity) using something like this.



Suppose we want to find the height that the pendulum will rise. There will be a potential energy at the top of the swing. And from what we have done before we know that if we know the muzzle speed of the ball, we could find the height of the swing. We could use work to find the muzzle speed. We know how to find the work done by a spring, but let's try conservation of momentum instead.

So here is the strategy: We can find an expression for the initial velocity of the ball as it leaves the cannon using conservation of momentum. That expression will have the final velocity of the ball and pendulum after the collision in the equation. Then we can find the ball pendulum maximum height using conservation of energy and knowing initial velocity of the pendulum and ball.

We need to know the masses to do this. For our setup, I found the following:

$$\begin{array}{ll} m = 67.1\,\mathrm{g} & \text{mass of the ball} \\ M = 138.5\,\mathrm{g} & \text{Mass of the pendulum cup} \end{array} \tag{29.1}$$

Our basic equations are

$$\begin{aligned} \overrightarrow{P}_i &= \overrightarrow{P}_f \\ E_i &= E_f \end{aligned}$$

Let's start with conservation of momentum. We will need a before and after case for the collision alone. Notice that this is a totally inelastic collision, like Mr. Bland landing in the boat. Before the collision, we have

$$\begin{aligned} P_i &= p_{mi} + p_{Mi} \\ &= mv_{mi} + Mv_{Mi} \\ &= mv_{mi} \end{aligned}$$

and we realize that $v_{Mi} = 0$, the pendulum is not moving at first.

After the collision we have

$$\begin{aligned} P_f &= p_{mi} + p_{Mi} \\ &= (m + M)\, v_{sf} \end{aligned}$$

The ball and the pendulum move together. Now there are other forces acting on the ball and the pendulum, but we will invoke the impulse approximation for the short time of the collision. Then

$$\begin{aligned} P_{ix} &= P_{fx} \\ mv_{mi} &= (m + M)\, v_{sf} \end{aligned}$$

we can relate the two velocities to each other

$$v_{sf} = \frac{m}{(m + M)} v_{mi}$$

But we don't know v_{mi}.

Still we have more equations. So let's try conservation of energy. We want to find how high up the pendulum goes. That is the distance y. That shows up in U_g. Let's start our conservation of energy part just after the collision. So we will know v_{sf} (at least in terms of v_i). Start with

$$E_i = E_f$$

Which we write as

$$K_i + U_i = K_f + U_f$$

where now our initial case is just after the collision, so the initial energy part is made from the final values of the momentum part. So v_{si} for the momentum part is v_i for the energy part!

$$\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f$$

and it is the y_f that we want. Let's choose y_i to be our origin, so $y_i = 0$ and notice that at the top of the swing, $v_f = 0$. So let's use these zeros

$$\frac{1}{2}mv_i^2 + 0 = 0 + mgy_f$$

so

$$\frac{1}{2}mv_i^2 = mgy_f$$

Notice that the masses cancel.

$$\frac{1}{2}v_i^2 = gy_f$$

In this problem we want to find y_f so let's solve for it now

$$y_f = \frac{1}{2g}v_i^2$$

Now remember that v_{sf} for the momentum part is v_i for the energy part, so

$$y_f = \frac{1}{2g}v_{sf}^2$$

and we have an equation for v_{sf}

$$v_{sf} = \frac{m}{(m+M)}v_{mi}$$

so we can combine these to get

$$y_f = \frac{1}{2g}\left(\frac{m}{(m+M)}v_{mi}\right)^2$$

So if we know the initial velocity and the masses, we can find the height. Of course, if we were testing a new cannon, we could measure how high up the pendulum goes and determine the muzzle velocity. Let's try that for our case.

$$2gy = \left(\frac{m}{(m+M)}v_{mi}\right)^2$$

$$\sqrt{2gy} = \frac{m}{(m+M)} v_{mi}$$
$$\frac{(m+M)}{m}\sqrt{2gy} = v_{mi}$$
$$v_{mi} = \frac{(m+M)}{m}\sqrt{2gy}$$

In class we will try this, for now, let's say we know the length of the pendulum, and we measure the angle when the pendulum is at the top of it's swing.

$L = 0.30\,\mathrm{m}$	length of pendulum
$\theta = 58.5\,°$	deflection angle in radians

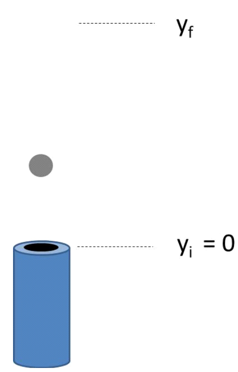
then, using some trig we can find that

$$\begin{aligned} y &= L\left(1-\cos\left(\theta\right)\right) \\ &= 0.143\,25\,\mathrm{m} \end{aligned}$$

then

$$\begin{aligned} v_{mi} &= \frac{(67.1\,\mathrm{g} + 138.5\,\mathrm{g})}{67.1\,\mathrm{g}}\sqrt{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(0.143\,25\,\mathrm{m})} \\ &= 5.134\,2\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

I checked this before class shooting the ball straight upward. The ball will fly up, and for a brief moment at the top of it's flight, the speed will be zero.



So $v_f = 0$. We will define the starting point as $x_i = 0$, $y_i = 0$. We can measure the maximum height

$$y_f = 1.335\,\mathrm{m}$$

For fun, here is how it goes!

$$
\begin{aligned}
E_i &= E_f \\
K_i + U_{gi} &= K_f + U_{gf} \\
\frac{1}{2}mv_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f
\end{aligned}
$$

The masses cancel

$$\frac{1}{2}v_i^2 + gy_i = \frac{1}{2}v_f^2 + gy_f$$

and use our zeros

$$\frac{1}{2}v_i^2 + 0 = 0 + gy_f$$

then

$$v_i^2 = 2gy_f$$

and finally

$$v_i = \sqrt{2gy_f}$$

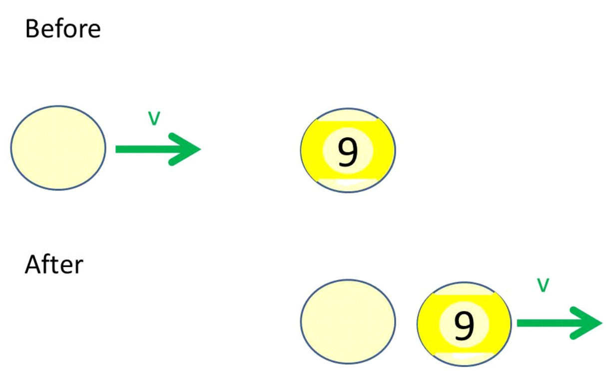
so if I know how high the ball goes, I will be able to find the initial velocity

$$
\begin{aligned}
v_i &= \sqrt{2\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)(1.335\,\mathrm{m})} \\
&= 5.1153\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

which is really close to what we got.

Elastic Collisions: Momentum *and* Energy conservation

We have found that we can solve some very sophisticated problems using the idea of conservation of momentum and conservation of energy. And we have even tried a problem where we used both together. Let's try another combined conservation of momentum and energy problem. Let's go back to our two pool balls. Suppose we know that the cue ball is moving at speed v_o in the x-direction and the 9 ball is not moving. But suppose we don't know either of the final velocities. Could we find them?



Let's assume we can neglect friction. Also notice that our momentum is all in the x-direction.

Start with the idea of conservation of momentum

$$\overrightarrow{P}_i = \overrightarrow{P}_f$$

and divide the problem into x-parts and y-parts.

$$P_{ix} = P_{fx}$$
$$P_{iy} = P_{fy}$$

but

$$P_{iy} = P_{fy} = 0$$

because our collision and motion are all in the x-direction. We are left with

$$P_{ix} = P_{fx}$$

The collision will be short, and we expect that during the collision, all other horizontal forces can be ignored. So we will use the impulse approximation.

The initial total momentum of the two ball system would be

$$P_{ix} = p_{cix} + p_{9ix}$$

and the final total momentum would be

$$P_{fx} = p_{cfx} + p_{9fx}$$

so we can write

$$p_{cix} + p_{9ix} = p_{cfx} + p_{9fx}$$

then

$$m_c v_{cix} + m_9 v_{9ix} = m_c v_{cfx} + m_9 v_{9fx}$$

and we know pool balls all have just about the same mass so

$$v_{cix} + v_{9ix} = v_{cfx} + v_{9fx}$$

and we know v_{ci} and v_{9i}

$$v_o + 0 = v_{cfx} + v_{9fx}$$

but we have two things we don't know v_{cfx} and v_{9fx} and we have only one equation.

But now conservation of energy can come to the rescue. If we can ignore external forces during the collision, we can say that

$$E_i = E_f$$

but we need the energy of the entire system, both balls.

$$E_{ci} + E_{9i} = E_{cf} + E_{9f}$$

or

$$\frac{1}{2}m_c v_{ci}^2 + mgy_{ci} + \frac{1}{2}m_9 v_{9i}^2 + mgy_{9i} = \frac{1}{2}m_c v_{cf}^2 + mgy_{cf} + \frac{1}{2}m_9 v_{9f}^2 + mgy_{9f}$$

This is the total energy equation for the entire system of two balls. Notice that $y_{ci} = y_{9i} = y_{cf} = y_{9f}$ so all of the potential energy terms cancel. Then, canceling the masses, and using our zeros and knowns

$$\frac{1}{2}v_o^2 + 0 = \frac{1}{2}v_{cf}^2 + \frac{1}{2}v_{9f}^2$$

or, canceling the $1/2$ factors.

$$v_o^2 = v_{cf}^2 + v_{9f}^2$$

Since we only have x-parts to our velocities

$$\begin{aligned} v_{cfx} &= v_{cf} \\ v_{9fx} &= v_{9f} \end{aligned}$$

so

$$v_o^2 = v_{cfx}^2 + v_{9fx}^2$$

and this is another equation with our unknowns that comes from our collision. So we should be able to solve for v_{cfx} and v_{9fx}. Let's try solving both of our equations for v_{cfx}. From conservation of momentum (above) we know

$$v_o = v_{cfx} + v_{9fx}$$

so

$$v_{cfx} = v_o - v_{9fx}$$

and from conservation of energy

$$v_o^2 = v_{cfx}^2 + v_{9fx}^2$$

so

$$v_o^2 - v_{9fx}^2 = v_{cfx}^2$$

so let's square our first equation and set it equal to the second because it has v_{cfx}^2

$$v_{cfx}^2 = (v_o - v_{9fx})^2$$

so

$$(v_o - v_{9fx})^2 = v_o^2 - v_{9fx}^2$$

then

$$v_o^2 - 2v_o v_{9fx} + v_{9fx}^2 = v_o^2 - v_{9fx}^2$$

and we see that v_o^2 cancels

$$-2v_o v_{9fx} + v_{9fx}^2 = -v_{9fx}^2$$

and every term has a v_{9fx} in it, so we can cancel one v_{9fx}

$$-2v_o + v_{9fx} = -v_{9fx}$$

let's combine all the v_{9fx} terms

$$-2v_o = -2v_{9fx}$$

so

$$v_{9fx} = v_o$$

then we can put this back into the equation from conservation of momentum to solve for the cue ball final speed

$$v_{cfx} = v_o - v_{9fx}$$

$$\begin{aligned} v_{cfx} &= v_o - v_o \\ &= 0 \end{aligned}$$

And we can see that the last time we did this was not a special case. If we hit the 9 ball dead on, the cue will always stoop and the 9 ball will go off with the cue balls initial speed.

Reference frames and momentum

I'm sure that you are wondering, what if we consider relative motion and conservation and momentum? So suppose we do this problem again but consider that the pool table is on a cruse ship, and that the cruse ship is traveling in such a way that from the point of view of someone stationary on the shore, the cue ball is not moving and the 9 ball (and table) are moving with a speed of v_o the other direction.

Conservation of momentum still gives

$$m_c v_{cix} + m_9 v_{9ix} = m_c v_{cfx} + m_9 v_{9fx}$$

and conservation of energy still gives

$$\frac{1}{2} m_c v_{ci}^2 + \frac{1}{2} m_9 v_{9i}^2 = \frac{1}{2} m_c v_{cf}^2 + \frac{1}{2} m_9 v_{9f}^2$$

Let's deal with momentum first

$$v_{cix} + v_{9ix} = v_{cfx} + v_{9fx}$$

and this time $v_{cix} = 0$ and $v_{9ix} = -v_o$,

$$-v_o = v_{cfx} + v_{9fx}$$

and from energy, canceling the $1/2$ and mass factors, and again equating $v_{cfx} = v_{cf}$ and $v_{9fx} = v_{9f}$ and further realizing that $v_{cix} = v_{ci}$ and $v_{9ix} = v_{9i}$ because this is a totally one-dimensional problem, we can write

$$v_{cix}^2 + v_{9ix}^2 = v_{cfx}^2 + v_{9fx}^2$$

and again using $v_{cix} = 0$ and $v_{9ix} = -v_o$,

$$0 + v_o^2 = v_{cfx}^2 + v_{9fx}^2$$

and we solve again for v_{9fx}

$$-v_o = v_{cfx} + v_{9fx}$$
$$-v_o - v_{cfx} = v_{9fx}$$

$$v_o^2 - v_{cfx}^2 = v_{9fx}^2$$

let's square the first of these

$$(-v_o - v_{cfx})^2 = v_{9fx}^2$$

so

$$\begin{aligned} (-v_o - v_{cfx})^2 &= v_{9fx}^2 \\ v_o^2 + 2v_o v_{cfx} + v_{cfx}^2 &= v_o^2 - v_{cfx}^2 \end{aligned}$$

or, again canceling the v_o terms

$$2v_o v_{cfx} + v_{cfx}^2 = -v_{cfx}^2$$

so

$$2v_o v_{cfx} = -2v_{cfx}^2$$

or

$$2v_o = -2v_{cfx}$$

then

$$-v_o = v_{cfx}$$

and finally

$$\begin{aligned} -v_o - (-v_o) &= v_{9fx} \\ &= 0 \end{aligned}$$

And this is just what we would expect. In the moving reference frame, the balls have just switched velocities because of the relative motion. Conservation of Momentum should work in any inertial reference frame.

So far we have done quite a lot with our conservation of momentum and energy. But it turns out that our energy picture is not quite complete. What if you are an expert pool player and you put a spin on the cue ball? Is there energy involved with rotation?

30 Rotational Energy

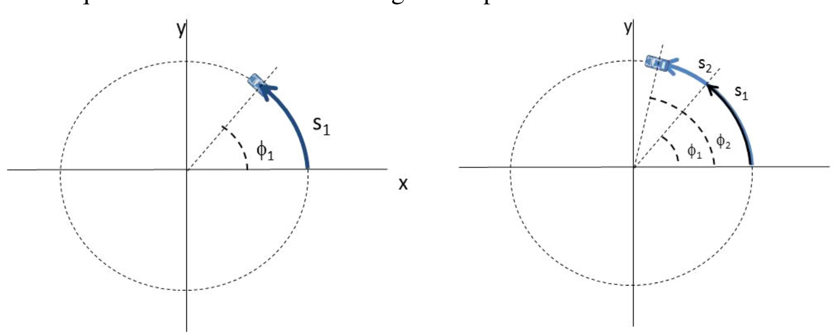
We have studied rotational motion for some time now. But think, does it take energy to spin up a DVD? Or does some energy go into making the wheels of a car turn? We are going to take on the idea of rotational kinetic energy in this lecture. But let's review our basic rotation equation set, and let's reconsider our particle model first.

Rotational Motion Equation Set Review

Here is our rotational motion equation set so far:

$$\Delta\phi = \phi_f - \phi_i$$
$$\Delta t = t_f - t_i$$
$$\omega_{ave} = \frac{\Delta\phi}{\Delta t}$$
$$\omega = \frac{d\phi}{dt}$$
$$\omega = \frac{v}{r}$$
$$\alpha = \frac{d\omega}{dt}$$
$$\alpha = \frac{a}{r}$$
$$a_c = \frac{v^2}{r}$$

The first equation is our definition of angular displacement.



The angular speed is how fast the angle ϕ changes

$$\omega = \frac{d\phi}{dt}$$

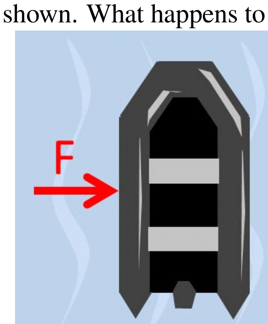
This is how fast something spins. The angular acceleration is how fast the angular speed changes

$$\alpha = \frac{d\omega}{dt}$$

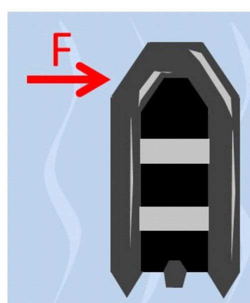
We have described this angular acceleration as how fast something "spins up."

Beyond particles

Let's return to our James Bland adventure and study Mr. Bland's boat. Suppose we push on the boat near the middle as shown. What happens to the boat?



This is not hard for us to predict now. The boat will accelerate to the right. But what if we push on the boat near the front of the boat?



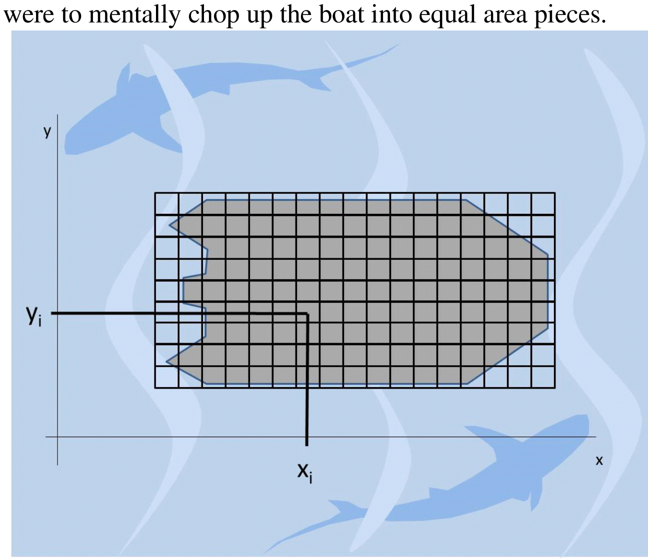
Our experience tells us that the boat will move to the right, but something else will happen. The boat will start to spin! Our particle model won't predict the spin. A free body diagram for the two situations might look like this

The left hand diagram is for the first case, and the right hand diagram is for the second case. Notice that there is no difference! So our force diagrams using the particle model are not enough to allow us to predict spinning motion. We need more physics!

Notice that we did get a particle-like behavior for the boat when we pushed on the "middle" of the boat. But we need a mathematical way to describe what the "middle" of an object is.

If you have experience with boats, you probably could guess that the place to push without the boat spinning is not the exact middle of the boat. That is because the boat motor is on the back of the boat, and it has more mass than other parts of the boat (and also sticks into the water, so it has more drag). The motor is more dense. It looks like we need to have a non-rotation location that is a weighted average of how much mass is on each side of that location weighted by how far that mass is away from our non-rotation point. Let's develop a way to do that average.

Suppose we were to mentally chop up the boat into equal area pieces.



We could number each piece from left to right and down to up so that each piece would have an x number and a y number. The piece shown would be $7th$ in the x-direction and the $4th$ in the y-direction. These numbers would tell us the geographic middle of the boat. That would be $9th$ pice in the x-direction and the $5th$ in the y-direction. How did we arrive at these numbers? We counted up the number of little areas, and divided by 2 to get the middle in each dimension. We want to do something like this to find our "middle" that includes mass. But we realize that the motor is to the left, and so we should weight those areas more. Here is what we can do. Let's weigh each piece of the boat by how much mass the piece has.

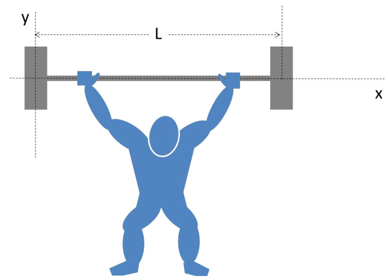
$$x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n} \tag{30.1}$$

$$y_{cm} = \frac{\Sigma m_i y_i}{\Sigma m_i} = \frac{m_1 y_1 + m_2 y_2 + \ldots + m_n y_n}{m_1 + m_2 + \ldots + m_n} \tag{30.2}$$

This will give us the location of a "middle" but it will be a middle that is offset from the geographic middle and is closer to where the majority of the mass is. We have weighted the location of each little area with the mass that we have at that area location. So the motor does get weighted more strongly than the front of the boat.

But if we just had the numerator, we would not have a location (units are wrong for one thing), so we have to get rid of the mass units. We do that by dividing by the total mass of the boat. The result is what we call the *center of mass* of the boat.

Let's try this for a simple object. Suppose a guy holds up a barbell. Each weight has a mass of $100\,\mathrm{kg}$. Where is x_{cm} for this barbell? assume the bar mass can be neglected.



Our formula is

$$x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}$$

We only have two masses, so

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

and we have one mass at $x = 0$, the left end of the bar, and one at $x = L$, the right end of the bar.

$$x_{cm} = \frac{m_1 (0) + m_2 (L)}{m_1 + m_2}$$

and we know that $m_1 = m_2 = m$ so

$$\begin{aligned} x_{cm} &= \frac{m_2(L)}{m_2 + m_2} \\ &= \frac{m_2(L)}{2m_2} \\ &= \frac{L}{2} \end{aligned}$$

which is not too much of a surprise, for our simple system we got the "middle" to be in the middle.

Suppose we chop up our boat or barbell into a bunch of pieces, but this time let's assume the pieces have equal mass. That will mean the spacing between the pieces won't be as uniform as it was last time. Still this will work, and our equations will be

$$x_{cm} = \frac{\Sigma x_i \Delta m}{M}$$
$$y_{cm} = \frac{\Sigma y_i \Delta m}{M}$$

where x_i is the distance from the origin to the i^{th} Δm piece in the x-direction and y_i is the distance from the origin to the i^{th} Δm piece in the y-direction. In this case we are using Δm to mean a small piece of mass. The M is the total mass

$$M = \sum_i \Delta m$$

Of course, where there is a sum, there could be an integral, because integrals are just sums over infinitesimally small pieces. Our sum would become an integral if we let Δm become very small

$$x_{cm} = \frac{1}{M}\int x dm$$
$$y_{cm} = \frac{1}{M}\int y dm$$

To see how these work, let's take on the barbell rod and assume it has enough mass so that it's mass is not negligible. It is just a cylinder of metal. So the amount of mass does not change as a function of x. If the bar has a total mass M, and a total length L, then a small part of the bar, say, Δx long, would have a mass of

$$\Delta m = \frac{M}{L}\Delta x$$

We would call the term M/L the *linear mass density* of the rod. So if the total mass is $15\,\mathrm{kg}$, and the total length is $2.19\,\mathrm{m}$. Then the linear mass density of our rod would be

$$\lambda = \frac{M}{L} = \frac{15\,\mathrm{kg}}{2.19\,\mathrm{m}} = 6.849\,3\frac{\mathrm{kg}}{\mathrm{m}}$$

We can see that a very small length of bar would have a very small mass

$$dm = \lambda dx$$

I only want to do the x_{cm} part of our barbell rod problem, so

$$x_{cm} = \frac{1}{M}\int xdm$$

and we have an equation for dm, we can substitute this into our x_{cm} equation

$$x_{cm} = \frac{1}{M}\int x\lambda dx$$

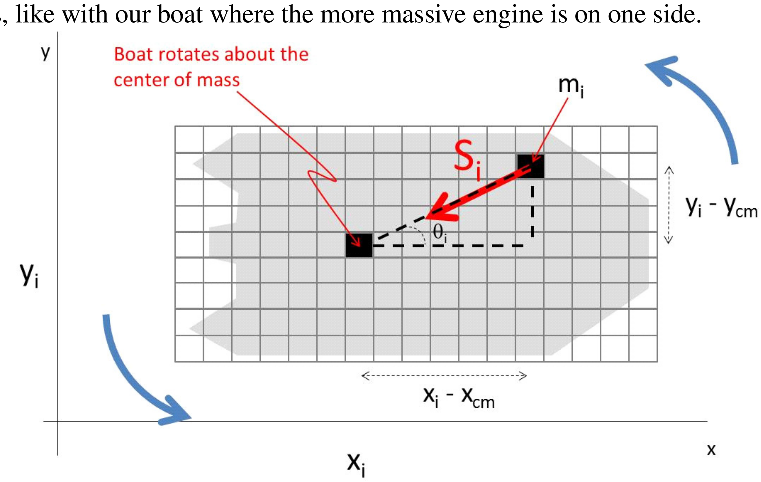
and we can integrate from $x = 0$ to $x = L$

$$x_{cm} = \frac{1}{M}\int_0^L x\lambda dx$$

We get

$$\begin{aligned} x_{cm} &= \frac{1}{M}\left(\left(\frac{1}{2}x^2\right)\Big|_0^L\right. \\ &= \frac{1}{2}\frac{L^2}{M}\lambda \\ &= \frac{1}{2}\frac{L^2}{M}\frac{M}{L} \\ &= \frac{1}{2}L \end{aligned}$$

which is, once again, what we would expect. The center of the bar is, well, in the center of the bar. But we will have to do this for objects where we don't know the exact center of mass, like with our boat where the more massive engine is on one side.



Let's try to find the center of mass for our boat using the motion of the boat. Take a close look at the forces involved in rotation. Each mass segment, say, m_i, must have a centripetal acceleration, so it must have a centripetal force. This force is the spring-like atomic bond forces that hold the object together. If we spin our boat, Mr. Bland might come flying out, because he is not bound to the boat, but the parts of the boat are bound together, and the forces can be modeled as springs. Since these spring forces provide the centripetal acceleration

$$\begin{aligned} S_i &= m_i a_c \\ &= m_i \frac{v_t^2}{r_i} \\ &= m_i \omega^2 r_i \end{aligned}$$

where r_i is measured from the center of mass

$$r_i = \sqrt{\left(x_i - x_{cm}\right)^2 + \left(y_i - y_{cm}\right)^2}$$

Then if we sum up all the internal force x-components we get

$$\begin{aligned} \sum_i S_{ix} &= \sum_i S_i \cos\theta_i \\ &= \sum_i m_i \omega^2 r_i \cos\theta_i \end{aligned}$$

and, if the boat does not fly apart, we know that all these internal forces must be balanced. so

$$\sum_i m_i \omega^2 r_i \cos\theta_i = 0$$

and we can see from the figure

$$\cos\theta_i = \frac{x_i - x_{cm}}{r_i}$$

so

$$\sum_i m_i \omega^2 r_i \left(\frac{x_i - x_{cm}}{r_i}\right) = 0$$

or

$$\sum_i m_i \omega^2 \left(x_i - x_{cm}\right) = 0$$

or finally

$$\sum_i \left(m_i x_i - m_i x_{cm}\right) \omega^2 = 0$$

Since ω is not equal to zero, then

$$\sum_i \left(m_i x_i - m_i x_{cm}\right) = 0$$

Remember this is a sum, and we can add things up in any order, so we could write our sum as

$$\sum_i m_i x_i - \sum_i m_i x_{cm} = 0$$

and we could use the distributive property of multiplication to take out x_{cm}

$$\sum_i m_i x_i - x_{cm} \sum_i m_i = 0$$

so that

$$\sum_i m_i x_i = x_{cm} \sum_i m_i$$

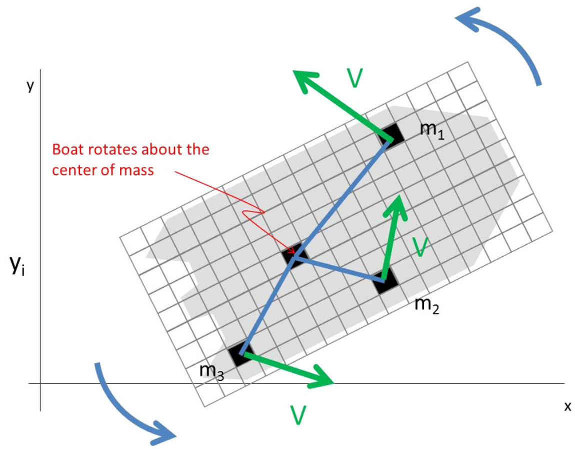
or

$$\frac{\sum_i m_i x_i}{\sum_i m_i} = x_{cm}$$

and this is just our equation for the x_{cm}. We could do the same thing for y_{cm}.

Kinetic energy for rotating objects

Now that we understand center of mass, let's consider the energy of a rotating object.



Each part of the object may have a different speed, and the total kinetic energy is the sum of the energies for each piece of the object. So

$$K_{total} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \cdots + \frac{1}{2}m_N v_N^2$$

and the speeds can be related to the angular speed

$$v_i = r_i \omega$$

where ω does not have a subscript because it is the same for each part of a spinning object. Then

$$K_{total} = \frac{1}{2}m_1 (r_1\omega)^2 + \frac{1}{2}m_2 (r_2\omega)^2 + \frac{1}{2}m_3 (r_3\omega)^2 + \cdots + \frac{1}{2}m_N (r_N\omega)^2$$

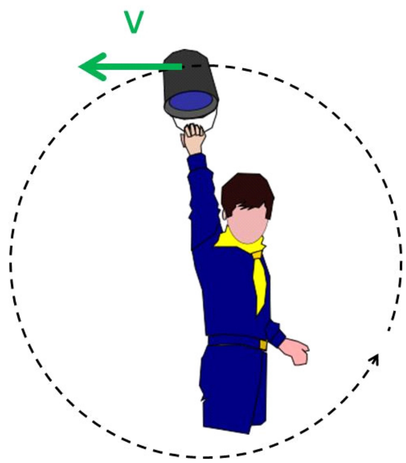
Since the ω's are all the same, then

$$K_{total} = \frac{1}{2}\omega^2 \left(m_1 r_1^2 + \frac{1}{2}m_2 r_2^2 + \frac{1}{2}m_3 r_3^2 + \cdots + \frac{1}{2}m_N r_N^2 \right)$$

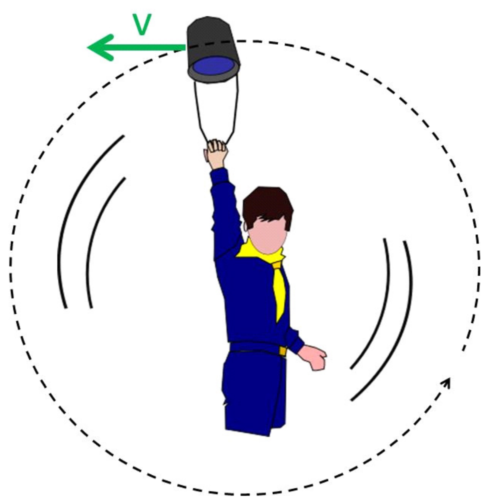
then

$$K_{total} = \frac{1}{2}\omega^2 \sum_i m_i r_i^2$$

Once again we have a weighted average of the mass and the distances, but this time each mass piece is weighted by it's displacement from the center of mass squared. This weighted average tells us something important. It contains all the terms that make the object piece hard to rotate. Think of swinging our bucket of water again.



If we fill the bucket with more water, then it is harder to get rotating. Also if the distance from the center of rotation increases, it is harder to get the bucket rotating.



Let's give this new term that tells us how hard it is to get something rotating a name and a symbol. We will call it the *moment of inertia* and give it the symbol $\mathbb{I}$. Note that we are using the fancy character "$\mathbb{I}$" but most books just use "I." The "I" is just fine, but in

PH220 we will use "I" to mean electric current. So I have chosen to make moment of inertial look different. You are free to use just 'I" if you wish.

$$\mathbb{I} = \sum_i m_i r_i^2$$

Then our kinetic energy for a rotating system is

$$K_{total} = \frac{1}{2}\omega^2 \mathbb{I}$$

or

$$K_{total} = \frac{1}{2}\mathbb{I}\omega^2$$

Of course, this is really just a sum of all the individual kinetic energies of all the parts of the object. So this is not new or different than the kinetic energy we know and love. But for rotating objects it can be more convenient two write the energy involved in the circular motion of the object parts in this way.

We can say that for a rotating system our mechanical energy might be

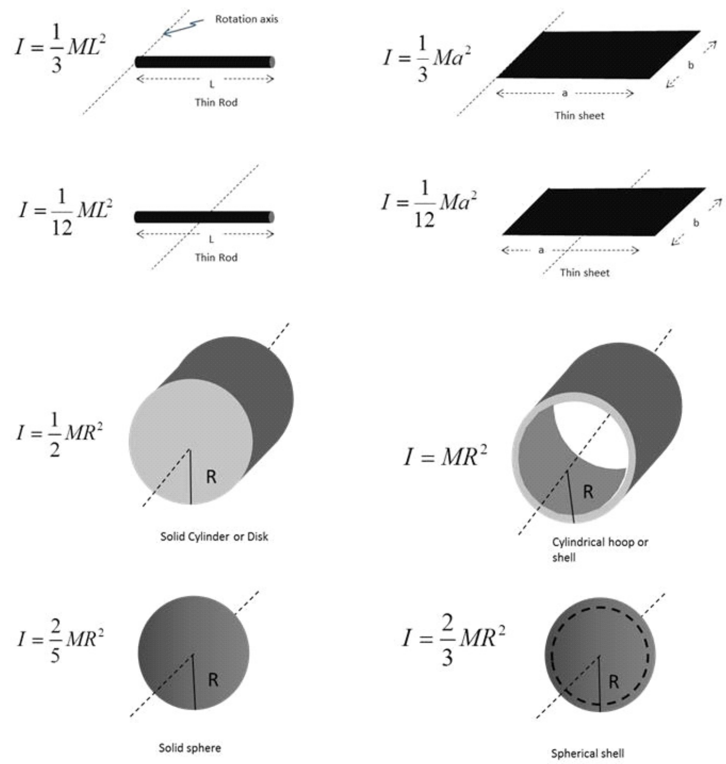
$$E_{mech} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\mathbb{I}\omega^2 + U_g$$

where the first term is the kinetic energy of the entire object moving as one piece, the second term is our new rotational kinetic energy due to the object rotating, and the final term is the gravitational potential energy (of course, if we have springs, we need to include some spring energy too, etc.). We could write this as

$$E_{mech} = K_t + K_r + U_g$$

where K_t is our old kinetic energy, but to distinguish it from rotational kinetic energy we will now call it *translational kinetic energy*. The term K_r is rotational kinetic energy, and U_g is still gravitational potential energy.

But, you might say, it looks tedious to find $\sum_i m_i r_i^2$ for different objects. And you might be right. We will try this next lecture. But the moment of inertia has been calculated for many common objects. So if you have a table of moments of inertia, you can just choose one from the table that matches your object, and find the rotational kinetic energy. In the table below, the dotted line shows the rotation axis for each rotating object. The moment of inertia is different for the same object if the object is rotated about a different part of the object. So we need to know the rotation axis so we can make sure we have the right moment for our problem


$I = \frac{1}{3} ML^2$
Rotation axis
L
Thin Rod
$I = \frac{1}{3} Ma^2$
b
a
Thin sheet
$I = \frac{1}{12} ML^2$
L
Thin Rod
$I = \frac{1}{12} Ma^2$
b
a
Thin sheet
$I = \frac{1}{2} MR^2$
R
Solid Cylinder or Disk
$I = MR^2$
R
Cylindrical hoop or shell
$I = \frac{2}{5} MR^2$
R
Solid sphere
$I = \frac{2}{3} MR^2$
R
Spherical shell

31 Torque

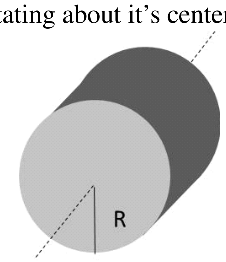
It takes a force to change motion, so it must take a force to cause rotational motion. It is time to get forces involved in producing rotational motion. We have a clue from thinking about Mr. Bland's boat that where we push might matter for rotational motion. We saw that how far the mass element was from the center of rotation mattered for our boat and bucket problems in the last lecture. We combined "how far" and "how much mass" into one term called the *moment of inertia* and we even built a table of common objects and their moments of inertia.

In this lecture, let's see how to calculate a moment of inertia, and then, in the next two lectures, we will combine the ideas of force and moment of inertia together to be able to see how to include forces into rotational dynamics problems.

Finding the moment of Inertia

We learned about moment of inertia last time. And we have a table of moment's of inertia for several useful mass distributions in our table. We have rotating wheels (solid cylinders and hollow cylinders) and balls and sticks. But we should see how we can find the moment of inertia for an arbitrary shape. To do this, let's pick a shape we know from the table, and find the moment of inertia so we can see how it is done in general for any shape. If we use a shape that we know, we can check our answer when we are done. Let's pick a solid cylinder (a wheel) rotating about it's center of mass.

$$I = \frac{1}{2}MR^2$$



Solid Cylinder or Disk

Our basic equation for moment of inertial for a single mass element is

$$\mathbb{I} = \sum_i m_i r_i^2$$

but let's rewrite this by rearranging the terms.

$$\mathbb{I} = \sum_i r_i^2 m_i$$

and let's envision splitting up our wheel into little equal masses. Our equation calls these little masses m_i but I want to rename them Δm_i because they are little pieces of the whole mass. Notice that this is not a change in mass or a difference between masses. I am using the Δ symbol to say that the mass is "little." So we would call Δm a "little bit of mass."

$$\mathbb{I} = \sum_i r_i^2 \Delta m_i$$

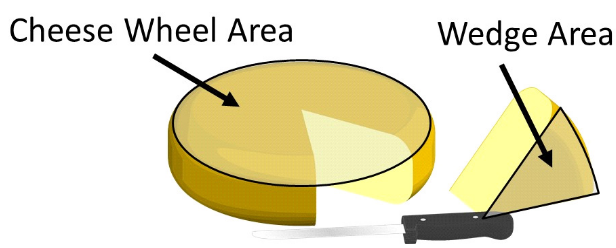
I also want to assume that my wheel has a uniform density

$$\eta = \frac{M_{\text{whole wheel}}}{A_{\text{whole wheel}}}$$

This may be a new concept. We take the total mass of the whole wheel, M, and divide the total mass by the area of the whole wheel. If this ratio is constant, I can break off any part of the wheel and the ratio of mass to area for the piece will be the same as it is for the whole wheel.

$$\begin{aligned}\frac{\Delta m}{\Delta A} &= \frac{M}{A} \\ \Delta m &= \frac{M}{A}\Delta A\end{aligned}$$

What we are really saying is that our object is all made out of the same material. We use concepts like this in buying food products all the time. For example, think of a cheese wheel.



The cheese is all the same material. So if we find the mass of the wheel and divide by the area of the wheel, we get a number that represents a density of the cheese. Now if we cut out a wedge of cheese, we could see what the area of the wedge is to find the mass. If the whole mass divided by the area is

$$\eta = \frac{m_{whole}}{A_{whole}}$$

and we know it's all the same type of cheese, then we know

$$\eta = \frac{m_{wedge}}{A_{wedge}}$$

so

$$\frac{m_{wedge}}{A_{wedge}} = \frac{m_{whole}}{A_{whole}}$$

which gives the mass of the wedge to be

$$m_{wedge} = A_{wedge}\frac{m_{whole}}{A_{whole}}$$

This might be a great way to sell cheese!

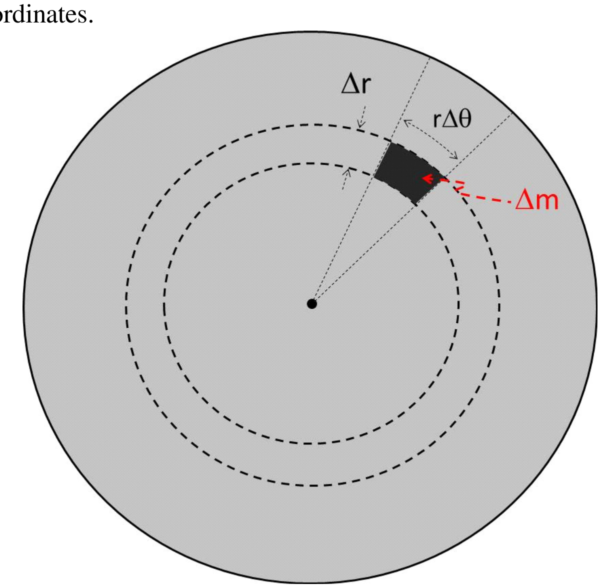
But we can use the same approach to find out how much mass there is in our piece of our rotating wheel. We could solve for Δm

$$\Delta m = \frac{M}{A}\Delta A$$

We gave the symbol η (and "eta" or "n-looking-thing") to the ratio of M to A so we could write this as

$$\Delta m = \eta \Delta A$$

This means that a small amount of mass Δm can be written as the density of the wheel material multiplied by how much of the wheel we have in our piece. We could even find a way to express our small area ΔA in terms of the dimensions of the wheel. Let's use cylindrical coordinates.



A small box-like area would have a small area of nearly

$$\Delta A = \ell \times w$$

where ℓ is the length of the box shape and w is the width of the box shape. But in cylindrically coordinates we could make the length

$$\ell = \Delta r$$

be the displacement in the $\hat{r}$ direction along the side of the black box shape. Notice that the width then would really be a small arclength

$$w = r\Delta\theta$$

so

$$\Delta A = \Delta r \times r\Delta\theta$$

Then

$$\begin{aligned}
\mathbb{I} &= \sum_i r_i^2 \Delta m_i \\
&= \sum_i r_i^2 \eta \Delta A \\
&= \sum_i r_i^2 \eta \Delta r r_i \Delta\theta
\end{aligned}$$

but of course this is just an approximation. Our ΔA is not exactly box shaped. But if we made the little area *very* small, then the approximation of being box shaped would be better. A very small piece of area would be

$$dA = dr \times rd\theta$$

where our $\Delta's$ have turned into smaller $d's$ to show that the quantities are much smaller. For these very small distances, the curviness of the lines is not very important, so a square area is a good approximation for dA. We could find a mass for the very small dA

$$\begin{aligned}
dm &= \eta dA \\
&= \eta dr r d\theta
\end{aligned}$$

we have used dm because our small piece of mass is now a very small piece so we use a d instead of a Δ. Then our moment of inertia could be written as an integral

$$\mathbb{I} = \int r^2 dm$$

We drop the subscripts because we understand when we integrate over every possible part of the mass so every dm bit is included up to and including the limits of the integration. Then using our expression for dm

$$\mathbb{I} = \int r^2 \eta dA$$

or

$$\mathbb{I} = \int r^2 \eta \left(r dr d\theta \right)$$

and finally

$$\mathbb{I} = \eta \int r^3 dr d\theta$$

But now we realize that we have something new! We have a dr *and* a $d\theta$. That is two integration variables. So what do we do? The answer is easy[13], we just integrate twice. Once over dr and once over $d\theta$. Start with

$$\int_0^{2\pi} r^3 d\theta = (\text{result})$$

then take the result and

$$\eta \int_0^R (\text{result})\, dr$$

We write this more compactly like this

$$\mathbb{I} = \eta \int_0^R \left(\int_0^{2\pi} r^3 dr \right) d\theta$$

but it means first integrate over θ and then integrate the result over r. Let's try this. Note that nothing inside the parenthesis depends on θ so r^3 and dr are constants as far as the θ integral is concerned. We can take them out

$$\mathbb{I} = \eta \int_0^R r^3 dr \int_0^{2\pi} d\theta$$

then

$$\mathbb{I} = \eta \int_0^R r^3 dr \left(\theta|_0^{2\pi}\right.$$

or

$$\mathbb{I} = \eta \int_0^R r^3 dr \, (2\pi - 0)$$

Now the 2π does not depend on r, so we can take it to the front

$$\mathbb{I} = \eta 2\pi \int_0^R r^3 dr d\theta$$

The remaining integral is an integral we can do!

$$\mathbb{I} = 2\pi\eta \left(\left. \frac{r^4}{4} \right|_0^R \right.$$

or

$$\mathbb{I} = 2\pi\eta \left(\frac{R^4}{4} - 0 \right)$$

and finally

$$\mathbb{I} = 2\pi\eta \frac{R^4}{4}$$

Now remember that

$$\eta = \frac{M}{A} = \frac{M}{\pi R^2}$$

so for our wheel

$$\mathbb{I} = 2\pi \left(\frac{M}{\pi R^2} \right) \frac{R^4}{4}$$

or

$$\mathbb{I} = \frac{1}{2} M R^2$$

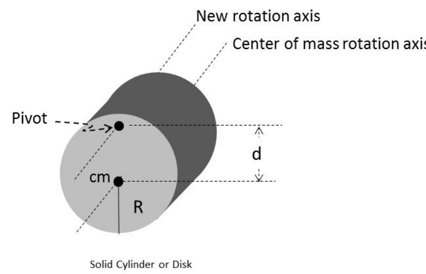
which is what our table has. We can use this integration technique for any shape we have so long as we know the material does not change density. In fact, if we let the density

[13] Take another math class!

$\eta(r,\theta)$ change with position, we can even use this technique for any object (but sadly, we won't in PH121).

Parallel axis theorem

Suppose we know the moment of inertial for some object, say, our wheel. But suppose we want to rotate it around a different point than the center of mass.

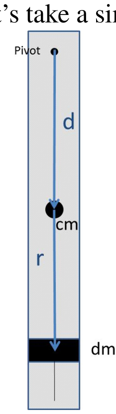


It turns out that as long as the axis is parallel to the direction the axis would have been if the object rotated about the center of mass we can easily calculate the moment of inertial about the new pivot using the center of mass equation from our table. The way to do this is

$$\mathbb{I} = \mathbb{I}_{cm} + d^2 M$$

where M is the mass of the object and d is the distance from the center of mass axis to the new axis of rotation.

To show that this really will work, let's take a simple bar,



and consider rotating it about the pivot shown. We can use our new integral equation for moment of inertia

$$\mathbb{I} = \int r_m^2 dm$$

The r_m is the distance from the pivot to a piece of mass dm. Then for the piece of mass marked in the figure,

$$r_m = d + r$$

Then the moment of inertial would be

$$\mathbb{I} = \int (d + r)^2 \, dm$$

and we can expand out the squared term

$$\mathbb{I} = \int \left(d^2 + 2dr + r^2\right) dm$$

and divide up the integrals into parts

$$\mathbb{I} = \int d^2 dm + \int 2dr dm + \int r^2 dm$$

Note that d is a constant, so we can take it out of the integrals

$$\mathbb{I} = d^2 \int dm + 2d \int r dm + \int r^2 dm$$

The first term has the integral of dm

$$\begin{aligned} \int dm &= (m|_0^M \\ &= M \end{aligned}$$

so

$$\mathbb{I} = d^2 M + 2d \int r dm + \int r^2 dm$$

The last integral is just the moment of inertial about the center of mass

$$\mathbb{I} = d^2 M + 2d \int r dm + \mathbb{I}_{cm}$$

And from our definition of center of mass we remember that

$$r_{cm} = \frac{1}{M} \int r dm$$

so the middle integral

$$\int r dm = (M r_{cm})$$

then

$$\mathbb{I} = d^2 M + 2d \left(r_{cm} M\right) + \mathbb{I}_{cm}$$
$$\mathbb{I} = d^2 M + 2d r_{cm} M + \mathbb{I}_{cm}$$

now if we choose our zero point, $r = 0$, to be the center of mass,

$$r_{cm} = 0$$

then

$$\mathbb{I} = d^2 M + 0 + \mathbb{I}_{cm}$$

or

$$\mathbb{I} = \mathbb{I}_{cm} + d^2 M$$

This may not seem exciting, but it is. It means that if we know the moment of inertial about the center of mass, we can easily find out the moment of inertial about any other axis! well, any other axes that is parallel to the axis we know. We can take the moment of inertia table and extend it's use without the need to do our integration. Let's try it. The moment if inertial for a long, thin rod about it's center of mass is

$$\mathbb{I} = \frac{1}{12}ML^2$$

Suppose we want the moment of inertia for the rod spinning around one end. That would be

$$\mathbb{I}_{end} = \mathbb{I}_{cm} + d^2M$$

and

$$d = \frac{L}{2}$$

in this case so

$$\mathbb{I}_{end} = \frac{1}{12}ML^2 + \left(\frac{L}{2}\right)^2 M$$

$$\mathbb{I}_{end} = \frac{1}{12}ML^2 + \frac{1}{4}L^2M$$

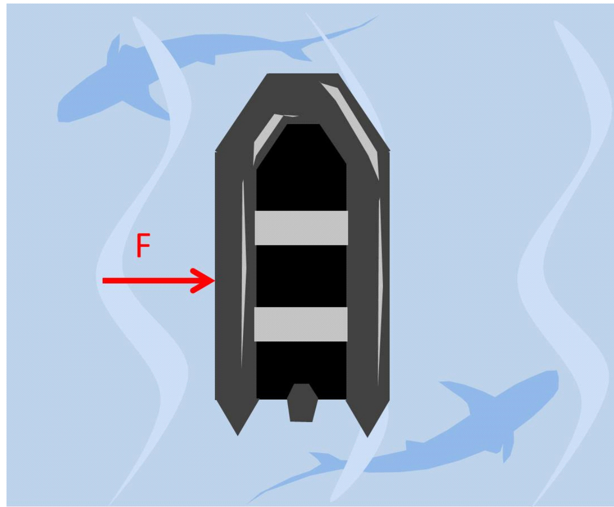
$$\begin{aligned} I_{end} &= \frac{1}{12}ML^2 + \frac{3}{12}L^2M \\ &= \frac{4}{12}ML^2 \\ &= \frac{1}{3}ML^2 \end{aligned}$$

just as our table tells us. Using this, we could find the moment of inertial for the rod rotating around a point, say, a quarter of the length from one end.

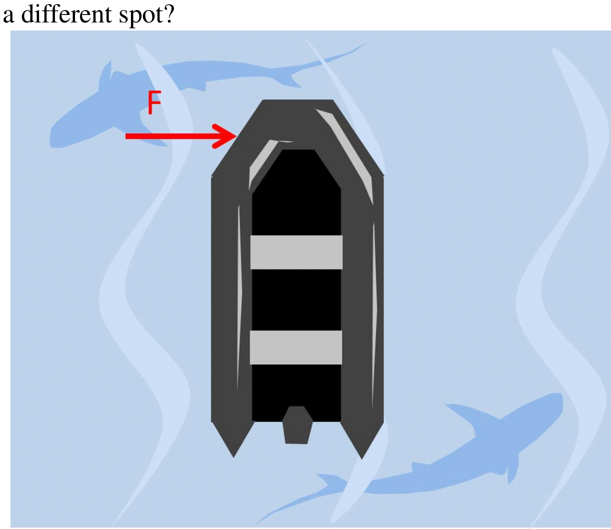
This way of finding a new moment of inertia from the moment about the center of mass has a name, and the name shows a limitation. It is called the *parallel axis theorem.* And this is the limitation, the two axes of rotation must be parallel for it to work.

Torque

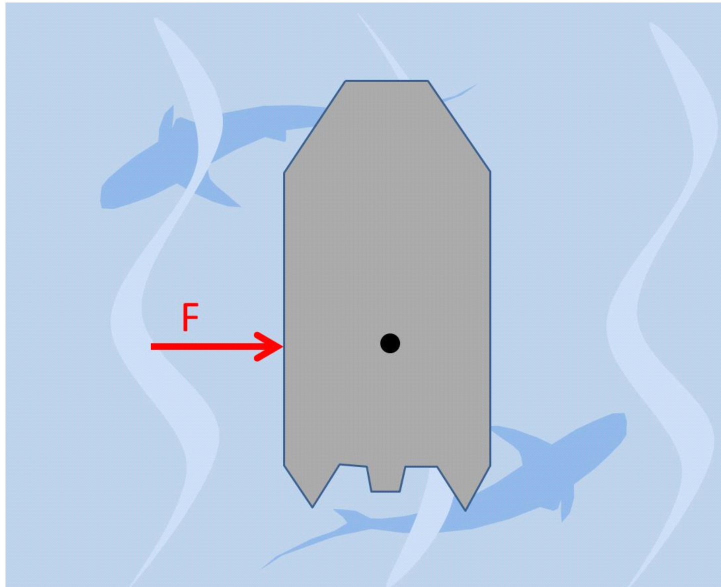
In studying motion with rotation included, we need to revisit forces. A force acting on an object will make the object accelerate.



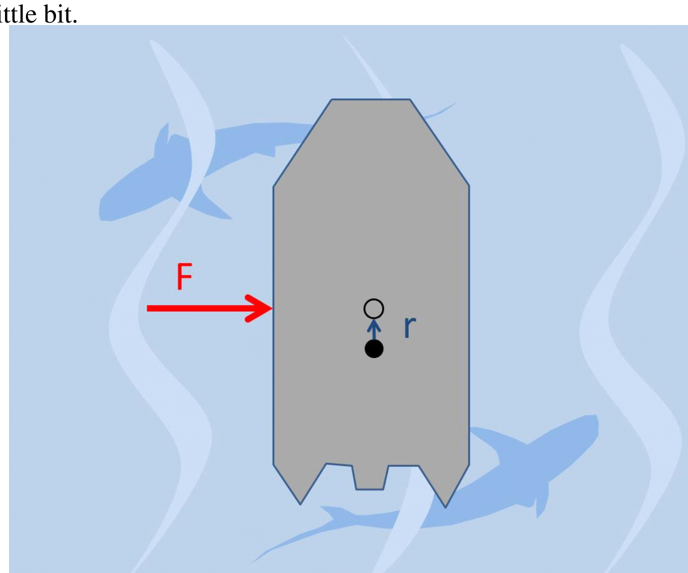
The boat in the figure would accelerate to the right. But what would happen if we pushed on the boat in a different spot?



Experience tells us that the boat will now rotate as well as move to the right. To predict the effect of a force, we will have to go beyond our particle model if we want to know about rotation! We need to know where the push happens with respect to the center of mass. If we push right at the center of mass, no rotation.



But if we push even a small distance away from the center of mass, the boat will rotate at least a little bit.

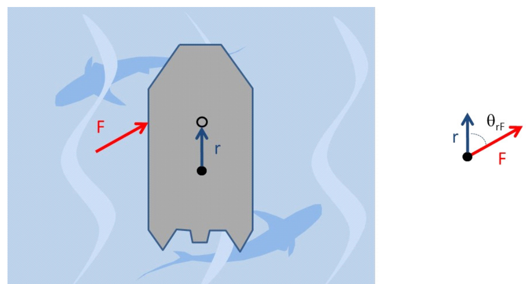


The farther the distance from the center of mass, the more the rotation. Of course, our boat will rotate about it's center of mass. We call the point of rotation a *pivot*. So for the boat case the pivot is the center of mass of the boat that we have been drawing.

But from our boat experience, to describe the motion of the boat including the boat's rotation, we need a quantity that will account for not only how hard we push, but also how far away from the pivot we push. That term is

$$\tau = rF\sin\theta_{rF}$$

Notice that it does have the magnitude of the force in it. It also has the distance, r, from the pivot. And it also has the sine of the angle between the $\overrightarrow{r}$ and $\overrightarrow{F}$ directions.



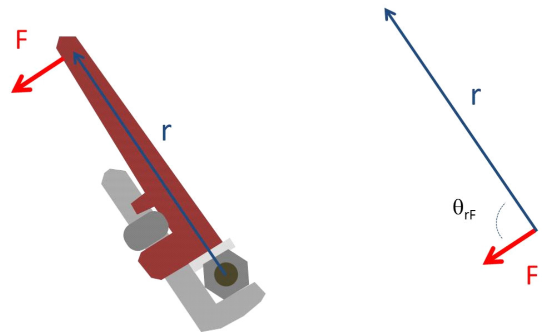
So our new quantity has all the parts we need to find the motion of the boat. We give this new quantity the symbol τ which is the Greek letter "tau," or "t-looking-thing," and we call this combination of distance and force *torque.*

If we think about it, the dependence on the angle makes sense. If we push against the side of the boat perpendicular to the axis of the boat, we would get more rotation than if we push at some angle.

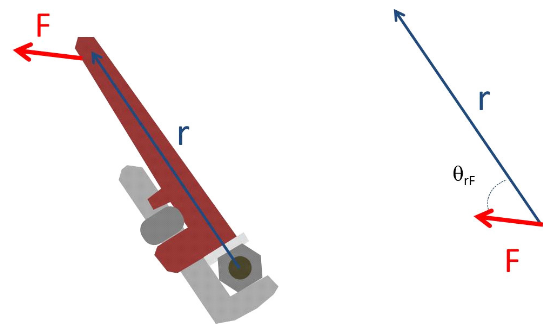
Of course, I could get a group of friends (or villains, or whatever) to all help push on the boat. The net torque is like a net force. We add up the contribution for each push on the boat.

$$\begin{aligned}\tau_{net} &= \tau_1 + \tau_2 + \tau_3 + \cdots + \tau_N \\ &= \sum_i \tau_i\end{aligned}$$

We do have to take into consideration which way the friends push. Let's make an agreement that if we plot the $\overrightarrow{r}$ and $\overrightarrow{F}$ and if we go counter clockwise from $\overrightarrow{r}$ to $\overrightarrow{F}$ we will call the torque positive. If we go clockwise from $\overrightarrow{r}$ to $\overrightarrow{F}$ we will say the torque is negative. It might be easier to see this with another system, say, a wrench and a nut.



Notice that in going from $\overrightarrow{r}$ to $\overrightarrow{F}$ we go counter clockwise. This is positive torque. If you have experience with wrenches, you will recognize that if we push at a different angle it would be less effective.



We would have less torque. The sine of the angle is important.

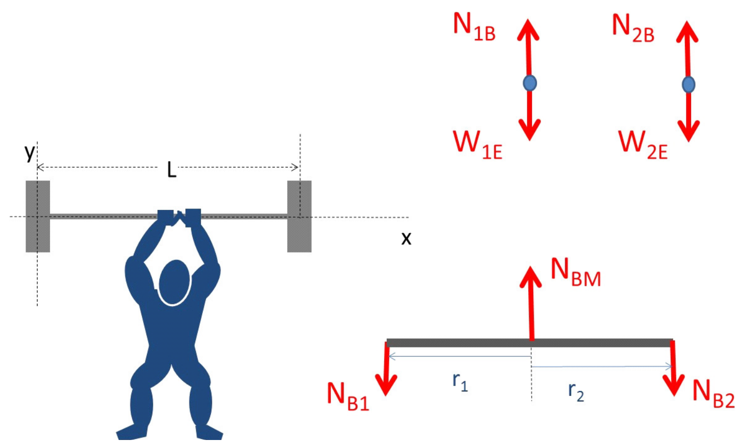
Let's have our boat friends all push the boat so it spins in a counter clockwise manner, then all the torques will be positive.

$$\begin{aligned}\tau_{net} &= r_1F_1\sin\theta_{rF1} + r_2F_2\sin\theta_{rF2} + r_3F_3\sin\theta_{rF3} + \cdots + r_NF_N\sin\theta_{rFN} \\ &= \sum_i r_iF_i\sin\theta_{rFi}\end{aligned}$$

where each person pushes with their own force at their own location and at their own angle.

Let's try a problem with torque. Let's invite our weight lifter to lift his barbell with his hands in the middle (don't really do this!). What is the net torque on the barbell?

Notice that we can not draw a free-body diagram using our particle model. We have to draw a diagram with a box the shape of the bar. We have to draw a diagram that shows where the forces act. This is called an *extended free body diagram.* From our extended free body diagram it is clear that weight 1 is likely to make the barbell rotate in a counter-clockwise direction and weight 2 is likely to make the barbell rotate in a



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clockwise direction. We will say that the torque from N_{B1} will be positive and the torque from N_{B2} will be negative.

Let's choose the pivot point to be the middle of the bar, that is the center of mass, then let's choose a coordinate system with the origin at the center of the bar so that

$$r_{cm} = 0$$

We then can write out

$$\tau_{net} = \sum_i \tau_i$$

using

$$\tau = rF \sin\theta_{rF}$$

We get

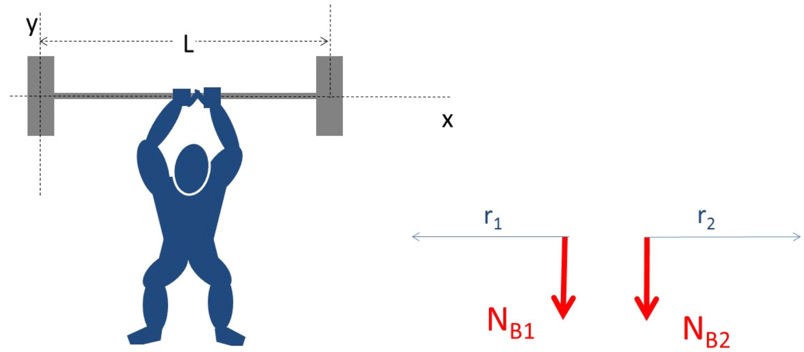
$$\begin{aligned}\tau_{net} &= r_1 N_{B1} \sin\theta_{rW_1} - r_2 N_{B2} \sin\theta_{rW_2} + r_{cm} N_{BM} \sin\theta_{rN} \\ &= r_1 N_{B1} \sin\theta_{rW_1} - r_2 N_{B2} \sin\theta_{rW_2} + 0 \\ &= r_1 m_1 g \sin\theta_{rW_1} - r_2 m_2 g \sin\theta_{rW_2}\end{aligned}$$

Notice that the angles θ_{rW_1} and θ_{rW_2} are both $90\,^{\circ}$. These are angles between two vectors, in this case the weights and the position vectors. Note that they are not the angles from the x-axis! We know the $\sin(90) = 1$ so

$$\begin{aligned}\tau_{net} &= r_1 m_1 g - r_2 m_2 g \\ &= r_1 W_{1E} - r_2 W_{2E}\end{aligned}$$

Since the two weights are the same, this gives

$$\tau_{net} = 0$$



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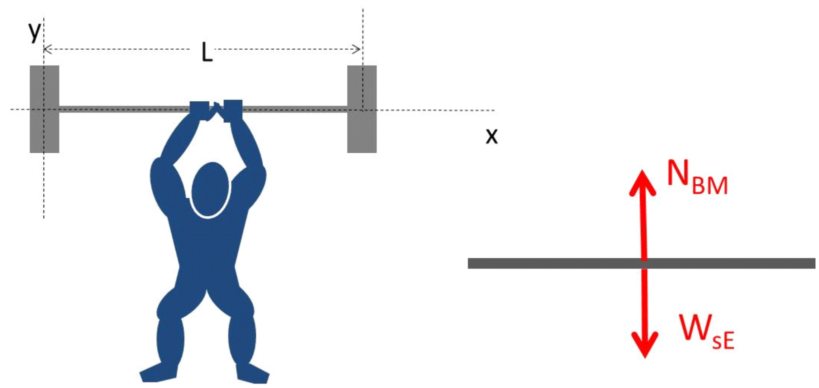


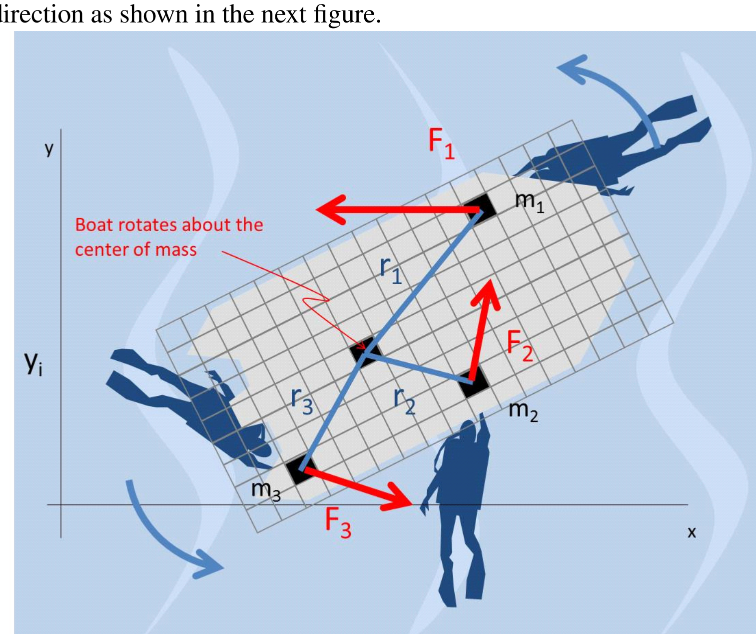
Now we can see why the barbell balances. A normal force at the center of mass will be opposed by the weight of the entire barbell (where s is for the barbell "system"), acting on the center of mass. There is no net force. And we have shown that there is no net torque. We will add to our definition of equilibrium that things in equilibrium will have zero net torque.

$$\begin{aligned} \overrightarrow{F}_{net} &= 0 \\ \tau_{net} &= 0 \end{aligned}$$

Armed with the idea of torque, moment of inertia, and angular acceleration, we can do rotational dynamics problems. And that is just what we will do in the next lecture.

32 Dynamics of Rotation

Now that we understand torque, we can fully take on rotational problems. Suppose the intrepid Mr. Bland is being attacked by divers from below. Each grabs onto his boat from underneath. But they are not very well organized, so each pushes the boat in a different direction as shown in the next figure.

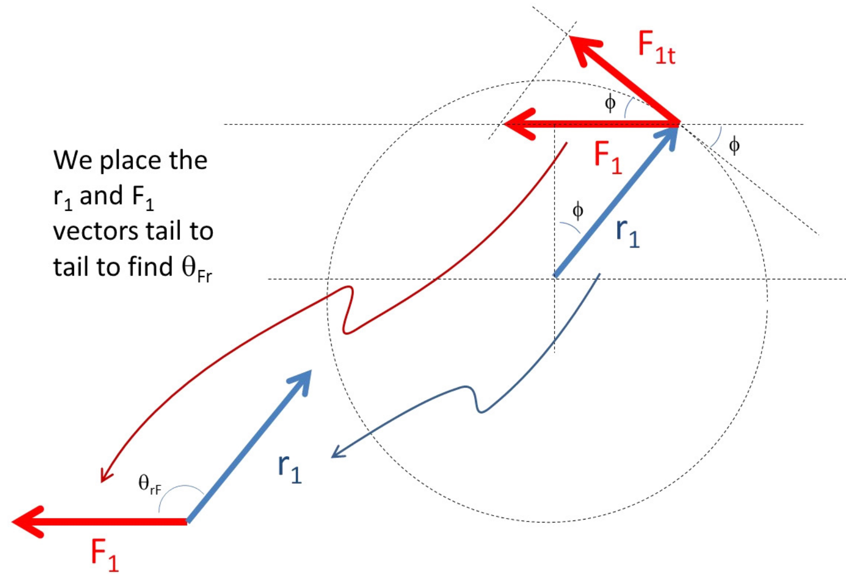


It's clear that the boat will go in a circle. Each diver will exert a torque on the boat. We could find the net torque!

Remember that torque is a force applied at a distance from the pivot point. For our boat, the pivot point will be the center of mass. Let's take one of the torques

$$\tau_1 = r_1 F_1 \sin\theta_{rF1}$$

Let's look at this geometrically for a moment



Notice that the angle

$$\theta_{rF} = 90^\circ + \phi$$

then

$$\begin{aligned}\sin\theta_{rF} &= \sin(90^\circ + \phi)\\ &= \cos\phi\end{aligned}$$

Now notice that up at the top, ϕ is the angle between F_1 and the tangent line that passes through the tip of the r_1 vector. The tangential component of F_1 would be

$$F_{1t} = F_1\cos\phi$$

or

$$F_{1t} = F_1\sin\theta_{rF}$$

so we can write our torque as

$$\tau_1 = r_1F_{1t}$$

and we know

$$F_{1t} = m_1a_{1t}$$

This a_{1t} is the part of the acceleration that can make part m_1 of the boat speed up as it goes around in a circle (a_{1r} makes part m_1 stay in the circle), so we could write our force using $a_t = r\alpha$

$$F_{1t} = m_1r_1\alpha$$

and the torque would be

$$\begin{aligned}\tau_1 &= r_1m_1r_1\alpha\\ &= m_1r_1^2\alpha\end{aligned}$$

It is also true that

$$\begin{aligned}\tau_2 &= m_2 r_2^2 \alpha \\ \tau_3 &= m_3 r_3^2 \alpha\end{aligned}$$

so the net torque is

$$\tau_{net} = \sum_i m_i r_i^2 \alpha$$

and we know that for an isolated piece of mass a diatance r_i from a pivot point

$$\mathbb{I} = \sum_i m_i r_i^2$$

then

$$\tau_{net} = \mathbb{I}\alpha$$

And this last equation is great! It relates the angular acceleration to the net torque and the moment of inertia..

$$\alpha = \frac{\tau_{net}}{\mathbb{I}}$$

Think about this for a minute. Back when we studied Newton's second law, we found that

$$a = \frac{F_{net}}{m}$$

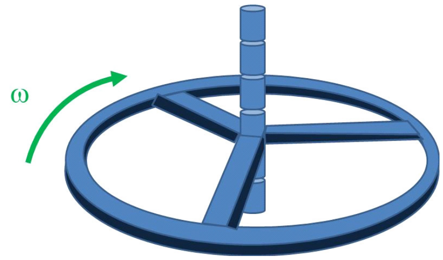
The force was how hard we push, and the mass was how hard it was to make the object go.

Now we have

$$\alpha = \frac{\tau_{net}}{\mathbb{I}}$$

which combines how hard we push an object with the direction of the push and the distance from the center of mass of the object with how hard the object is to rotate. We have fixed Newton's second law so that it includes rotation!

Let's try a problem. You are on a ring-designed space station that rotates to create the feel of having gravity. The centripetal acceleration needs to be g to do this. The ring has a large radius of $5000.0\,\mathrm{m}$.You, the chief engineer, start with the station not spinning. You apply a gentle angular acceleration of $5.12 \times 10^{-7}\,\mathrm{rad/\,s^2}$ to the structure to spin up the rotation. Let's suppose that we can ignore the rest of the station and just talk about the ring rotating about its center of mass. Suppose the ring has a mass of $4.196 \times 10^6\,\mathrm{kg}$. What torque is required to spin-up the station?



We know

$$\begin{aligned}\alpha &= 5.12 \times 10^{-7}\,\mathrm{rad/s^2} \\ R &= 5000.0\,\mathrm{m} \\ M &= 4.196 \times 10^{6}\,\mathrm{kg}\end{aligned}$$

Our basic equations

$$\alpha = \frac{\tau_{net}}{\mathbb{I}}$$

From our table

$$\mathbb{I}_{ring} = MR^2$$

Then we could write

$$\alpha = \frac{\tau_{net}}{\mathbb{I}}$$

as

$$\tau_{net} = \mathbb{I}\alpha$$

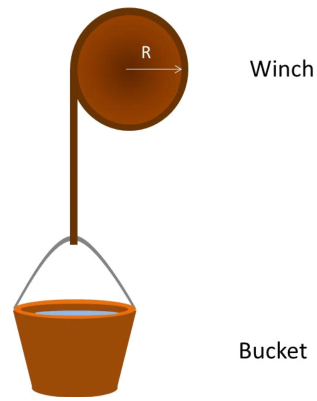
then

$$\tau_{net} = MR^2\alpha$$

$$\begin{aligned}\tau_{net} &= \left(4.196 \times 10^{6}\,\mathrm{kg}\right)\left(5000.0\,\mathrm{m}\right)^2\left(5.12 \times 10^{-7}\,\mathrm{rad/s^2}\right) \\ &= 5.3709 \times 10^{7}\,\mathrm{N\,m} \\ &= 5.37 \times 10^{7}\,\mathrm{N\,m}\end{aligned}$$

We should make two observations. The radians are a unit, but have no dimensions. So we can drop them in the final units for our calculations. We are left with $\mathrm{N\,m}$. It might be tempting to write this as an energy unit. Work is $\mathrm{N\,m}$ as well. But torque is really not the same thing as energy, so we will keep the torque units as just $\mathrm{N\,m}$. *We won't use Jules as the unit of torque* because torque is really not an energy.

The standard problem to do to demonstrate how to do rotational dynamics goes back to our pioneer ancestors. Suppose we have an old fashioned water well where you draw the water with a bucket. The bucket is on a rope and the rope goes around a cylindrical piece of wood and winds on the cylinder. Let's call this the winch.



Also suppose the winch cylinder has a mass of $1.5\,\mathrm{kg}$ and a radius of $2.0\,\mathrm{cm}$. Let's also assume that the bucket starts from rest hanging just at the top of the well. Then you let it go, falling on the rope into the well. It takes $0.85\,\mathrm{s}$ for the bucket to reach the water below. How deep is the well? We know

$$\begin{aligned} m_w &= 1.5\,\mathrm{kg} \\ r_w &= 0.02\,\mathrm{m} \\ m_b &= 2.0\,\mathrm{kg} \\ \Delta t &= 0.85 \end{aligned}$$

We could guess that the bucket will have a constant acceleration, so this is probably a kinematics problem. We could start there.

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\ v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \end{array}$$

For the bucket, x-direction motion is not important, so let's use the y-set of equations. We could define the starting point of the bucket to be

$$y_i = 0$$

and we know

$$v_{bi} = 0$$

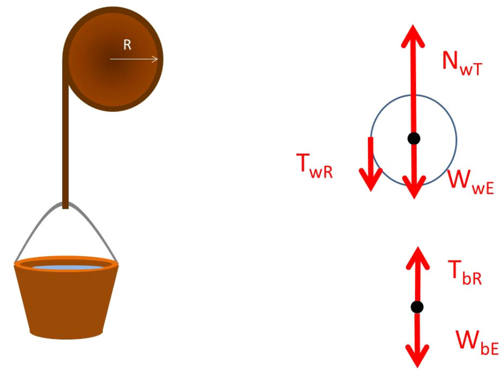
because the bucket starts from rest. Marking what we know and using our zeros gives

$$\begin{aligned} y_{bf} - 0 &= 0 + \tfrac{1}{2}a_{by}\underline{\Delta t}^2 \\ v_{bfy} &= 0 + a_{by}\underline{\Delta t} \\ v_{bfy}^2 &= 0 + 2a_{by}\left(y_{bf} - 0\right) \end{aligned}$$

and we are almost there.

$$y_{bf} = \frac{1}{2}a_{by}\underline{\Delta t}^2$$

We need a_{by}. But we have learned how find acceleration, we use the tool we call Newton's second law! We can try this now to find a_{by}. But we are sophisticated enough to realize that we have rotation, so we might need extended free body diagrams.



And here are our basic equations for Newton's second law (including our rotational newton's second law)

$$a = \frac{F_{net}}{m}$$
$$\alpha = \frac{\tau_{net}}{\mathbb{I}}$$
$$\tau = rF\sin\theta$$

Let's start by writing out Newton's second law and Newton's second law for rotation.

$$\begin{aligned} F_{net_y} &= -m_b a_{by} \\ &= T_{bR} - W_{bE} \end{aligned}$$

There are no x-direction forces for the bucket. The bucket is not likely to rotate, so $\alpha_b = 0$ and $\tau_b = 0$ and we can write the torque equation for the bucket as

$$\begin{aligned} \tau_b &= \mathbb{I}_b \alpha_b \\ 0 &= 0 \end{aligned}$$

which is not terribly helpful.

Now for the winch. We immediately identify $T_{wR} = T_{bR}$ since, of course, our pioneer ancestors had a massless rope (we are using massless rope approximation, but it is not too bad, usually the rope used in a well had far less mass than the bucket of water).

Now we add our torque equation

$$\tau = \mathbb{I}\alpha$$

we need the moment of inertia for a cylinder rotating about it's center of mass (assuming our ancestors built a good winch). We could get this by integrating, but I think we will use the table for this one

$$\mathbb{I}_w = \frac{1}{2}MR^2$$

then

$$\tau_w = \frac{1}{2}m_w R^2 \alpha$$

Now we need to realize we have a constraint. The rope can't slip on the winch because it is attached to the winch and wrapped around it. So the rope's acceleration and the bucket's acceleration must be equal to the tangential acceleration of the winch cylinder.

$$a_{by} = a_{wt}$$

And

$$a_{wt} = R\alpha$$

so

$$\alpha = \frac{a_{wt}}{R}$$

Then the torque on the winch would be

$$\tau = \frac{1}{2}m_w R^2 \left(\frac{a_{wt}}{R}\right)$$

It's also true that

$$\tau = rF\sin\theta_{rF}$$

which in our case is

$$\tau = RT_{wR}\sin\theta_{RT}$$

we can see that in the angle between the tension force and the displacement from the center of the winch would be $\theta_{RT} = 90\,^\circ$ so

$$\tau = RT_{wR}$$

Setting these two expressions for the torque on the winch equal to each other gives

$$RT_{wR} = \frac{1}{2}m_w R^2 \left(\frac{a_{wt}}{R}\right)$$

Now we need to go back to Newton's second law for the bucket

$$-m_b a_{by} = T_{bR} - W_{bE}$$

so using our massless string approximation we have

$$T_{bR} = T_{wR} = -m_b a_{by} + W_{bE}$$

then by substitution,

$$R\left(-m_b a_{by} + W_{bE}\right) = \frac{1}{2}m_w R^2 \left(\frac{a_{wt}}{R}\right)$$

or, dividing through by R we have

$$-m_b a_{by} + W_{bE} = \frac{1}{2}m_w a_{wt}$$

Solving for W_{bE} gives

$$-m_b a_{by} - \frac{1}{2} m_w a_{wt} = -W_{bE}$$

Now, using our constraint $a_{by} = a_{wt}$

$$\left(m_b + \frac{1}{2} m_w\right) a_{wt} = W_{bE}$$

so

$$a_{wt} = \frac{W_{bE}}{\left(m_b + \frac{1}{2} m_w\right)}$$

and we have the tangential acceleration of the winch, but this is also the acceleration of the rope and bucket. And this acceleration is not changing. None of the terms in a_{wt} will change as the bucket goes downward. So from this point our problem is a constant acceleration problem. Since $a_{wt} = a_{by}$ we are ready to complete our kinematic equation (remember our acceleration is downward!)

$$\begin{aligned} y_{bf} &= \frac{1}{2} a_{by} \underline{\Delta t}^2 \\ &= \frac{1}{2} \left(-\frac{W_{bE}}{\left(m_b + \frac{1}{2} m_w\right)} \right) \underline{\Delta t}^2 \end{aligned}$$

or just

$$\Delta y = \frac{1}{2} \left(-\frac{m_b g}{\left(m_b + \frac{1}{2} m_w\right)} \right) \Delta t^2$$

and we know all these pieces! The depth will be

$$\begin{aligned} \Delta y &= \frac{1}{2} \left(-\frac{(2.0\,\mathrm{kg}) \left(9.8 \frac{\mathrm{m}}{\mathrm{s}^2}\right)}{\left(2.0\,\mathrm{kg} + \frac{1}{2}(1.5\,\mathrm{kg})\right)} \right) (0.85\,\mathrm{s})^2 \\ &= -2.5747\,\mathrm{m} \end{aligned}$$

Notice that we used torque, Newton's second law and Newton's second law for rotation, *and* kinematics. The last part is what makes this dynamics. we used forces and torques to find acceleration, and knowing acceleration we can find how the object moves.

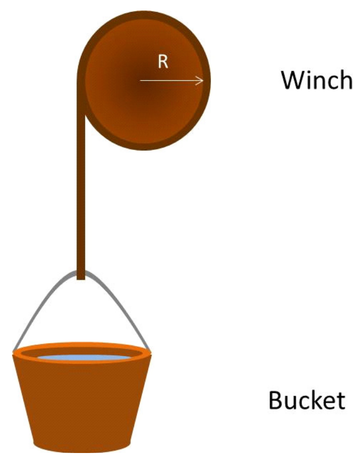
33 Equilibrium Revisited

We studied equilibrium before. We defined static equilibrium as the condition of having no net force.

$$\overrightarrow{F}_{net} = 0$$

with our object in our reference frame not moving.

But now we realize that something with no net force could have a net torque. Think of the winch in the last lecture. The winch did not accelerate, but it did rotationally accelerate!



In this lecture we want to extend our definition of static equilibrium. Our winch is really rotating, and it is rotationally accelerating. This is not entirely static, since it *is* moving rotationally. We can easily fix our definition for static equilibrium by requiring the net torque to be zero as well as the net force.

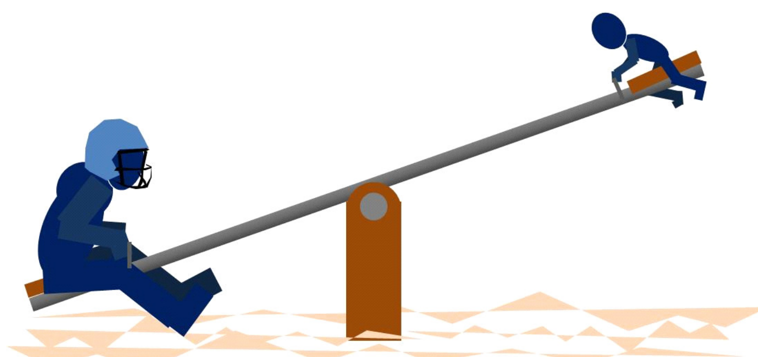
$$\begin{aligned} \overrightarrow{F}_{net} &= 0 \\ \tau_{net} &= 0 \end{aligned}$$

Now an object in static equilibrium will be static–not moving (within our reference frame).

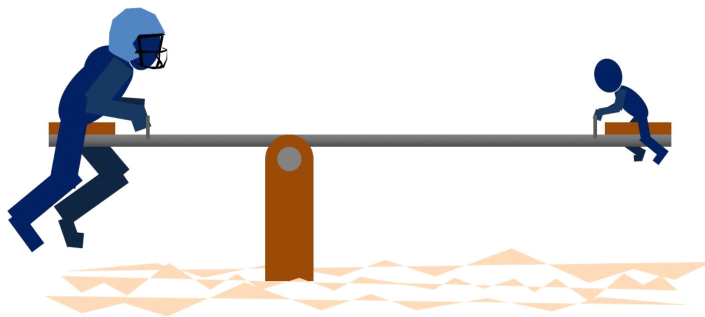
Let's try a problem: Let's invite back our linebacker and six-year-old child. And let's have them try to play on a teeter-totter. This is an old-fashioned toy that is hard to find in the United States. Here are some kids in Afghanistan playing on one.

The design is simple. You have a beam balanced on a pivot point. A person sits on each side. Each person's weight causes a torque on the beam. Either person can cause an additional torque by kicking off of the ground, allowing both people to have a ride.

Our line backer (subscript L for "linebacker") and child (subscript 6 for "six-year-old") try one of these things. We will also need a subscript B for "bar," S for the pivot "support," and E for "Earth." The line backer and six-year-old get on opposite sides of the device

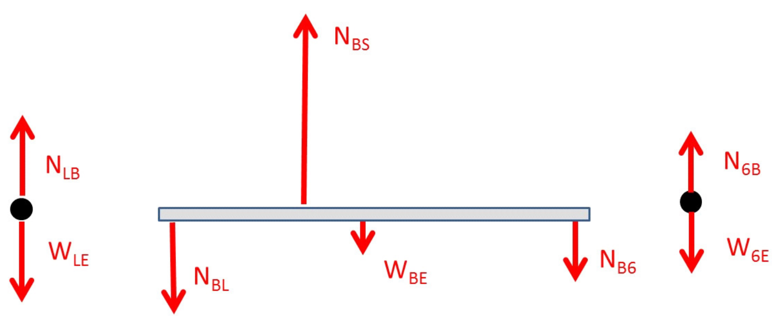


but at first it won't work. The linebacker is to heavy and the child ends up stuck in the air. A good teeter-totter is adjustable so you can move the pivot point. The linebacker does this and now the teeter-totter can balance.



Suppose our linebacker has a mass of $111.2\,\mathrm{kg}$ and the child has a mass of $27\,\mathrm{kg}$. Also suppose the beam is $4.0\,\mathrm{m}$ long and the linebacker and child sit right on the ends of the beam. Where should the pivot have to be so that the beam can balance if the beam has a mass of $15\,\mathrm{kg}$?

We will need free-body diagrams. The bar will have to have an extended free body diagram because it can rotate.



We know

$$
\begin{aligned}
m_L &= 111.2\,\mathrm{kg} \\
m_6 &= 27\,\mathrm{kg} \\
L &= 4.0\,\mathrm{m} \\
m_B &= 15\,\mathrm{kg}
\end{aligned}
$$

and our basic equations for static equilibrium are now

$$\begin{aligned}\sum \overrightarrow{\tau} &= 0 \\ \sum \overrightarrow{F} &= 0\end{aligned}$$

$$\tau = rF\sin\theta_{rF}$$

$$\overrightarrow{F}_{net} = m\overrightarrow{a}$$

We should use newton's second for the linebacker and the six-year-old

$$\begin{aligned}F_{netL} &= m_L a_L \\ &= N_{LB} - W_{LE} \\ F_{net6} &= m_6 a_6 \\ &= N_{6B} - W_{6E}\end{aligned}$$

and, as they balance we know, $a_L = 0$ and $a_6 = 0$ so

$$\begin{aligned}N_{AB} &= W_{AE} \\ N_{CB} &= W_{CE}\end{aligned}$$

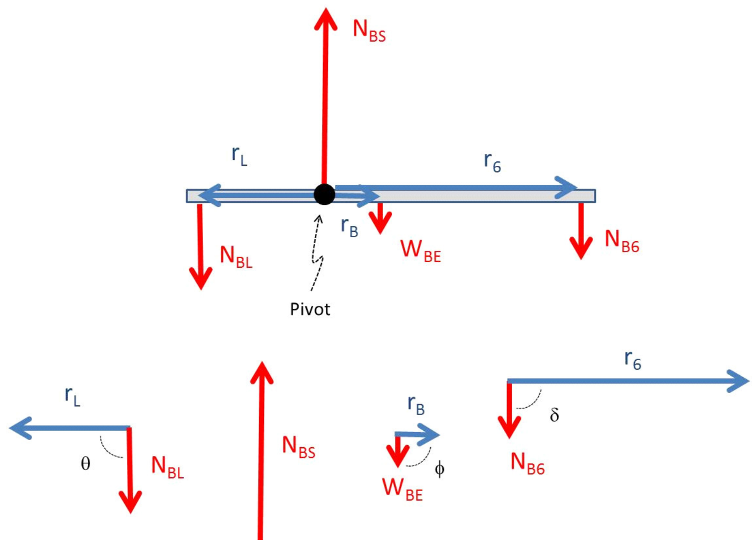
then for the board

$$\begin{aligned}F_{net_y} &= m_B a_B \\ &= N_{BS} - N_{BL} - W_{BE} - N_{B6}\end{aligned}$$

and it is also true that the board is not accelerating since we are in static equilibrium, so

$$0 = N_{BS} - N_{BL} - W_{BE} - N_{B6}$$

Now for torque, let's choose the pivot as the, well, pivot, right where the support holds up the bar. If that is our pivot choice, then we can define distances from that pivot point as shown.



We can use our rotational Newton's second law and our torque basic equation

$$\tau = rF\sin\theta_{rF}$$

to find

$$\begin{aligned}\sum\tau &= \mathbb{I}\alpha \\ &= r_L N_{BL}\sin\theta + (0)\,N_{BP}\sin(?) - r_B W_{BE}\sin\phi - r_6 N_{B6}\sin\delta = 0\end{aligned}$$

For each of the angles I used a different letter, but each is the angle between the displacement from the pivot and the force vectors. Notice that there must be a torque for each force acting on the bar, and we have to look at the angles carefully and determine the signs. Also notice that the normal force N_{BS} that acts on the pivot is going to be multiplied by zero because the distance from the pivot is zero for this force. So we don't need to worry about the angle for this force. For static equilibrium we now know $\alpha = 0$ so

$$0 = r_L N_{BL} - r_B W_{BE} - r_6 N_{B6}$$

Notice that we have two equations relating the forces on the bar to each other now!

$$0 = N_{BS} - N_{BL} - W_{BE} - N_{B6}$$

and

$$0 = r_L N_{BL} - r_B W_{BE} - r_6 N_{B6}$$

one from Newton's second law for the bar, and one from Newton's second law for rotation. We also have

$$\begin{aligned} N_{LB} &= W_{LE} = m_L g \\ N_{6B} &= W_{6E} = m_6 g \end{aligned}$$

from Newton's second law for the linebacker and the six-year-old. We can use these last two equations to fill in the bar equations using force pairs.

$$\begin{aligned} 0 &= N_{BS} - m_L g - W_{BE} - m_6 g \\ 0 &= r_L m_L g - r_B W_{BE} - r_6 m_6 g \end{aligned}$$

and

$$W_{BE} = m_B g$$
$$0 = N_{BS} - m_L g - m_B g - m_6 g$$
$$0 = r_L m_L g - r_B m_B g - r_6 m_6 g$$

Notice that r_B must be the distance from the pivot to the center of mass of the bar. And that is what we want. There are some constraints that we know from the figure

$$L = r_L + r_6$$

and

$$\frac{L}{2} = r_B + r_L$$

We can solve for r_6

$$r_6 = L - r_L$$

and for r_L

$$\frac{L}{2} - r_B = r_L$$

and put this into our second bar equation

$$0 = r_L m_L g - r_B m_B g - (L - r_L)\, m_6 g$$

The $g's$ cancel so

$$0 = r_L m_L - r_B m_B - (L - r_L)\, m_6$$
$$0 = \left(\frac{L}{2} - r_B\right) m_L - r_B m_B - \left(L - \left(\frac{L}{2} - r_B\right)\right) m_6$$

and solve for r_B (this will take some algebra!). Let's distribute the m_L in the first term on the left.

$$0 = \frac{L}{2} m_L - r_B m_L - r_B m_B - \left(L - \left(\frac{L}{2} - r_B\right)\right) m_6$$

Now let's combine the L's in the last term on the right

$$0 = \frac{L}{2} m_L - r_B m_L - r_B m_B - \left(\frac{2}{2} L - \frac{L}{2} + r_B\right) m_6$$

$$0 = \frac{L}{2}m_L - r_B m_L - r_B m_B - \left(+\frac{L}{2} + r_B\right) m_6$$

and distribute the m_6 in the last term on the right.

$$0 = \frac{L}{2}m_L - r_B m_L - r_B m_B - \frac{L}{2}m_6 - r_B m_6$$

Now let's keep everything with an r_B on the left, and move every term without and r_B to the right.

$$-r_B m_L - r_B m_B - r_B m_6 = \frac{L}{2}m_6 - \frac{L}{2}m_L$$

Take out an r_B from every term on the right,

$$(-m_L - m_B - m_6)\, r_B = \frac{L}{2}m_6 - \frac{L}{2}m_L$$

and tidy up the right side

$$-\left(m_L + m_B + m_6\right) r_B = \frac{L}{2}\left(m_6 - m_L\right)$$

now dived both sides by $(m_L + m_B + m_6)$. This gives us r_B

$$r_B = \frac{\left(\frac{L}{2}\right)\left(m_L - m_6\right)}{\left(m_L + m_B + m_6\right)}$$

Putting in numbers gives

$$\begin{aligned} r_B &= \frac{\left(\frac{4\,\mathrm{m}}{2}\right)\left(111.\,2\,\mathrm{kg} - 27\,\mathrm{kg}\right)}{\left(111.\,2\,\mathrm{kg} + 15\,\mathrm{kg} + 27\,\mathrm{kg}\right)} \\ &= 1.\,099\,2\,\mathrm{m} \end{aligned}$$

So the linebacker had to move the bar about a meter off it's midpoint to get the teeter-totter to balance. We could find how far the linebacker is from the pivot.

$$\frac{L}{2} = r_B + r_L$$

$$r_L = \frac{L}{2} - r_B$$

$$\begin{aligned} r_L &= \frac{4\,\mathrm{m}}{2} - 1.\,099\,2\,\mathrm{m} \\ &= 0.900\,8\,\mathrm{m} \end{aligned}$$

The linebacker is very close to the pivot as we would expect, about a meter from the pivot point. We could also find how far the six-year-old would be from the pivot.

$$r_6 = L - r_L$$

$$\begin{aligned} r_6 &= 4\,\mathrm{m} - 0.900\,8\,\mathrm{m} \\ &= 3.\,099\,2\,\mathrm{m} \end{aligned}$$

The six-year-old is much farther away from the pivot as expected. We should check our numbers. Let's plug them back in our newton's second law equation for rotation.

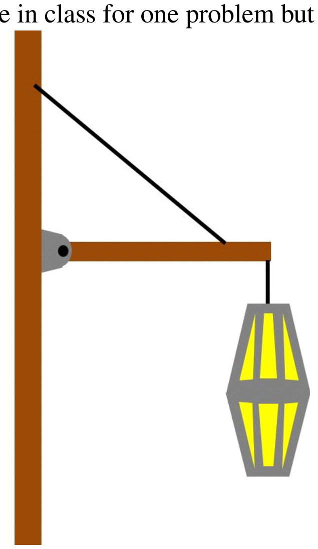
$$0 = r_L m_L g - r_B m_B g - r_6 m_6 g$$

$$\begin{aligned} 0 &= (0.900\,8\,\mathrm{m})\,(111.\,2\,\mathrm{kg}) - (1.\,099\,2\,\mathrm{m})\,(15\,\mathrm{kg}) - (3.\,099\,2\,\mathrm{m})\,(27\,\mathrm{kg}) \\ &= 0.003\,\mathrm{m\,kg} \end{aligned}$$

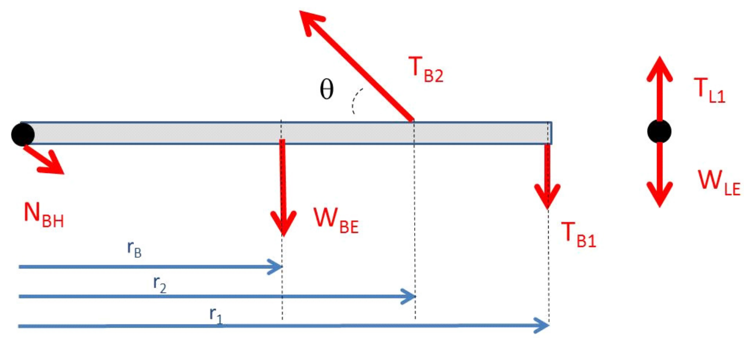
So to two digits we are good. And that is all our significant figures allow, so to within the accuracy of our calculation, this worked!

This is a long problem! But it is a powerful problem. With this type of analysis we can build bridges and buildings that don't collapse! If you were following closely you will have noticed that we never used the equation we got from Newton's second law (the non-rotational one). But I didn't know that we would not need it until I finished the problem. There is no way to know in advance! So write out **all of Newton's second law** *and* **Newton's second law for rotation** even if you are not sure you need all these equations. You won't know for sure until the problem is done.

We will likely only have time in class for one problem but here is another example.



A light weighing $250\,\mathrm{N}$ is suspended from a beam with a weight of $50\,\mathrm{N}$. The beam is connected to the wall with a hinge, so it can rotate about it's left end. The beam is $2.00\,\mathrm{m}$ long. A guy wire is attached $1.75\,\mathrm{m}$ from the hinge at an angle of $40.0\,^{\circ}$ to help support the beam and light. Find the tension in the guy wire.



We know

$$\begin{aligned} W_{LE} &= 250\,\mathrm{N} \\ W_{BE} &= 50\,\mathrm{N} \\ r_2 &= 1.75\,\mathrm{m} \\ r_1 &= 2.00\,\mathrm{m} \\ r_{cm} &= r_1/2 = 1.00\,\mathrm{m} \\ \theta &= 40.0\,^\circ \end{aligned}$$

and our basic equations are

$$\begin{aligned} \sum \overrightarrow{\tau} &= 0 \\ \sum \overrightarrow{F} &= 0 \end{aligned}$$
$$\tau = rF\sin\theta_{rF}$$
$$\overrightarrow{F}_{net} = m\overrightarrow{a}$$

Again let's start with Newton's second law, this time we need the x and y-components.

$$\begin{aligned} F_{netL_y} &= m_L a_{L_x} \\ &= T_{1L} - W_{LE} \end{aligned}$$

and

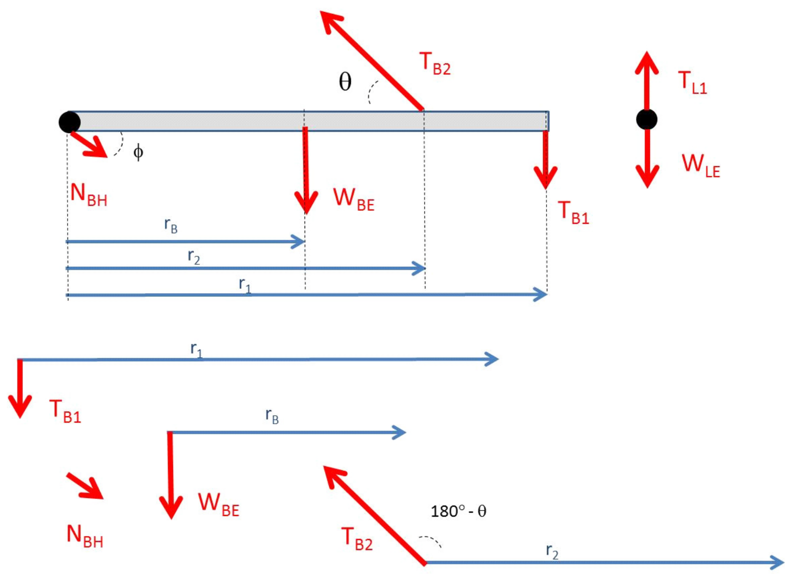
$$F_{netL_x} = 0$$

and for the support beam

$$\begin{aligned} F_{netb_y} &= m_B a_{by} \\ 0 &= -N_{BH}\sin\phi - W_{BE} + T_{B2}\sin\theta - T_{B1} \end{aligned}$$

$$\begin{aligned} F_{netb_y} &= m_B a_{by} \\ 0 &= -N_{BH}\cos\phi + T_{B2}\cos\theta \end{aligned}$$

Let's also use Newton's second for rotation. Here is the extended free body diagram again, with displacements from the pivot marked.



Notice that the vectors have been redrawn to show the angles between each r, F set. I really think this helps. We can find the signs using these as well

$$
\begin{aligned}
\tau_{net} &= \mathbb{I}\alpha \\
0 &= (0)\,N_{BH} - r_B W_{BE} + r_2 T_{B2} \sin\left(180^\circ - \theta\right) - r_1 T_{B1}
\end{aligned}
$$

so we have a three-equation-set like in our last problem, Two from Newton's second law and one from Newton's second for rotation.

$$
\begin{aligned}
N_{BH} \cos\phi &= T_{B2} \cos\theta \\
N_{BH} \sin\phi &= W_{BE} - T_{B2} \sin\theta + T_{B1}
\end{aligned}
$$

$$0 = -r_B W_{BE} + r_2 T_{B2} \sin\left(180^\circ - \theta\right) - r_1 T_{B1}$$

From the light we know that

$$T_{1R} = W_{LE}$$

and we know W_{LE}, so

$$
\begin{aligned}
N_{BH} \cos\phi &= W_{LE} \cos\theta \\
N_{BH} \sin\phi &= W_{BE} - W_{LE} \sin\theta + T_{B1}
\end{aligned}
$$

$$0 = -r_B W_{BE} + r_2 T_{B2} \sin\left(180^\circ - \theta\right) - r_1 W_{LE}$$

Let's solve the last for T_{B2}

$$r_B W_{BE} + r_1 W_{LE} = +r_2 T_{B2} \sin\left(180^\circ - \theta\right)$$

$$\frac{r_B W_{BE} + r_1 W_{LE}}{r_2 \sin\left(180^\circ - \theta\right)} = T_{B2}$$

and we know all of these

$$\begin{aligned}\frac{(1.00\,\mathrm{m})\,(50\,\mathrm{N})+(2.00\,\mathrm{m})\,(250\,\mathrm{N})}{(1.75\,\mathrm{m})\sin\left(180^\circ-40.0^\circ\right)} &= T_{B2}\\ &= 488.94\,\mathrm{N}\\ &= 489\,\mathrm{N}\end{aligned}$$

If you were to build this system, you would need to buy wire that can withstand this force (and more, incase of wind or snow).

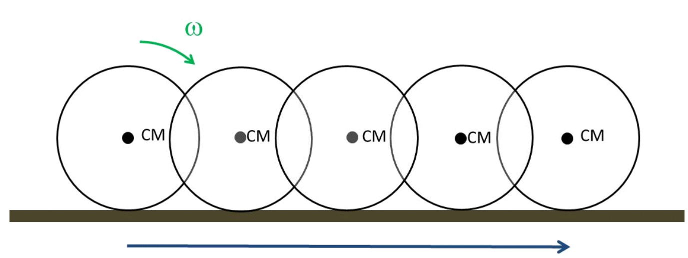
The problems are long, but not really hard. And they are the start of designing working systems for things people really build.

34 Wheels, Angular Momentum

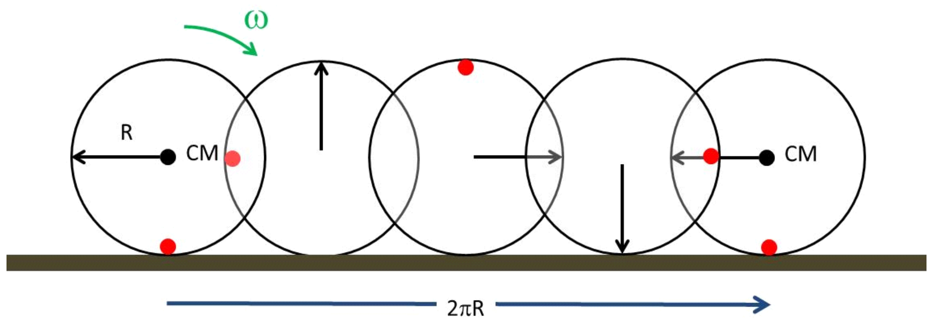
We learned earlier that having mass makes an object hard to stop. And also moving fast makes an object hard to stop. We called the combination of mass and velocity momentum. But wheels have mass, and if they are spinning fast, it is hard to stop the spin. Do spinning things have momentum? Let's review wheels a bit, and then see if we can find a momentum in spinning.

Summary of rotating wheels

There is a special case of rotational motion that we have used in our problems. That special case is a moving wheel rotating without slipping. This is the normal way a car tire moves, for example. We now know that we could describe the motion of the wheel as a whole with the motion of the center of mass of the wheel. Here is a motion diagram for the wheel.



The motion of the wheel as a whole is clearly represented by the motion of the center of mass of the wheel. But this center of mass representation does not show the wheel's rotation. If we mark a place on the wheel and follow the motion of the marked place, say, the point on the wheel that starts out next to the ground,



we find that the wheel will travel a distance $x = 2\pi R$ before the marked wheel part will again be on the ground. This is our constraint of not slipping. Every time the wheel turns, the wheel travels one circumference. The time it takes a wheel to complete one revolution is called the period, T. So the speed of the wheel as a whole is

$$\begin{aligned} v_{cm} &= \frac{\Delta x}{\Delta t} \\ &= \frac{2\pi R}{T} \end{aligned}$$

Recall that for a spot on the wheel the tangential speed is given by

$$v_t = r\omega$$

and if we choose a spot on the edge of the wheel then

$$v_t = R\omega$$

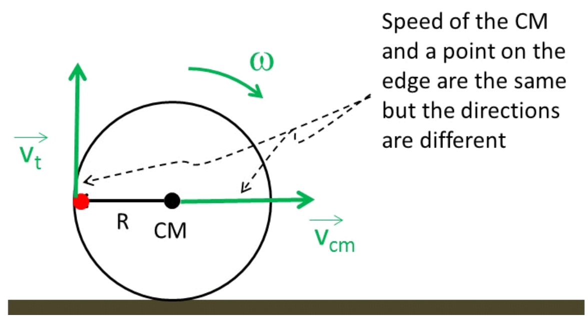
So the angular speed for our rotating wheel would be

$$\omega = \frac{2\pi \,\mathrm{rad}}{T}$$

and we can see that our non-slipping constraint can be written as

$$\begin{aligned} v_{cm} &= \frac{2\pi}{T}R \\ &= \omega R \\ &= v_t \end{aligned}$$

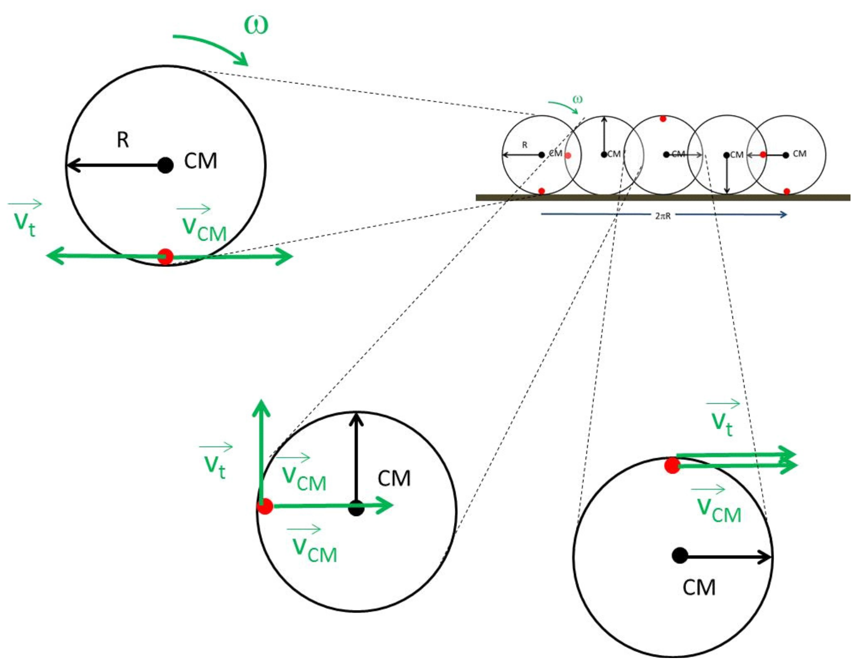
that is, the tangential speed of a point on the edge of the wheel has the same speed as the speed of the center of mass of the wheel. Notice that the velocities of the center of mass and of a point on the edge of the wheel are not the same because their directions are not the same.



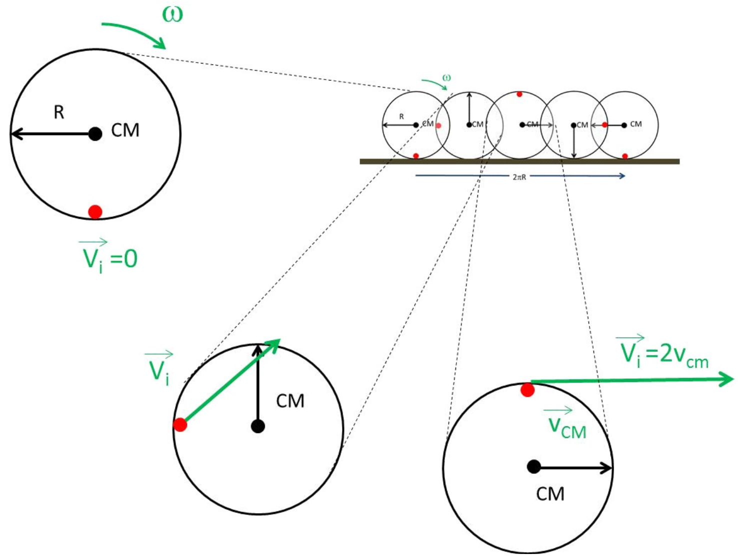
Of course, the net velocity of any point, say, our red point, is the vector sum of these two velocities. This is because the wheel is both rotating *and* moving as a whole. So unless the wheel splits apart, each point on the wheel must stay with the wheel. So if we call the speed of the red point v_i then

$$\overrightarrow{v}_i = \overrightarrow{v}_{cm} + \overrightarrow{v}_t$$

We can plot this for various positions of our red dot.



We can see that there will be two special locations for our red dot that will be interesting. One is when the red dot is right at the top of the wheel. Then the red dot net speed will be twice the center of mass speed. But even more interesting is when the red dot is at the bottom of the wheel so that it is right on the pavement. Then the red dot speed will momentarily be zero!



This may seem strange, but really this is just what we should expect if the wheel is not slipping on the ground. The part of the wheel represented by our red dot must not move in the x-direction relative to the ground when it is at the bottom of the wheel if the wheel does not slip. Of course this lasts for just a split second for any given marked wheel part. But for that split second, the marked point has a velocity relative to the ground of zero!

We would expect that with both rotation and translation (movement of the CM) we would have both rotational *and* translational kinetic energy

$$\begin{aligned} K_{total} &= K_{rot} + K_t \\ &= \frac{1}{2}\mathbb{I}\omega^2 + \frac{1}{2}mv^2 \end{aligned}$$

and this is true.

An interesting way to see this is to consider that for the split second that the red dot part of the wheel is on the ground, it could be our axis of rotation. After all, it is stationary for that split second. Then we could use the parallel axis theorem

$$\mathbb{I}_P = \mathbb{I}_{cm} + MR^2$$

to find the moment of inertia about the red dot! Let's use this in our rotational kinetic energy equation

$$\begin{aligned} K_{rot} &= \frac{1}{2}\mathbb{I}\omega^2 \\ &= \frac{1}{2}\left(\mathbb{I}_{cm} + MR^2\right)\omega^2 \\ &= \frac{1}{2}\mathbb{I}_{cm}\omega^2 + \frac{1}{2}MR^2\omega^2 \end{aligned}$$

but remember

$$\omega = \frac{v_t}{R}$$

and $v_t = v_{cm}$ so

$$\omega = \frac{v_{cm}}{R}$$

and

$$K_{rot} = \frac{1}{2}\mathbb{I}_{cm}\omega^2 + \frac{1}{2}MR^2\left(\frac{v_{cm}}{R}\right)^2$$

so

$$K_{rot} = \frac{1}{2}\mathbb{I}_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

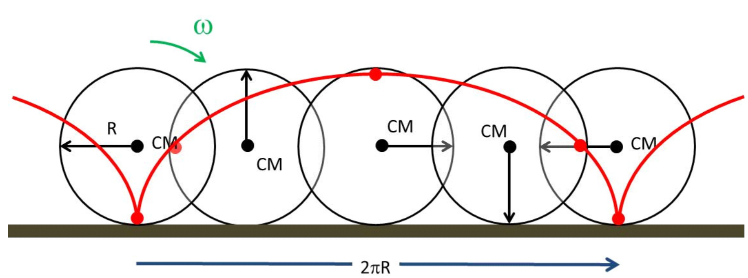
which is just what we thought. The kinetic energy of the wheel is a combination of translational kinetic energy

$$K_r = \frac{1}{2}Mv_{cm}^2$$

and rotational kinetic energy about the center of mass

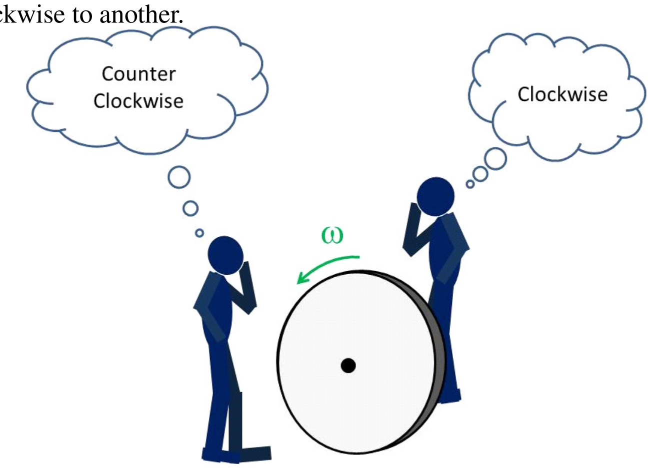
$$K_{cm} = \frac{1}{2}\mathbb{I}_{cm}\omega^2$$

It's interesting to plot the position of our red dot part of the wheel. Notice that as the wheel rotates the marked spot will make a curve. This curve is called a *cycloid*.



Rotation and direction

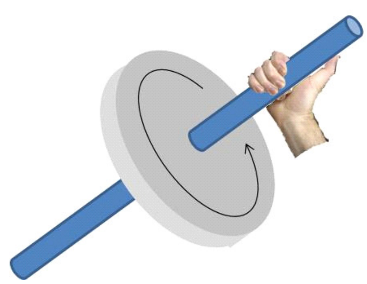
So far we have used clockwise rotation to mean a negative ω and counter-clockwise to mean a positive ω. But you might object to this! What for one person is clockwise is counter-clockwise to another.



We need a way to be sure that we can tell if an object's angular speed should be positive or negative. To do this, we will assign a direction to our angular velocity, ω. To find that direction, we will formulate a rule.

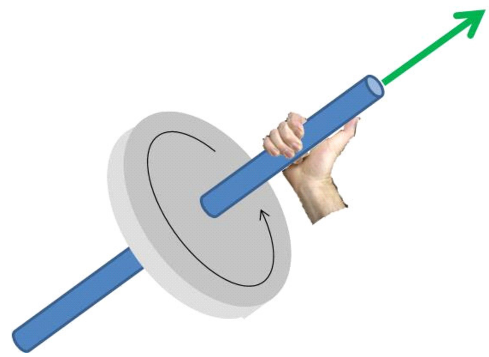
Right hand rule #1:

Suppose we have a rotating object as shown.

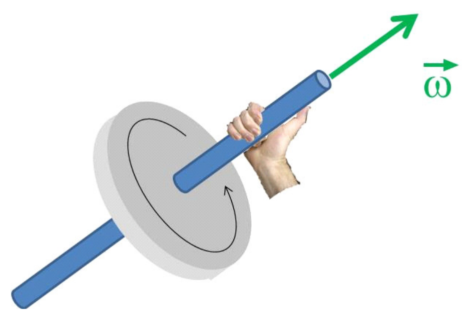


The object is rotating in a counter-clockwise fashion, so we would say that ω is positive.

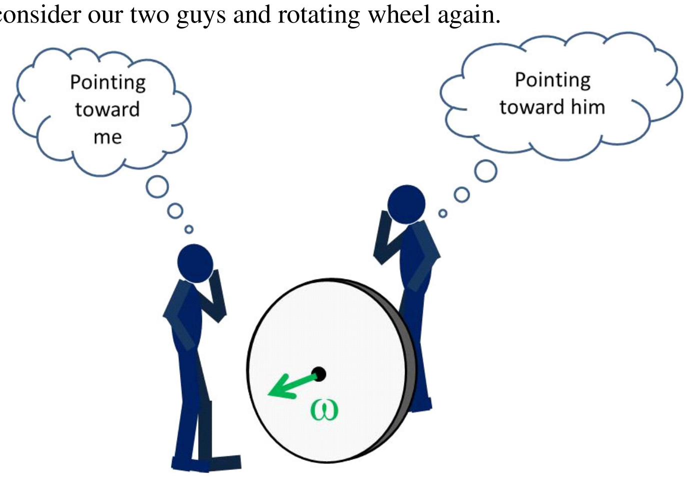
Imagine grabbing the axis of rotation with your right hand. Your thumb would stick out along the axis. Let's imagine a vector going in the direction of your thumb.



We could use this vector to beat our clockwise/counter-clockwise problem. Let's say the angular speed has this direction. To be sure, nothing is going in this direction except your thumb! But let's assign to our angular velocity this "thumb" direction.



Now let's consider our two guys and rotating wheel again.



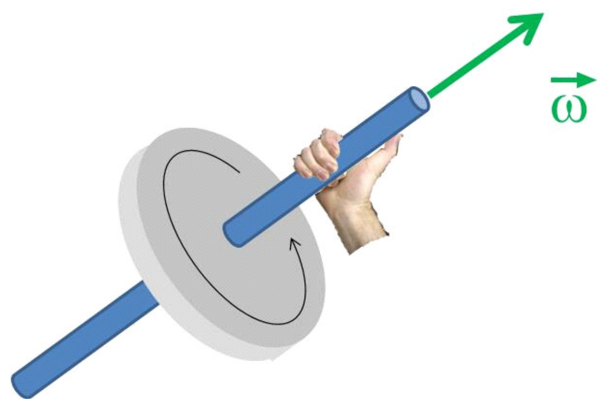
Both guys can use their right hand and get the same "thumb direction" for the vector assigned to ω. Now there is no confusion. Of course we can't really say that a particular "thumb direction" is positive or negative. The direction assigned to ω is a direction in three-dimensional space. So it is not simply positive or negative any more.

It's really important to understand that nothing goes in the "thumb direction" for our angular velocity. Choosing this odd "thumb direction" is only a way to keep track of which way an object is rotating that does not have the confusion of clockwise/counter-clockwise type distinctions.

We will call this method of finding the direction assigned to angular velocity a *right hand rule*.

Here is the right hand rule again:

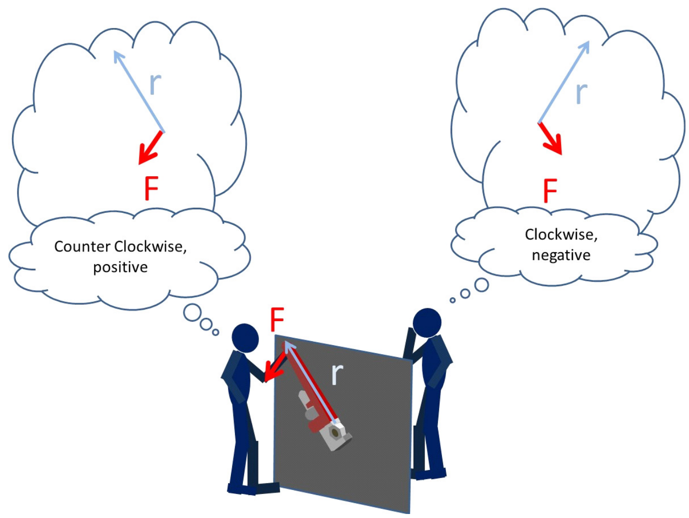
We assign a direction to angular speed that is given by imagining that you grab the axis of rotation with your right hand so that your fingers seem to curl the same way the object is rotating. Then your thumb gives the direction of $\vec{\omega}$



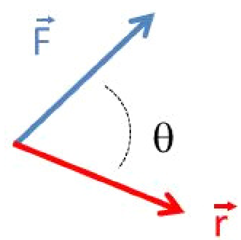
Remember that you curl the fingers of your *right hand* (sorry left handed people, you have to use your right hand for this) in the direction of rotation. Then your thumb points in the direction of the vector.

Right hand rule #2

We also used clockwise and counter-clockwise to tell the sign of torque.



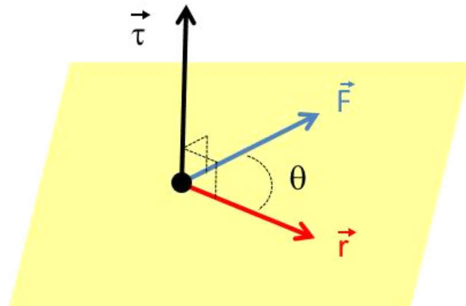
We can make this easier and less likely to create confusion by assigning a vector to a torque that would not be different if we look at the situation from behind.



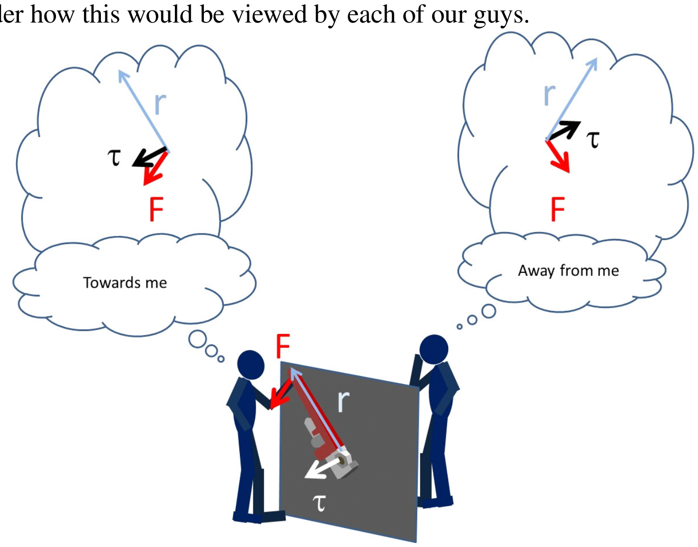
Consider the torque equation

$$\tau = rF\sin\theta_{rF}$$

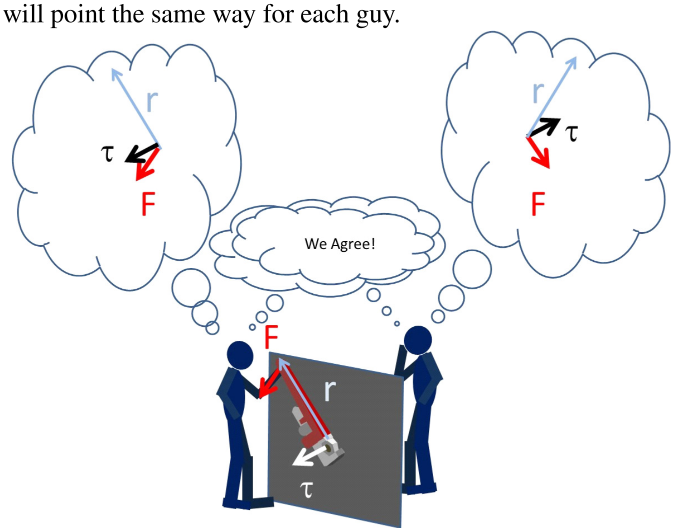
We found whether the torque was positive or negative by looking at our r-direction and seeing which way we would turn as we go from r to the F-direction. If it was counter clockwise, we called it positive. We could do the same sort of thing for torque by assigning a direction to the torque. Let's say that we assign a vector in the direction shown in the next figure



and consider how this would be viewed by each of our guys.



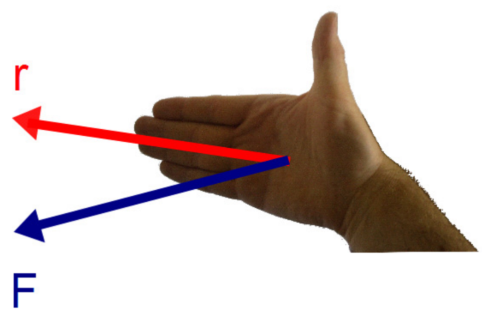
Like with our angular velocity, this new vector is perpendicular to the actual rotation. This vector will point the same way for each guy.



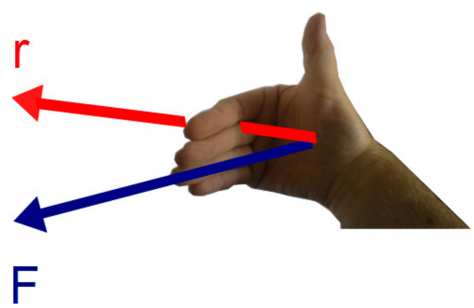
And once again, nothing goes the direction that our new vector points. But we can say that the torque has this direction! And using this odd made-up direction rule, we are never confused about whether the rotation is counter clockwise or clockwise.

Let's use our right hand again for another easy way to find this direction that we have assigned to torque. This is a little more complicated that our first right hand rule, so here are some steps:

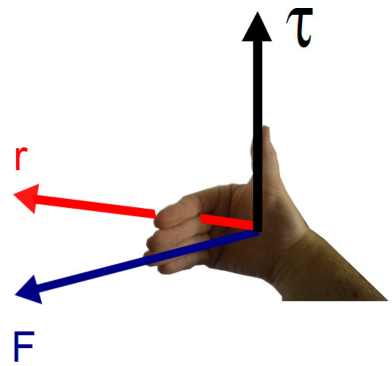
1. Put your fingers of your right hand in the direction of $\tilde{\mathbf{r}}$



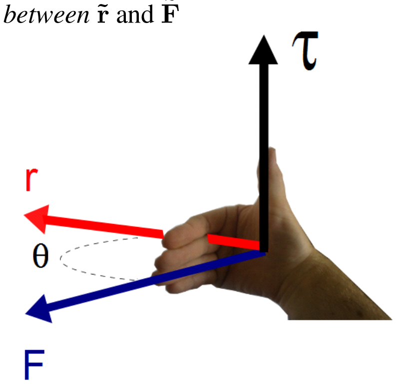
2. Curl them toward $\tilde{\mathbf{F}}$



3. The direction of your thumb is the torque direction



4. The angle θ is the angle *between* $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{F}}$



This gives the direction of the torque. And the magnitude of the torque is

$$\tau = rF\sin\theta_{rF}$$

Another new math moment

It turns out that this procedure for finding a torque vector is useful in many places, so mathematicians have given it a special symbolic way to be written.

The torque vector is written symbolically as

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$$

This might look like our old grade school multiply sign, **but it is not!** It is called a *vector cross product* and it means just what we have said. Take two vectors, $\overrightarrow{A}$ and $\overrightarrow{B}$. Then the vector cross product of $\overrightarrow{A}$ and $\overrightarrow{B}$ is written as

$$\overrightarrow{A} \times \overrightarrow{B}$$

and it has a magnitude of

$$\left|\overrightarrow{A}\times\overrightarrow{B}\right| = AB\sin\theta_{AB}$$

and the direction is given by our right hand rule #2.

Notice that the direction our right hand rule gives is perpendicular to both $\overrightarrow{A}$ and $\overrightarrow{B}$. For a cross product, the order matters. Think of our torque,

$$\overrightarrow{r}\times\overrightarrow{F}$$

is not going to point the same way as

$$\overrightarrow{F}\times\overrightarrow{r}$$

Use your right hand rule and you will see that these two point opposite directions! Mathematically we express this as

$$\overrightarrow{A}\times\overrightarrow{B} = -\left(\overrightarrow{B}\times\overrightarrow{A}\right)$$

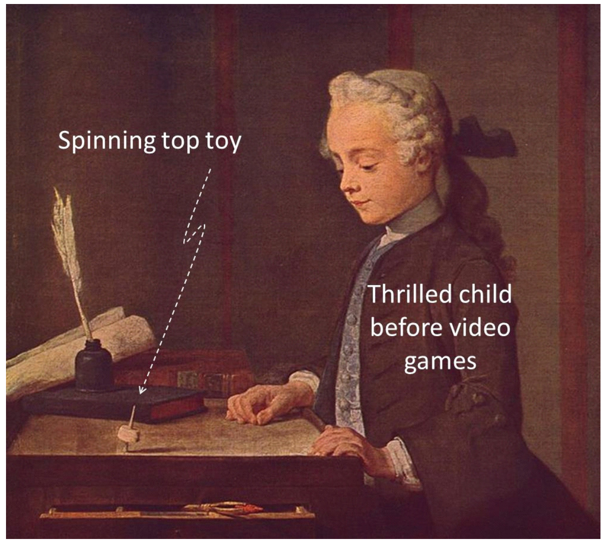
Note again that nothing goes in this torque direction. It is just a way to agree the direction of the torque so we don't have the problems associated with clock-wise or counter-clock-wise directions.

Angular momentum

We have learned that motion does not change unless a force acts on it. We called the combination of things that make an object's motion hard to change momentum

$$\overrightarrow{p} = m\overrightarrow{v}$$

The more mass we have, the harder it is to stop the object. And the faster an object goes, the harder it is to stop. Consider an old toy, the top.



You start a top spinning, and it keeps spinning. It is as though there was some momentum tied up in spinning. And there is!

Let's review momentum to see if we can extend our definition of momentum to rotating systems. When we first studied momentum we started with Newtons' second law.

$$\overrightarrow{F}_{net} = m\overrightarrow{a}$$

and the definition of acceleration

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$$

We put these together to write Newton's second law as

$$\overrightarrow{F}_{net} = m\frac{d\overrightarrow{v}}{dt}$$

And in order to allow the mass to change, we took it inside the derivative.

$$\overrightarrow{F}_{net} = \frac{d\left(m\overrightarrow{v}\right)}{dt}$$

and we called

$$\overrightarrow{p} = m\overrightarrow{v}$$

so

$$\overrightarrow{F}_{net} = \frac{d\left(\overrightarrow{p}\right)}{dt}$$

Let's try the same thing with torque. We can now write our rotational view of Newton's second law as

$$\overrightarrow{\tau}_{net} = \mathbb{I}\overrightarrow{\alpha}$$

and

$$\alpha = \frac{d\overrightarrow{\omega}}{dt}$$

so

$$\overrightarrow{\tau}_{net} = \mathbb{I}\frac{d\overrightarrow{\omega}}{dt}$$

Again we could take the mass-like term into the derivative

$$\overrightarrow{\tau}_{net} = \frac{d\left(\mathbb{I}\overrightarrow{\omega}\right)}{dt}$$

And we can see the momentum-like quantity. For linear momentum, m and $\overrightarrow{v}$ make the object hard to stop. For rotational motion we can see that $\mathbb{I}$ and $\overrightarrow{\omega}$ would make an object hard to stop spinning. These together must be our rotational momentum!

But we call this rotational momentum *angular momentum.* We give angular momentum the symbol L (for angu(L)ar) and notice that with $\overrightarrow{\omega}$ as part of our angular momentum, L must now be a vector. So

$$\overrightarrow{L} = \mathbb{I}\overrightarrow{\omega}$$

The direction of the angular momentum must be the same as the direction of the angular velocity. So the direction will be given by our right hand rule #1 (RHR1). It's still true that nothing goes in the $\overrightarrow{L}$ direction. We are still just avoiding the problems with clock-wise and counter-clock-wise as directions.

But there is another way to write angular momentum, and it is instructive. Let's recall that $v_t = r\omega$ so we could write our angular momentum magnitude as

$$L = \mathbb{I}\frac{v_t}{r}$$

and for a single particle

$$\mathbb{I} = mr^2$$

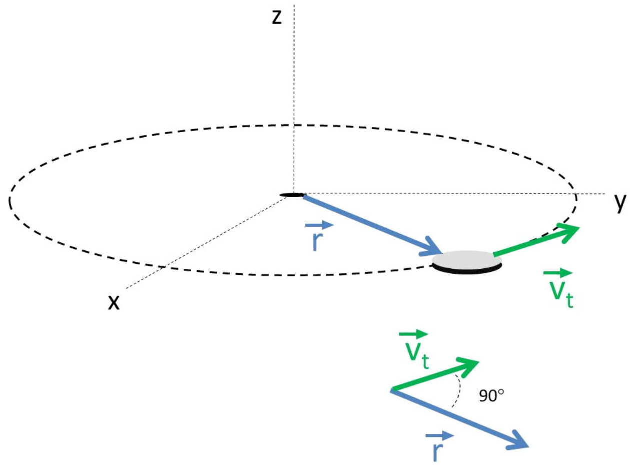
so

$$\begin{aligned} L &= mr^2\frac{v_t}{r} \\ &= mrv_t \end{aligned}$$

or

$$\begin{aligned} L &= rmv_t \\ &= rp \end{aligned}$$

This works for particles going in circles



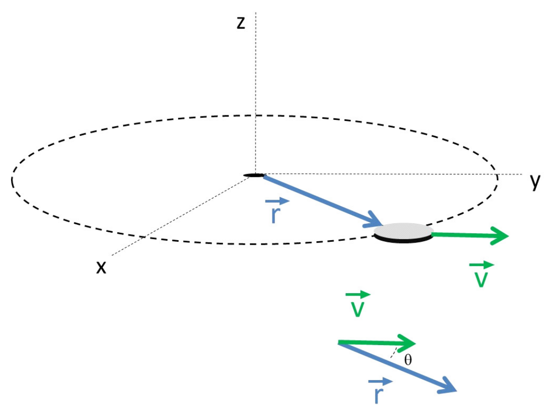
Consider that going in a circle is a very special case of motion. Notice that the angel between $\overrightarrow{r}$ and $\overrightarrow{v}_t$ is $90\,^{\circ}$. And the sine of $90\,^{\circ}$ is 1. So it could be that

$$L = rp\,(1)$$

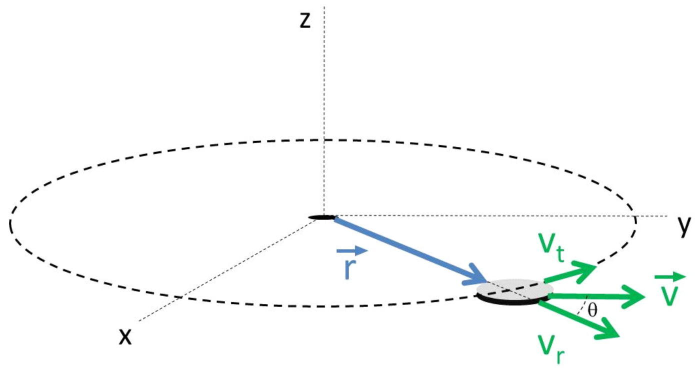
should be written as

$$\overrightarrow{L} = rp\sin\theta_{rp}$$

and if we look at cases other than perfect circular motion, we will see that this is the case



The angle looks like it should matter. Notice that for a particle moving in some random direction we would have a tangential and radial part to the velocity



By writing $\overrightarrow{v}$ in components we can see that

$$v_t = v \sin \theta_{rp}$$
$$v_r = v \cos \theta_{rp}$$

And it is the tangential component of the speed that is related to rotating, so we expect .

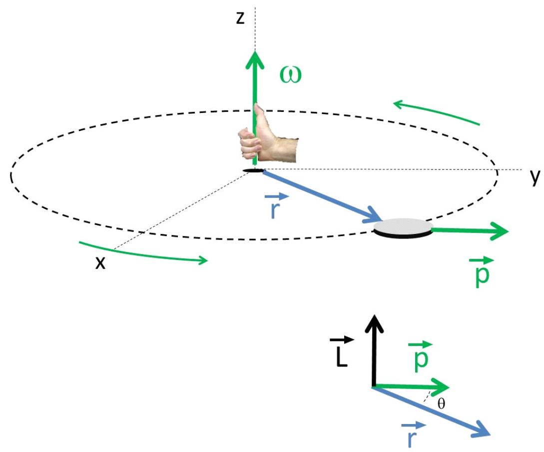
$$L = rpv_t$$
$$= rmv \sin \theta_{rp}$$

Angular momentum seems to be related to the tangential component of our particle's velocity. That makes some sence. The tangential componet tells us about the rotation of our particle and is related to the angular speed of the particle.

Now, notice that $\overrightarrow{r}$ is a vector, and $\overrightarrow{v}$ is a vector, and we have the sine of the angle between $\overrightarrow{r}$ and $\overrightarrow{v}$. This looks like a vector cross product!

$$\overrightarrow{L} = \overrightarrow{r} \times m\overrightarrow{v}$$
$$= \overrightarrow{r} \times \overrightarrow{p}$$

But we need to check, does the direction given by $\overrightarrow{r} \times \overrightarrow{p}$ match the direction of $\mathbb{I}\overrightarrow{\omega}$? We can use our right hand rule #1 for $\mathbb{I}\overrightarrow{\omega}$



and we will find $\overrightarrow{\omega}$ points straight up in the $\hat{k}$-direction. Now let's use the process we used for torque to find the direction of $\overrightarrow{r} \times \overrightarrow{p}$. That is right hand rule #2. We start with our fingers in the $\overrightarrow{r}$ just like we did with torque. Then we bend them into the $\overrightarrow{p}$ direction. Then our thumb gives the direction of $\overrightarrow{r} \times \overrightarrow{p}$. And we see that this is also straight up. It works!

Before we leave this idea of angular momentum, let's relate it to torque. Since we know that forces cause motion, so

$$\overrightarrow{F}_{net} = \frac{d\overrightarrow{p}}{dt}$$

that is, forces causes a change in momentum. We should expect torque to cause a change in angular momentum.

Then

$$\begin{aligned}
\overrightarrow{\tau}_{net} &= \frac{d\left(\overrightarrow{L}\right)}{dt} \\
&= \frac{d\left(\mathbb{I}\overrightarrow{\omega}\right)}{dt} \\
&= \frac{d\left(\overrightarrow{r} \times \overrightarrow{p}\right)}{dt}
\end{aligned}$$

and we have another problem. How do we take the derivative of a cross product?

Fortunately this is easy. We just use the product rule (but we have to keep our vectors in the same order, remember $\overrightarrow{r} \times \overrightarrow{p} = -\left(\overrightarrow{p} \times \overrightarrow{r}\right)$.

$$\begin{aligned}\overrightarrow{\tau}_{net} &= \overrightarrow{r} \times \frac{d\overrightarrow{p}}{dt} + \frac{d\overrightarrow{r}}{dt} \times \overrightarrow{p} \\ &= \overrightarrow{r} \times \frac{d\overrightarrow{p}}{dt} + \overrightarrow{v} \times \overrightarrow{p}\end{aligned}$$

and $|\overrightarrow{v} \times \overrightarrow{p}| = vp\sin\theta_{vp}$ but v and p are in the same direction, so this term is zero.

$$\begin{aligned}\overrightarrow{\tau}_{net} &= \overrightarrow{r} \times \frac{d\overrightarrow{p}}{dt} \\ &= \overrightarrow{r} \times \overrightarrow{F} \\ &= rF\sin\theta_{rF}\end{aligned}$$

This is, indeed, what torque is, so we have come full circle. Our definition of angular momentum works!

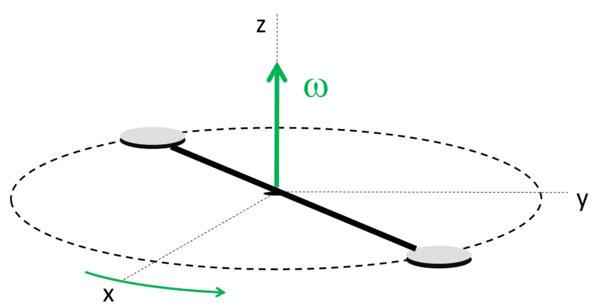
$$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$$

This is all the things that make the spinning motion hard to change.

So far, though, we have just dealt with particles. What do we do for an extended object? We have done this many times before, We just divide up the spinning object into parts, and find the angular momentum for each part. Then we sum up the angular momentum for all the parts

$$\overrightarrow{L} = \sum_i \overrightarrow{L}_i$$

Let's try a problem. Often angular momentum us used to stabilize a satellite system. Suppose you design a satellite that consists of two sensor systems attached to a long (low mass) rod. The system rotates with an angular speed of $\omega = 3.15\,\mathrm{rad/\,s}$. If the rod has a length of $30\,\mathrm{m}$ and each sensor package has a mass of $400.\,\mathrm{kg}$. What is the angular momentum of the system?



This is an angular momentum problem (no surprise)

We know

$$\begin{aligned} \omega &= 3.15\frac{\text{rad}}{\text{s}} \\ D &= 30\,\text{m} \\ m &= 400\,\text{kg}. \end{aligned}$$

and our new basic equations are

$$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$$
$$\overrightarrow{L} = I\overrightarrow{\omega}$$
$$\overrightarrow{\tau}_{net} = \frac{d\overrightarrow{L}}{dt}$$

$$\mathbb{I} = \sum_i m_i r_i^2$$

The bar mass can be neglected, and we know ω, so if we can find $\mathbb{I}$ we will have a solution.

$$\begin{aligned} \mathbb{I} &= \sum_i m_i r_i^2 \\ &= m\left(\frac{D}{2}\right)^2 + m\left(\frac{D}{2}\right)^2 \\ &= \frac{2mD^2}{4} \\ &= \frac{1}{2}mD^2 \end{aligned}$$

so

$$L = \left(\frac{1}{2}mD^2\right)\omega$$

$$\begin{aligned} L &= \left(\frac{1}{2}(400\,\text{kg})(30\,\text{m})^2\right)\left(3.15\frac{\text{rad}}{\text{s}}\right) \\ &= 5.67 \times 10^5\,\frac{\text{m}^2}{\text{s}}\,\text{kg} \end{aligned}$$

So we can do problems with angular momentum. But you are probably asking, is angular momentum conserved, and if so, can we do hard problems in an easy way with conservation of angular momentum like we did with conservation of (linear) momentum? That is the subject of our next lecture!

35 Conservation of Angular Momentum and Newton's Law of Gravitation

Conservation of angular momentum

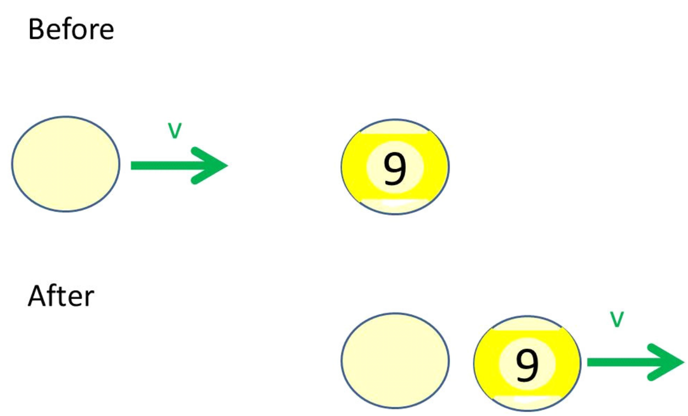
When we studied momentum we found that, if there are no net outside forces,

$$\Sigma \overrightarrow{F}_{ext} = 0$$

momentum was conserved.

$$P_i = P_f$$

And when momentum was conserved we could use this fact to solve motion problems. Remember the pool ball problem?



We started with the idea of conservation of momentum

$$P_i = P_f$$

and we considered that the collision would have a short duration, and that during the collision, all other horizontal forces would be negligible, (impulse approximation). Then the initial total momentum of the two ball system was

$$P_i = p_{ci} + p_{9i}$$

and the final total momentum was

$$P_f = p_{cf} + p_{9f}$$

so conservation of momentum gave us

$$p_{ci} + p_{9i} = p_{cf} + p_{9f}$$

or

$$m_c v_{ci} + m_9 v_{9i} = m_c v_{cf} + m_9 v_{9f}$$

Since both balls had the same mass, so mass canceled out. we found that

$$v_{ci} + 0 = 0 + v_{9f}$$

or

$$v_{ci} = v_{9f}$$

The two balls have switched velocities. Conservation of momentum gave us a very easy answer to a rather difficult force problem. We should ask ourselves, should we expect angular momentum to be conserved? and if so, will it allow us to solve complicated torque problems in an easy way?

The answer to both questions is yes! Let's demonstrate with a problem.

Suppose we put you on a rotating platform. And further suppose we put some large weights in your hands and then spin you up to an initial angular speed of ω_i. Will your angular speed be faster, slower, or the same if you draw in your hands (and the weights)?

We know that

$$L_i = \mathbb{I}_i \omega_i$$

and if we assume that you draw in your hands quickly (impulse approximation again)

then

$$L_f = \mathbb{I}_f \omega_f$$

and

$$0 = L_f - L_i$$

since the (rotational) impulse will be zero becuse there is no net external torque. Then

$$L_i = L_f$$

Then we can write

$$\mathbb{I}_i \omega_i = \mathbb{I}_f \omega_f$$

and solve for

$$\omega_f = \frac{\mathbb{I}_i}{\mathbb{I}_f} \omega_i$$

so if $\mathbb{I}_i / \mathbb{I}_f$ is bigger than one, you will spin faster, if it is less than one, you will spin slower, and if it is exactly one you will spin the same. The moment of Inertia does not change for most of your body (your head and kidneys are not in new positions, for example). But the moment of inertia of the masses in your hands does change. That moment of Inertial would be

$$\mathbb{I}_{masses} = 2 m_{mass} r^2$$

(where the 2 is because you have two arms and two masses) and you change the distance from the pivot to the mass when you bring the masses into your body. So the moment of inertia of yours spinning person-mass system does change. Suppose your arms are half a meter long, and that the masses are each $5\,\mathrm{kg}$. Then

$$\begin{aligned} \mathbb{I}_i &= 2\,(5\,\mathrm{kg})\,(0.5\,\mathrm{m})^2 \\ &= 2.5\,\mathrm{m}^2\,\mathrm{kg} \end{aligned}$$

and the final moment of inertial is about

$$\begin{aligned} \mathbb{I}_f &= 2\,(5\,\mathrm{kg})\,(0.2\,\mathrm{m})^2 \\ &= 0.4\,\mathrm{m}^2\,\mathrm{kg} \end{aligned}$$

then your final angular speed would be

$$\begin{aligned} \omega_f &= \frac{2.5\,\mathrm{m}^2\,\mathrm{kg}}{0.4\,\mathrm{m}^2\,\mathrm{kg}} \omega_i \\ &= 6.25 \omega_i \end{aligned}$$

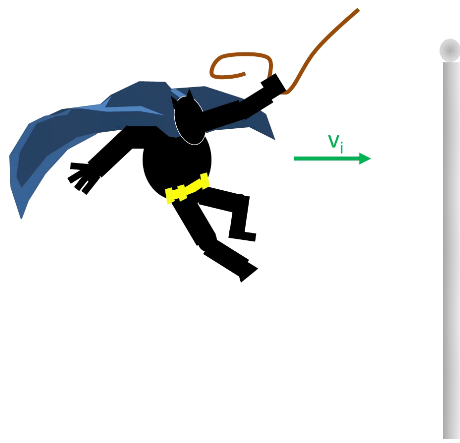
which means you are spinning over six times as fast. Of course, in this problem there was no net external torque

$$\Sigma \overrightarrow{\tau}_{ext} = 0$$

and this will be a requirement for conservation of angular momentum.

Notice that this problem would be very difficult to do if we used torques. You would have to find the torque on the masses, and would have to know how the torque changed in time as you brought the masses in toward your body. But using conservation of angular momentum it was relatively easy!

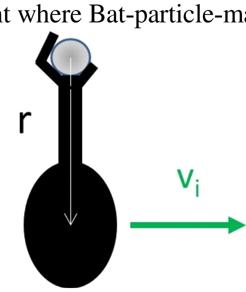
Let's try another problem.



The intrepid super-hero Bat-particle-man is swinging through the city. As he passes a pole, he reaches out and grabs the pole with his $0.70\,\mathrm{m}$ arm while letting go of the rope with the other arm. Bat-particle-man spins around the pole. If Bat-particle-man's initial speed is $10\,\mathrm{m/\,s}$ and his mass is $140\,\mathrm{kg}$, how fast will he spin around as he clings to the pole?

$$\begin{aligned}\vec{L} &= \vec{r} \times \vec{p} \\ &= \vec{r} \times m\vec{v}\end{aligned}$$

We can see that right at the point where Bat-particle-man grabs the pole



his is mostly a mass passing by the pole a distance r away (because the majority of his mass is in his middle, we will treat Bat-particle-man as a particle except for the arm that is holding him onto the pole).

Then

$$\begin{aligned} L_i &= rmv_i \sin\theta_{rv} \\ &= rmv_i\,(1) \end{aligned}$$

and we know all the values

$$\begin{aligned} L_i &= (0.70\,\mathrm{m})\,(140\,\mathrm{kg})\,(10\,\mathrm{m/\,s}) \\ &= 980.0\frac{\mathrm{m}^2}{\mathrm{s}}\,\mathrm{kg} \end{aligned}$$

we know after he grabs the pole, his angular momentum will be the same, so

$$L_i = L_f$$

and we can write L_f using

$$L_f = \mathbb{I}\omega_f$$

If we treat Bat-particle-man as, well, a particle, then

$$\mathbb{I} \approx mr^2$$

so

$$L_f = mr^2\omega_f$$

and

$$\begin{aligned} \omega_f &= \frac{L_f}{mr^2} = \frac{L_i}{mr^2} \\ &= \frac{980.0\frac{\mathrm{m}^2}{\mathrm{s}}\,\mathrm{kg}}{(140\,\mathrm{kg})\,(0.70\,\mathrm{m})^2} \\ &= 14.286\frac{\mathrm{rad}}{\mathrm{s}} \\ &= 2.3\frac{rev}{\mathrm{s}} \end{aligned}$$

Bat-particle-man will spin around the pole about twice a second. He will have to introduce some friction by clamping his hand tightly to the pole to slow his rotation. This clamping of his hand introduces another force, and makes it so he loses angular momentum.

Notice that, considering the pole as a pivot point, our super hero had angular momentum as he was traveling past the pole. There was a rotational component to his linear motion! And we used that angular momentum component to cause the rotation about the pole once he grabbed the pole.

Again this would be a hard problem to solve using torques. But was not too bad using conservation of angular momentum.

Newton's Law of gravitation

We have already studied Newton's law of gravitation, but it's worth taking a closer look. Especially as it relates to orbits. Phone calls, texts, data streams, the internet, and most forms of communication today rely on satellites. Orbital motion effects our daily lives. So let's look at gravitation and how orbits work.

G, g, and m

We know the equation

$$W = G\frac{m_m m_E}{r_{mE}^2} \tag{35.1}$$

In Newton's words:

> *Every particle in the universe attracts every other particle in the universe with a force that is proportional to the masses of both particles and inversely proportional to the square of the distance between the particles.*

In our formulation of the equation subscript m is for "mover" and the subscript E is for "environment." The distance r_{m_E} is the distance from the center of the mover object to the center of the environmental object.

The direction of the force is along the line connecting the centers of mass of the two particles. The constant G is a factor included to keep us in units we can use and to make the expression exact (instead of just a proportionality). It's value is

$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}} \tag{35.2}$$

But let's revisit gravitation.

Weight

We called the gravitational force acting on the object the weight of the object.

$$W = m_m g \tag{35.3}$$

where g is the acceleration due to gravity. Using Newton's law of gravitation, we wrote the weight as

$$W = G\frac{M_E m_m}{r_{mE}^2} \tag{35.4}$$

where M_E is the mass of the Earth. Given this, we can solved for g

$$\begin{aligned} m_m g &= G\frac{M_E m_m}{r_{mE}^2} \\ g &= G\frac{M_E}{r_{mE}^2} \end{aligned}$$

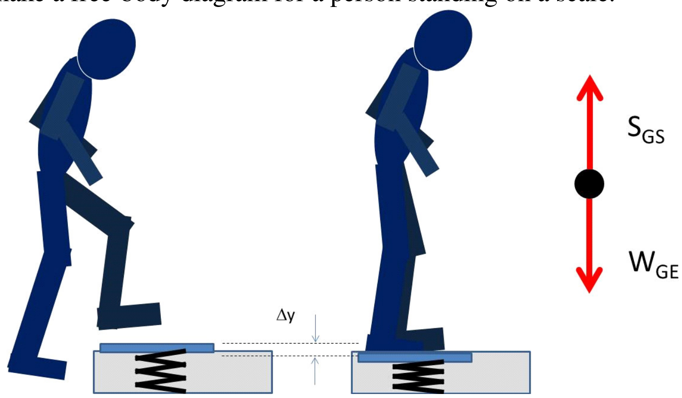
And we found that we should have written this as

$$g\left(r_{mE}\right) = G\frac{M_E}{r_{mE}^2}$$

since g depends on r_{mE}^2 . So $g\left(r_{mE}\right)$ has one value at sea level, and another value in a weather balloon at high altitude. What does this tell us about weight ($W = mg$)? We see that weight is not constant for our mass, m. It varies with position relative to the Earth's center.

But we should think about scales that measure weight, how do they work?

A bathroom scale usually has a spring that is compressed when we step on the scale. So we could make a free-body diagram for a person standing on a scale.



Then

$$\begin{aligned} F_{net_y} &= ma_y \\ &= S_{Gs} - W_{GE} \end{aligned}$$

and

$$S_{Ge} = -k\Delta y$$

so

$$ma_y = k\Delta y - W_{GE}$$

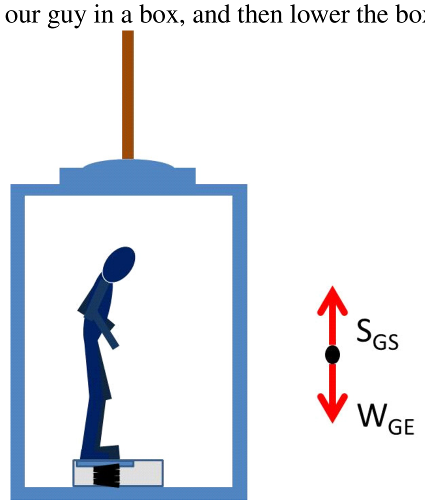
and hopefully your bathroom scale is not accelerating in the y-direction as you stand on it, so

$$k\Delta y = W_{GE}$$

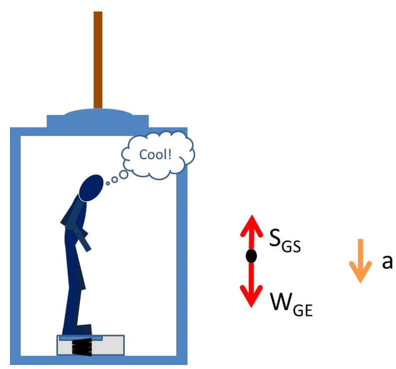
or

$$\Delta y = \frac{W_{GE}}{k}$$

So if the manufacturer of the scale knows the stiffness of the spring they built into the scale, they can make the scale reading directly proportional to how far down you compress the spring. The scale can transform Δy into weight. But will the scale always work? Suppose we put our guy in a box, and then lower the box on a cable.



So long as the acceleration is still zero (the box is lowered with a constant speed) everything still works and the scale reads the weight just fine. But what if we accelerate the box downward?



now our newtons's second law gives

$$-ma_y = k\Delta y - W_{GE}$$

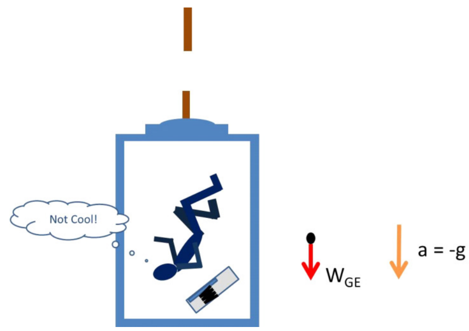
so that the spring Δy will be

$$k\Delta y = W_{GE} - ma_y$$

$$\Delta y = \frac{W_{GE} - ma_y}{k}$$

and we can see that the scale will read less than the actual weight of the guy. This is why you feel "lighter" as an elevator goes down.

In the extreme case, we could cut the cable and see that the scale has no displacement, so it reads zero weight!



Some books define the reading of the scale as the person's weight. That complicates things in our Newton's second law calculations. But now we know that our definition of weight could complicate things in real life. Of course there is a fix for this, don't weight things in accelerating elevators! But if you do try to use a scale in a reference frame that is accelerating, you have to use Newton's second law to calculate the actual weight. For example, if we know the box is accelerating on the cable downward with an acceleration of $-a_y$, then the scale would be wrong but we could calculate the weight of the by solving for

$$W_{apparent} = k\Delta y$$

this is what the scale reads, so

$$-ma_y = k\Delta y - W_{GE}$$

becomes

$$-ma_y = W_{apparent} - W_{GE}$$

and

$$W_{GE} = ma_y + W_{apparent}$$

that is, take the mass of the guy and multiply by the acceleration of the box/guy system, Then add the result to the reading on the scale. That would give the true weight.

Where did G come from?

We understand where g comes from and how it changes with distance, but where did the

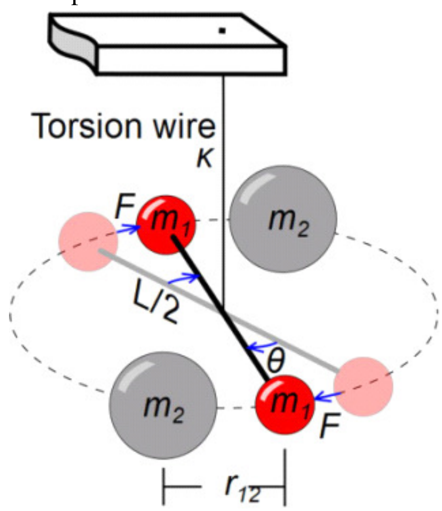
G term come from in our Newton's gravitational force equation?

$$F_g = G\frac{m_m m_E}{r_{mE}^2}$$

where the two masses involved (say, you, the mover, and the Earth, the environmental object) are m_m and m_E and the distance between the two masses is r_{mE} (e.g. the distance from the center of the Earth to the center of you). The constant G is a constant that puts the force into nice units that are convenient for us to use, like newtons (N). It has a value of

$$G = 6.67428 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}$$

You might ask, how do we know this? The answer is that Newton and others performed experiments. Newton's law of gravitation is empirical, meaning that it came from experiment. Lord Cavendish used a clever device to verify this law. He suspended two masses from a wire. Then he placed two other masses near the suspended masses.



He knew the torsion constant of the wire (how much it resists being twisted). A twisted wire acts much like a spring, with a restoring force that pushes back against the twist. Then by observing how far the suspended masses moved, he could work out the strength of the gravitational force. This is called a torsion balance.

Principle of Equivalence

We have actually done something very profound in our gravitational problems, and we did it without thinking about it. We said that the mover mass from Newton's second law

$$F_{net} = m_m a$$

that makes an object hard to push through space is the same is the same mover mass that creates the force of gravity.

$$F_g = G\frac{m_m m_E}{r_{mE}^2}$$

As far as we know, there is no requirement that this be the case! But this idea has been tested over and over again and it seems to be true. When Newton made this assumption, though, it was not obvious. The idea that the inertial mass that makes objects hard to move is also the same mass that makes gravitational forces is called the *principle of equivalence.* If you are lucky enough to get to take PH279, you will revisit the question of the principle of equivalence. But for us, all mass is the same, just mass.

36 Gravitational Energy

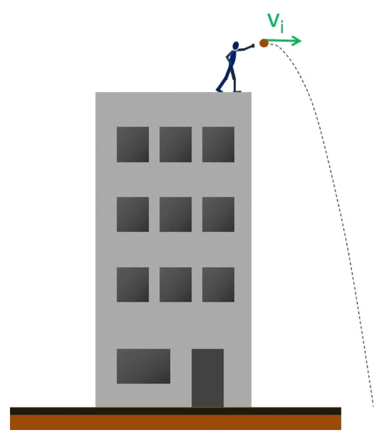
We already know about gravitational potential energy for the flat Earth approximation, but now we know that flat Earth approximation has some limits. Before we end our study of motion, let's look more carefully at gravitational potential energy for orbiting objects.

Gravitational Potential Energy

We studied the potential energy of objects near the Earth's surface. And we found that

$$U_g = m_m g y$$

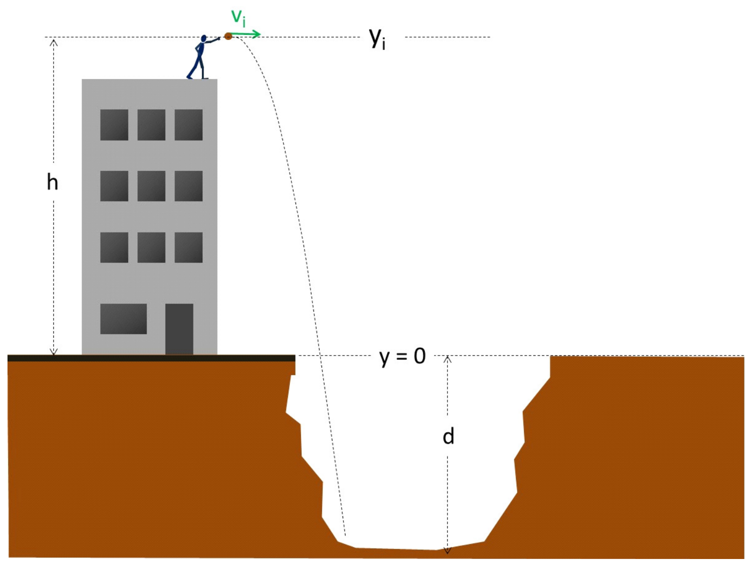
where m_m is the mass of the moving object, g is the acceleration due to gravity, and y is how high the object is compared to a $y = 0$ point. If you recall, we got to pick that $y = 0$ point. It could be any height. Suppose we have a guy throwing a ball off a building.



We could pick $y = 0$ to be at the top of the building, or at the ground level, or anywhere else!. Suppose we pick $y = 0$ at ground level. Then the initial potential energy of the ball would be

$$\begin{aligned} U_g &= m_m g y_i \\ &= m_m g h \end{aligned}$$

where h is the height of the building plus a little more because the guy has some height too. But further suppose that there is a deep hole near the building.



We can see that at the bottom of the hole the ball will have a potential energy of

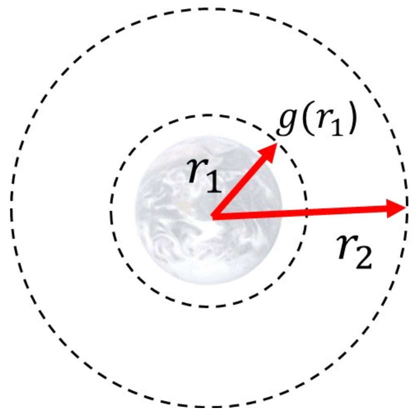
$$\begin{aligned} U_g &= m_m g y_f \\ &= m_m g(-d) \\ &= -m_m g d \end{aligned}$$

The potential energy is negative, but this is only because of our choice of $y = 0$ point.

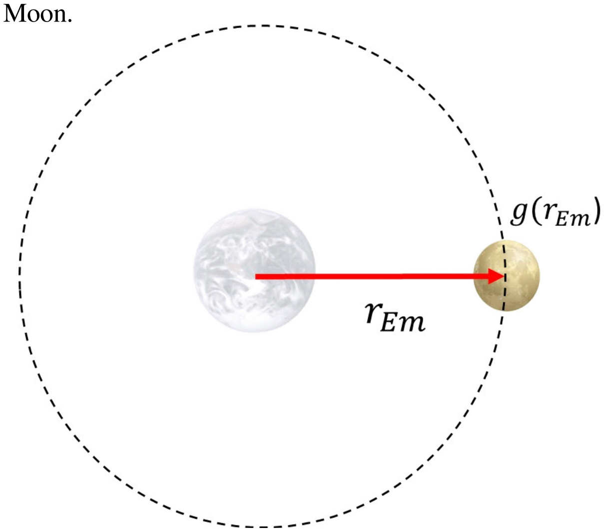
This all works fairly well so long as we take small objects near the much larger Earth. But let's consider objects farther away from the Earth's surface, or larger objects like the moon. For these objects, $m_m g y$ is not enough to describe the potential energy. The reason is that if we are far away from the center of the Earth we will notice that the Earth's gravitational acceleration is not a uniform $-g$. We already know g is a function of the distance, and that $g(r_{Em})$ diminishes with distance, r_{Em} between the centers of mass of the two objects.

$$g(r_{Em}) = G\frac{M_E}{r_{Em}^2}$$

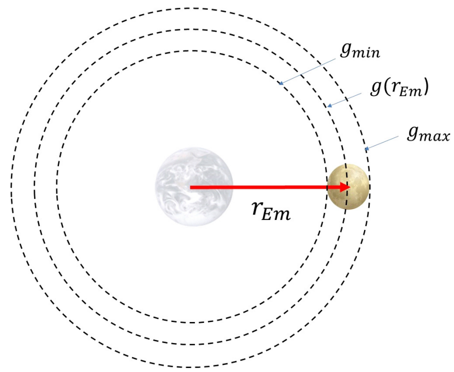
So we could envision a constant value of $g(r_{Em})$ existing for each elevation above the Earth. Sort of like spherical shells of constant $g(r)$ for each r.



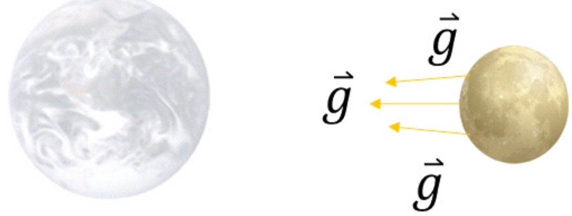
So, if an object is large, it will feel the change in the gravitational acceleration over its (the object's) large volume. The Moon, for example, would have a value of $g\left(r_{EM}\right)$ where r_{Em} is the Earth-Moon distance from center of mass of the Earth to center of mass of the Moon.



But some of the Moon's mass is closer to the Earth than r_{EM}, and some of the Moon's mass is farther from the Earth than r_{Em} and for something the size of the Moon, the change can't be neglected.



Worse yet, the Moon is so large that the direction of $g\left(r_{Em}\right)$ will change across the moon.



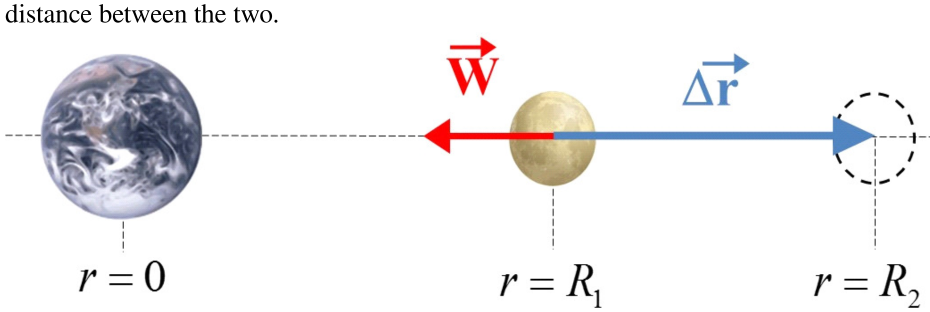
We need a better equation for gravitational potential energy that works for a round Earth. We have the tools to improve our potential energy calculation for this situation (solving at least some of the problems we have noted). We know that a change in potential energy is just an amount of work

$$\Delta U_g = -w_g = -\int \overrightarrow{\mathbf{W}} \cdot d\overrightarrow{\mathbf{r}}$$

The magnitude of the gravitational force is

$$W = G\frac{M_E m_m}{r_{Em}^2}$$

where M_E is the mass of the Earth, m_m is the mass of the mover object, and r_{Em} is the distance between the two.



The gravitational force is radial (along the radius, or in the $-\hat{r}$ direction) , so

$$\begin{aligned}\overrightarrow{\mathbf{W}} \cdot d\overrightarrow{\mathbf{r}} &= W dr \cos\theta_{Wr} \\ &= -W dr\end{aligned}$$

for the configuration we have shown, and we can perform the integration. Say we move the object a distance Δr away were

$$\Delta r = R_2 - R_1$$

and Δr is large, comparable to the size of the Earth or larger. Then

$$\begin{aligned}\Delta U_g &= -\int_{R_1}^{R_2} \left(-G\frac{M_E m_m}{r^2}\right) dr \\ &= G M_E m_m \int_{R_1}^{R_2} \frac{dr}{r^2}\end{aligned}$$

where R is the distance from the center of the Earth to the center of our object.

$$\begin{aligned}\Delta U_g &= G M_E m_m \int_{R_1}^{R_2} \frac{dr}{r^2} \\ &= G M_E m_m \left[-\frac{1}{r}\right|_{R_1}^{R_2} \\ &= G M_E m_m \left[-\frac{1}{R_2} - \left(-\frac{1}{R_1}\right)\right] \\ &= -G M_E m_m \left[\frac{1}{R_2} - \frac{1}{R_1}\right] \\ &= -G\frac{M_E m_m}{R_2} + G\frac{M_E m_m}{R_1}\end{aligned}$$

We recall that we need to set a zero point for the potential energy. Before, when we used the approximation $m_m g y$ we could choose $y = 0$ anywhere we wanted. But now we see an obvious (but strange) choice for the zero point of the potential energy. If we let $R_2 \to \infty$ then the first term in our expression will be zero. Likewise, of we let $R_1 \to \infty$ the second term will be zero. It looks like as we get infinitely far away from the Earth, the potential energy naturally goes to zero! Mathematically this makes sense. But we will have to interpret what this choice of zero point means.

First, let's see how much work it would take to move the moon out of obit and move it farther away. Say, from $R_{1,}$ the present orbit radius, to $R_2 = 2R_1$, or twice the original orbit distance. Then

$$\begin{aligned}
\Delta U_g &= U_2 - U_1 = -G\frac{M_E m_m}{2R_1} + G\frac{M_E m_m}{R_1} \\
&= G\frac{M_E m_m}{R_1}\left(-\frac{1}{2}+1\right) \\
&= \left(\frac{1}{2}\right)G\frac{M_E m_m}{R_1}
\end{aligned}$$

The change is positive just like it was when we moved our ball up from the bottom of the hole in the ball-building-hole example earlier. We gained potential energy as we went farther from the Earth's surface. That makes sense! That is analogous to increasing y in $m_m g y$. The potential energy also gets larger if the mass of our object (like the moon or a satellite) gets larger. Again that makes sense because in our more familiar approximation the potential energy increases with mass. So this new form for our equation for potential energy seems to work.

But what does it mean that the potential energy is zero infinitely far away? Recall that a change in potential energy is an amount of work

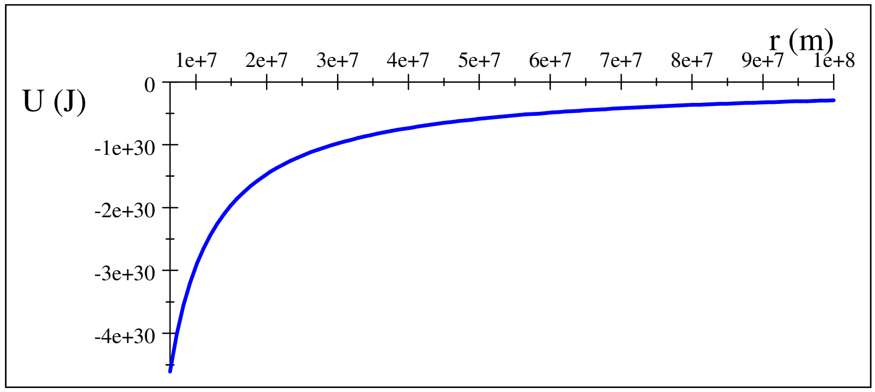
$$w = -\Delta U$$

Usually we will consider the potential energy to be the amount of work it takes to bring the test mass m_m from infinitely far away (our zero point!) to the location where we want it. It is how much energy is stored by having the object in that position. Like how much energy is stored by putting a mass high on a shelf. For example we could bring the moon in from infinitely far away. Then

$$\begin{aligned}
\Delta U_g &= U_2 - U_1 = -G\frac{M_E m_m}{R_2} + G\frac{M_E m_m}{\infty} \\
U_2 &= -G\frac{M_E m_m}{R_2}
\end{aligned}$$

This is how much potential energy the moon has as it orbits the Earth because it is high, above the Earth. But notice, this is a negative number! What can it mean to have a negative potential energy?

We use this convention to indicate that the mover mass, m_m is bound to the Earth. It would take an input of energy to get the moon free from the gravitational pull of the Earth. Here is the Moon potential energy plotted as a function of distance.



We can see that you have to go an infinite distance to overcome the Earth's gravity completely. That makes sense from our force equation. The force only goes to zero infinitely far away. When we finally get infinitely far away, there will be no potential energy due to the gravitational force because the gravitational force will be zero.

Of course, there are more than just two objects (Earth and Moon) in the universe, so as we get farther away from the Earth, the gravitational pull of, say, a galaxy, might dominate. So we might not notice the weak pull of the Earth as we encounter other objects.

We should show that this form for the potential energy due to gravity becomes the more familiar $m_m g y$ if our distances are small compared to the Earth's radius. This is our flat Earth approximation, so what we wish to do is to show that our more correct version of gravitational potential energy reduces to mgh if the conditions of the flat Earth approximation are valid.

Let our distance from the center of the Earth be $R_2 = R_E + y$ where R_E is the radius of the Earth and $y \ll R_E$. This is the condition for the flat Earth approximation to be true. Then

$$\begin{aligned} U &= -G\frac{M_E m_m}{R_2} \\ &= -G\frac{M_E m_m}{R_E + y} \end{aligned}$$

We can rewrite this as

$$\begin{aligned} U &= -G\frac{M_E m_m}{R_E\left(1 + \frac{y}{R_E}\right)} \\ &= -G\frac{M_E m_m}{R_E}\left(1 + \frac{y}{R_E}\right)^{-1} \end{aligned}$$

Since y is small y/R_E is very small and we can approximate the therm in parenthesis using the binomial expansion (another new math piece, you might recognize this as a special form of a Taylor series if you have taken higher math classes, but most of us we will accept it to be true and let the math department teach this one in the distant future).

$$(1 \pm x)^n \approx 1 \mp nx \qquad \text{if } x \ll 1$$

then we have

$$\left(1 + \frac{y}{R_E}\right)^{-1} \approx 1 - (-1)\frac{y}{R_E} \qquad \text{if } \frac{y}{R_E} \ll 1$$

and our potential energy is

$$U = -G\frac{M_E m_m}{R_E}\left(1 + \frac{y}{R_E}\right)$$

then

$$\begin{aligned} U &= -G\frac{M_E m_m}{R_E} + G\frac{M_E m_m y}{R_E^2} \\ &= U_o + m_m\left(G\frac{M_E}{R_E^2}\right)y \end{aligned}$$

If we realize that U_o is the potential energy of the object at the surface of the Earth, then the change in potential energy as we lift the object from the surface to a height y is

$$\begin{aligned} \Delta U &= \left(U_o + m_m\left(G\frac{M_E}{R_E^2}\right)y - \left(U_o + m_m\left(G\frac{M_E}{R_E^2}\right)(y_o)\right)\right) \\ &= m_m\left(G\frac{M_E}{R_E^2}\right)\Delta y \end{aligned}$$

All that is left is to realize that

$$\left(G\frac{M_E}{R_E^2}\right)$$

is just g

$$g = \left(G\frac{M_E}{R_E^2}\right)$$

so we have

$$\Delta U = m_m g \Delta y$$

and there is no contradiction. But we should realize that this is an approximation. The more accurate version of our potential energy is.[14]

$$U_2 = -G\frac{M_E m_m}{R_2}$$

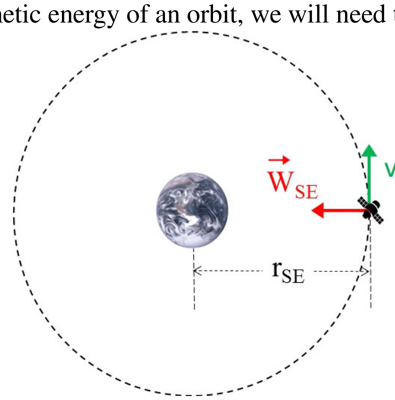
[14] If you were paying close attention, you might have noticed that we did not solve all of the problems with this new potential energy formula. We did solve the problem of a curved Earth. But there remains the problem of part of the Moon being farther away and part being closer. We would need to integrate over the Moon's mass to solve that problem.

Kinetic energy of an orbit

Kinetic energy is the energy of moving.

$$K = \frac{1}{2}mv^2$$

so to understand the kinetic energy of an orbit, we will need the orbital speed.



Let's find this in terms of the masses using Newton's law of gravitation. The satellite is the mover, and the Earth is the environmental object

$$\begin{aligned} W_{SE} &= F_g = G\frac{M_E m_m}{r_{Em}^2} \\ &= G\frac{M_E m_s}{r_{Es}^2} \end{aligned}$$

and from Newton's second law we know

$$F_{net_r} = m_s a_r$$

but we only have one force so

$$m_s a_r = G\frac{M_E m_s}{r_{Es}^2}$$

and we can see from the figure that W_{SE} is toward the center of the orbit so it is centripetal, so

$$a_r = a_c = \frac{v_t^2}{r_{Es}}$$

or

$$m_s \frac{v_t^2}{r_{Es}} = G\frac{M_E m_s}{r_{Es}^2}$$

and some things cancel

$$v_t^2 = G\frac{M_E}{r_{Es}}$$

so

$$v_t = \sqrt{G\frac{M_E}{r_{Es}}}$$

then the kinetic energy of the satellite would be

$$\begin{aligned} K &= \frac{1}{2}m_s\left(v_t\right)^2 \\ &= \frac{1}{2}m_s G\frac{M_E}{r_{Es}} \\ &= G\frac{M_E m_s}{2r_{Es}} \end{aligned}$$

But notice that

$$U_g = -G\frac{M_E m_s}{r_{Es}}$$

So the kinetic energy of our satellite must be

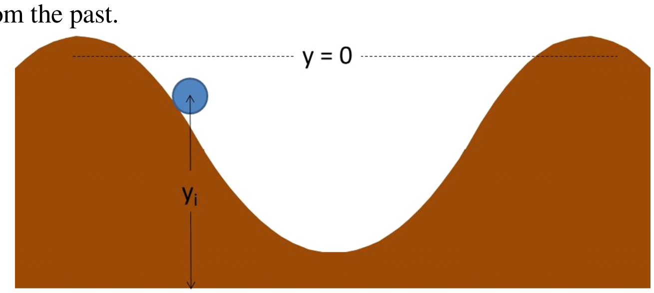
$$K = -\frac{1}{2}U_g$$

Now this is not always true for a satellite! So far we have only considered circular orbits. For other obits this relationship between the gravitational potential energy and the kinetic energy won't hold true. So what we have found is a constraint on K and U_g for *circular* orbits.

But if we restrict ourselves to circular orbits then the total mechanical energy for our satellite must be

$$\begin{aligned} E_{mech} &= K + U_g \\ &= G\frac{M_E m_s}{2r_{Es}} - G\frac{M_E m_s}{r_{Es}} \\ &= -G\frac{M_E m_s}{2r_{Es}} \\ &= \frac{1}{2}U_g \end{aligned}$$

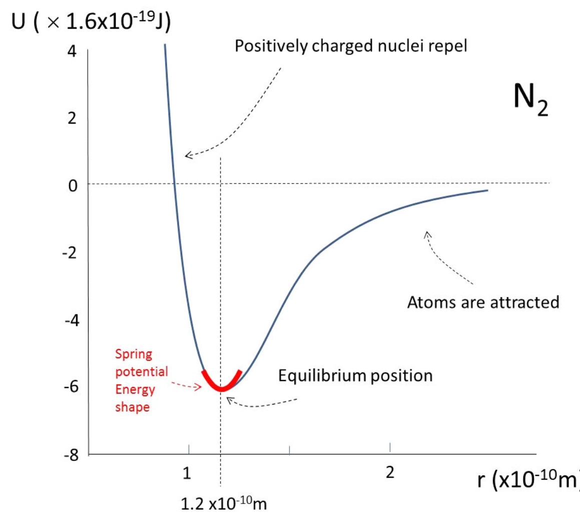
But remember that U_g is negative so the total energy of the Earth-satellite system is negative. Once again we can identify this as a bound situation. Think of our potential energy diagrams from the past.



If we start a ball rolling from rest as shown in the previous figure, the ball will start with a potential energy that is negative. Since we started the ball from rest, the initial potential energy is the total mechanical energy for the ball. So the total energy will be negative, and we can see that this simply means that the ball will never leave the valley without some addition of energy. The ball is bound between the two hills because it does not have enough energy to escape.

Likewise, the satellite is bound to the Earth. It does not have enough energy to escape.

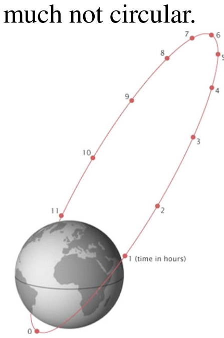
Let's go back to our graph of the potential energy of an N_2 molecule. We can now understand this graph better. Notice that at the equilibrium position the shape of the potential energy curve is like the shape of our hill/valley system.



And notice that the entire region around the equilibrium position has negative potential energy! This means that the two nitrogen atoms are bound together. It would take work to move them farther apart.

37 Kepler's Laws

We studied circular orbits in our last lecture. But, you may object, are all orbits circular? The answer is really a big no! Here is one popular choice for communication satellites called a molniya orbit. It is very much not circular.



Very few orbits are perfect circles. Kepler was the researcher who first set down the principles of orbits, so the orbital rules are called *Kepler's laws*. Kepler was working on the orbits of the planets around the Sun, so Kepler's laws are often given in terms of the planetary orbits. But let's consider them in terms of a mover (satellite) and an environmental object (Earth, or similar large object)

1. All orbiting mover objects move in elliptical orbits with the environmental object at one of the focal points.

2. A line drawn from the environmental object to any orbiting mover sweeps out equal areas in equal time intervals.

3. The square of the orbital period of any orbiting mover is proportional to the cube of the average distance from the mover object to the environmental object.

Let's tackle these in the order, 3, 1, and then 2.

Third law

Kepler's third law says that $T^2 \propto r^3$. Let's treat this as a problem and show that it is true for the case of a circular orbit.

First let's assume that we have a mover object, a satellite, in a circular orbit about an environmental object, the Earth. We know that there is only one force, the weight force or the force due to gravity pulling on our satellite.

$$\begin{aligned} F_{net_r} &= ma_r \\ &= W_{sE} \end{aligned}$$

and we recognize that the force is a centripetal force. Then let's use our equation for centripetal acceleration

$$a_c = \frac{v_t^2}{r}$$

So we have

$$m_s a_r = m_s \frac{v_t^2}{r_{Es}}$$

and for a circular orbit we know that the tangential speed is

$$v_t = \sqrt{G\frac{M_E}{r_{Es}}}$$

so

$$\begin{aligned} m_s a_r &= m_s \frac{G\frac{M_E}{r_{Es}}}{r_{Es}} \\ &= m_s \frac{GM_E}{r_{Es}^2} \end{aligned}$$

We can find the speed of our satellite another way. Let's use the time it takes for the satellite to go around the Earth. It is about

$$T = \frac{2\pi r_{Es}}{v_t}$$

so

$$v_t = \frac{2\pi r_{Es}}{T}$$

We can use this in our newton's second law

$$\begin{aligned} m_s a_r &= m_s \frac{v_t^2}{r_{Es}} \\ &= \frac{m_s \left(\frac{2\pi r_{Es}}{T}\right)^2}{r_{Es}} \\ &= \frac{4\pi^2 m_s r_{Es}^2}{r_{Es} T^2} \\ &= \frac{4\pi^2 m_s r_{Es}}{T^2} \end{aligned}$$

Let's combine these two equations for $m_s a_r$

$$\begin{aligned}\frac{4\pi^2 m_s r_{Es}}{T^2} &= \frac{G m_s M_E}{r_{Es}^2}\\ \frac{4\pi^2 r_{Es}^2}{T^2} &= \frac{G M_E}{r_{Es}}\\ \frac{1}{T^2} &= \frac{G M_E}{4\pi^2 r_{Es}^3}\end{aligned}$$

$$T^2 = \frac{4\pi^2}{G M_E} r_{Es}^3 \tag{37.1}$$

And this is a statement of Kepler's third law. We can see that Kepler's third law works for circular orbits.

First Law

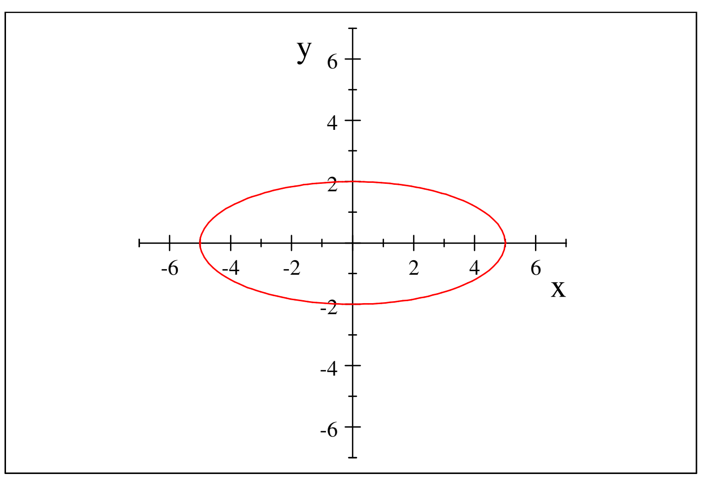
Kepler claimed that *all* orbits are elliptical. But we have studied circular orbits. Have we been wrong? It turns out that a circle is a special case of an ellipse. Let's look at ellipses a bit to see why.

The mathematical equation for an ellipse in xyz coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

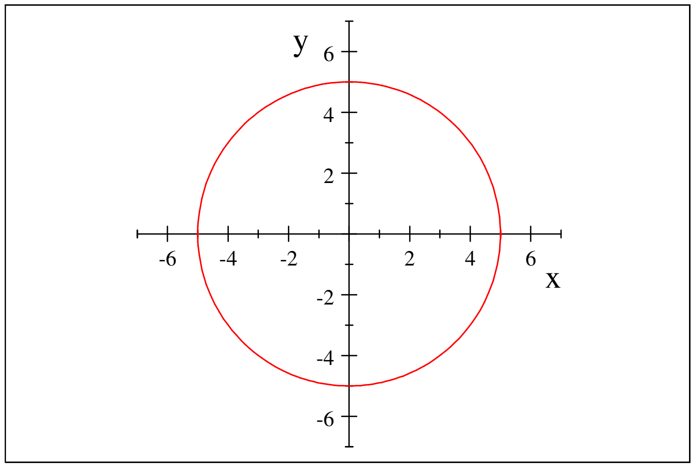
where a and b are called the semi-major and semi-minor axis. Here is a graph to see what this looks like for $a = 5\,\mathrm{m}$ and $b = 2\,\mathrm{m}$.

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$



But suppose $a = b = 5\,\mathrm{m}$, then what would we get?

$$\frac{x^2}{25} + \frac{y^2}{25} = 1$$



This is a circle! So indeed, a circle is an ellipse with the semi-major and semi-minor axis equal. You might object that all we have done is to show that we weren't wrong to use a circle. We certainly did not show that all orbits are ellipses. And it turns out that with our new math parts of differentiation, integration, dot product and cross product, we totally can show that if Newton's law of gravitation is true, then the orbits must be ellipses (well, really any conic section). This is given in the appendix (meaning that we won't actually learn how to do this in our class, but if you are interested, it's there for you to read!).

Second Law

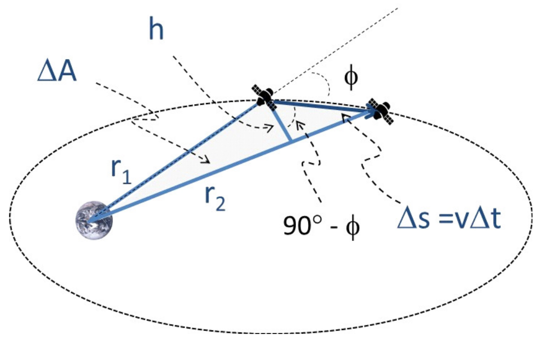
To see why Kepler's second law works, let's use our new angular momentum view. The satellite is moving with a displacement $\overrightarrow{r}$ (which now is not constant!) and has a momentum $\overrightarrow{p}$, so it has an angular momentum

$$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$$

The angular momentum will have a magnitude of

$$\begin{aligned} L &= rp\sin\phi_{rp} \\ &= rmv\sin\phi_{rp} \end{aligned}$$

Then let's look at the area swept out as the satellite moves. In the next figure ΔA is that area swept out between two positions of the satellite. Notice that the area is (very nearly) a triangle.



The area of the triangle is

$$\Delta A = \frac{1}{2}\left(base\right)\left(height\right)$$

only the base for us is the distance r from the environmental object (Earth) to the mover object (satellite). The height of the triangle is marked h. So

$$\Delta A = \frac{1}{2}r_2 h$$

but looking at the diagram,

$$\cos\left(90^\circ - \phi\right) = \frac{h}{\Delta s}$$

so

$$h = \Delta s \cos\left(90^\circ - \phi\right)$$

and we can use a trig identity to write this as

$$h = \Delta s \sin\left(\phi\right)$$

so

$$\Delta A = \frac{1}{2}r\Delta s \sin\phi$$

but

$$\Delta s = v\Delta t$$

where so long as Δt is not too big we can assume v is nearly constant, so

$$\Delta A = \frac{1}{2}r_2 v\Delta t \sin\phi$$

let's compare this to our angular momentum equation. The angle ϕ is the angle between the v direction and the r direction when the satellite is in the first position. And p is in the v direction, so

$$\phi = \phi_{rp}$$

then

$$\Delta A = \frac{1}{2}\Delta t\left(rv\sin\phi_{rp}\right)$$

Now let's return to our angular momentum equation

$$\begin{aligned} L &= r \times p \\ &= m_s\left(rv\sin\phi_{rp}\right) \end{aligned}$$

If we just had a m in the ΔA equation, we could write ΔA in terms of L, so let's multiply ΔA by m_s/m_s

$$\begin{aligned}\Delta A &= \frac{m_s}{m_s}\frac{1}{2}\Delta t\left(rv\sin\phi_{rp}\right)\\ &= \frac{1}{2m_s}\Delta t\left(m_s rv\sin\phi_{rp}\right)\\ &= \frac{1}{2m_s}\Delta t L\end{aligned}$$

then the area swept out per unit time would be

$$\frac{\Delta A}{\Delta t} = \frac{1}{2m_s}L$$

Now let's consider, would we expect that L would be conserved? If there is no interplanetary atmosphere, we would not expect a drag force. All we have is the gravitational force and for this problem it is an internal force for the Earth-sattelite system. If there is no net external force then there is no net external torques so L will be conserved and this means that L can't change. It is a constant. And the mass of the satellite is not changing, so $\Delta A/\Delta t$ can't change. Kepler's second law does work! The area swept out in a given time is constant.

But notice what Kepler's second law tells us.

$$\frac{\Delta A}{\Delta t} = \frac{1}{2m_s}\left(m_s rv\sin\phi_{rp}\right)$$

As the satellite in our strange molniya orbit gets farther away (r gets bigger) something else must get smaller to balance the increase in r. It is the orbital speed v that decreases. The satellite slows down as it gets farther away and speeds up as it gets closer to the environmental object, the Earth.

Orbital motion is wonderfully simple and yet a bit mathematically complex. To describe Kepler's laws more completely, it takes all we have learned; forces, conservation of energy, momentum, and angular momentum, along with the basics of position, velocity, and acceleration. Of course this is a multidimensional problem and vector notation is required. If you are brave and curious, read Appendix (A) which uses all these concepts to show that if we have our gravitational force

$$W = G\frac{m_m m_E}{r_{mE}^2}$$

that the orbits must be elliptical (or more accurately, conic sections). It is a beautiful piece of physics problem solving. But it is a *long* problem and likely too much for PH121. If you are going on in the adventure of learning physics, it is something that you soon would be able to do! If not, it gives an idea of what physics students do after PH121 (and PH123, and PH220, and PH279).

Retrospective

This is where PH121 ends. We have learned a lot about how an object moves. and that was our goal. But have we learned everything about the motion of an object?

Clearly the answer is no, or there would be no PH123 or PH220 etc. So what have we left out?

Well, objects can move together, like air molecules in wind or water molecules in waves on the ocean. We will study this in PH123. We also have electric and magnetic forces on charged objects. We will take on these forces and the motions they cause in PH220.

We have also left out motion of very fast objects, a topic we will study in PH279. And we have not dealt with the tiny motions inside atoms. This is also a topic for PH279. If you are staying with us for these classes, you will obtain a profound understanding of motion. But if you can't take these fascinating courses, you now know enough to solve many problems that occur in everyday life and everyday devices. To design these devices you will need more math and more course work based on that math. But we have all the concepts of everyday motion of single objects or simple systems in our heads now.

I hope these lectures have changed how you look at the world and all the moving objects in that world!

A Elliptical orbits

In all of our work in PH121 we used circular orbits. Kepler said orbits should be elliptical. And that is true. We won't go though this in class, but showing that Newton's law of gravitation implies an ellipse is a great way to show off our new mathematics of dot and cross products. So if you are comfortable with our new math, and curious to see how orbits work, read on.

Let's start with Newton's second law for our orbiting satellite again.

$$\begin{aligned}
\overrightarrow{W} &= -m_s \overrightarrow{a} \\
&= -m_s g\left(r\right)\hat{r} \\
&= -m_s\left(G\frac{M_E}{r_{Es}^2}\right)\hat{r}
\end{aligned}$$

We can write this as

$$-m_s \overrightarrow{a} - m_s\left(G\frac{M_E}{r_{Es}^2}\right)\hat{r} = 0$$

The subscripts may become burdensome, so we will drop them now, but remember that $r = r_{Es}$ is the distance from the satellite to the Earth center of mass to center of mass.

$$-m_s \overrightarrow{a} - m_s\left(G\frac{M_E}{r^2}\right)\hat{r} = 0$$

Conservation of Orbital Mechanical Energy

Now we are going to do something strange. For no apparent reason, lets compute the dot product of both sides of this equation

$$\overrightarrow{v}\cdot\left(m_s \overrightarrow{a} + m_s\left(G\frac{M_E}{r^2}\right)\hat{r}\right) = \overrightarrow{v}\cdot 0$$

then

$$\overrightarrow{v}\cdot m_s \overrightarrow{a} + \overrightarrow{v}\cdot m_s\left(G\frac{M_E}{r^2}\right)\hat{r} = 0$$

or

$$m_s \overrightarrow{v}\cdot\overrightarrow{a} + m_s\left(G\frac{M_E}{r^2}\right)\overrightarrow{v}\cdot\hat{r} = 0$$

Now we need to learn a little bit more about dot products mixed with derivatives. We have a position vector $\overrightarrow{r} = r\hat{r}$ The derivative of this position vector is

$$\begin{aligned}\frac{d\overrightarrow{r}}{dt} &= \frac{d}{dt}(r\hat{r})\\ &= r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}\end{aligned}$$

so if we take

$$\begin{aligned}\frac{d\overrightarrow{r}}{dt}\cdot\hat{r} &= \left(r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}\right)\cdot\hat{r}\\ &= r\frac{d\hat{r}}{dt}\cdot\hat{r} + \frac{dr}{dt}\hat{r}\cdot\hat{r}\\ &= 0 + \frac{dr}{dt}\end{aligned}$$

since $d\hat{r}/dt = 0$.

So

$$\frac{d\overrightarrow{r}}{dt}\cdot\hat{r} = \frac{dr}{dt}$$

and we recognize

$$\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt}$$

so we can write our orbit equation as

$$m_s\overrightarrow{v}\cdot\overrightarrow{a} + m_s\left(G\frac{M_E}{r_{Es}^2}\right)\frac{d\overrightarrow{r}}{dt}\cdot\hat{r} = 0$$

or even

$$m_s\overrightarrow{v}\cdot\overrightarrow{a} + m_s\left(G\frac{M_E}{r^2}\right)\frac{dr}{dt} = 0$$

We can do something similar for the first term We can recognize

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$$

and that $\overrightarrow{v} = v\hat{v}$ where $\hat{v}$ is a unit vector in the same direction $\overrightarrow{v}$. Then

$$\begin{aligned}\frac{d\overrightarrow{v}}{dt} &= \frac{d}{dt}(v\hat{v})\\ &= v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v}\end{aligned}$$

and

$$
\begin{aligned}
\overrightarrow{v} \cdot \overrightarrow{a} &= \overrightarrow{v} \cdot \frac{d\overrightarrow{v}}{dt} \\
&= \overrightarrow{v} \cdot \left(v\frac{d\hat{v}}{dt} + \frac{dv}{dt}\hat{v} \right) \\
&= \overrightarrow{v} \cdot v\frac{d\hat{v}}{dt} + \overrightarrow{v} \cdot \frac{dv}{dt}\hat{v} \\
&= v\hat{v} \cdot v\frac{d\hat{v}}{dt} + v\hat{v} \cdot \frac{dv}{dt}\hat{v} \\
&= 0 + v\frac{dv}{dt}\hat{v} \cdot \hat{v} \\
&= v\frac{dv}{dt}
\end{aligned}
$$

so our orbit equation becomes

$$m_s v\frac{dv}{dt} + m_s \left(G\frac{M_E}{r_{Es}^2} \right) \frac{dr}{dt} = 0$$

Now let's play a clever mathematical trick. Let's take the derivative of the kinetic energy with respect to time.

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}mv^2\right) &= \frac{1}{2}m\frac{d}{dt}\left(v^2\right) \\
&= \frac{1}{2}m\left(2v\frac{dv}{dt}\right) \\
&= mv\frac{dv}{dt}
\end{aligned}
$$

Notice that this is in our orbit equation! So then

$$m_s v\frac{dv}{dt} + m_s \left(G\frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

becomes

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) + m_s \left(G\frac{M_E}{r^2} \right) \frac{dr}{dt} = 0$$

We can play this trick again for the second term

$$
\begin{aligned}
\frac{d}{dt}\left(G\frac{M_E}{r}\right) &= GM_E\frac{d}{dt}\left(\frac{1}{r}\right) \\
&= GM_E\left(-\frac{1}{r^2}\frac{dr}{dt}\right)
\end{aligned}
$$

which once again we recognize this as part of our orbit equation so we can write

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) - m_s\frac{d}{dt}\left(G\frac{M_E}{r}\right) = 0$$

or

$$\frac{d}{dt}\left(\left(\frac{1}{2}mv^2\right) - m_s\left(G\frac{M_E}{r}\right)\right) = 0$$

which tells us that

$$\left(\frac{1}{2}mv^2\right) - m_s\left(G\frac{M_E}{r_{Es}}\right) = \text{constant}$$

That is, the mechanical energy is conserved since we recognize this as just

$$K + U_g = \text{constant}$$

And this makes sense. There are no energy loss mechanisms in our orbit. Our masses are particles (no tidal forces inside the objects, etc.) So we expect conservation of energy in forming our orbit.

Conservation of Orbital Angular Momentum

Now, let's do just what we did before only let's use a cross product with $\overrightarrow{r}$.

$$\begin{aligned}
\overrightarrow{r} \times \left(-m_s\overrightarrow{a} - m_s\left(G\frac{M_E}{r^2}\right)\hat{r}\right) &= \overrightarrow{r} \times 0 \\
-\overrightarrow{r} \times m_s\overrightarrow{a} - \overrightarrow{r} \times m_s\left(G\frac{M_E}{r^2}\right)\hat{r} &= 0 \\
m_s\overrightarrow{r} \times \overrightarrow{a} + m_sG\frac{M_E}{r^2}\overrightarrow{r} \times \hat{r} &= 0 \\
m_s\overrightarrow{r} \times \overrightarrow{a} + m_sG\frac{M_E}{r^2}r\hat{r} \times \hat{r} &= 0
\end{aligned}$$

The last term has $\hat{r} \times \hat{r}$. The angle between $\hat{r}$ and $\hat{r}$ must be zero (they are in the same direction) so

$$\hat{r} \times \hat{r} = (1)(1)\sin(0) = 0$$

and we are left with

$$m_s\overrightarrow{r} \times \overrightarrow{a} = 0$$

which really does note seem to helpful, but it is. Consider that

$$\overrightarrow{a} = \frac{d^2\overrightarrow{r}}{dt^2}$$

so

$$\begin{aligned}
\overrightarrow{r} \times \overrightarrow{a} &= \overrightarrow{r} \times \frac{d^2\overrightarrow{r}}{dt^2} \\
&= \overrightarrow{r} \times \frac{d^2(r\hat{r})}{dt^2}
\end{aligned}$$

Now consider the quantity

$$\frac{d}{dt}\left(\overrightarrow{r} \times \frac{d\overrightarrow{r}}{dt}\right) = \overrightarrow{r} \times \frac{d^2\overrightarrow{r}}{dt^2} + \frac{d\overrightarrow{r}}{dt} \times \frac{d\overrightarrow{r}}{dt}$$

The second term must be zero because the angle between any vector and itself must be zero and $\sin(0) = 0$, but the first term is just what we have in our equation! so our equation becomes

$$m_s \overrightarrow{r} \times \overrightarrow{a} = m_s \frac{d}{dt}\left(\overrightarrow{r} \times \frac{d\overrightarrow{r}}{dt}\right) = 0$$

which we can write as

$$\begin{aligned}
\frac{d}{dt}\left(\overrightarrow{r} \times m_s \frac{d\overrightarrow{r}}{dt}\right) &= 0 \\
\frac{d}{dt}\left(\overrightarrow{r} \times m_s \overrightarrow{v}\right) &= 0 \\
&= \frac{d}{dt}\left(\overrightarrow{L}\right) = 0
\end{aligned}$$

and, hurray! we have conservation of angular momentum for our general orbit!

Conic Section Equation

You may not be a thrilled as I was at this point, but what we have done is typical for physicists. We use the power of mathematics and some ingenuity to predict what motions will be. You might say, "but I would never think of taking cross and dot products seemingly randomly to find a result." This may be true now, but as you get used to using the mathematical tools an operation like this may become more obvious. In any case, recall that early physicists spent many years trying out ways to use their mathematical tools. So eventually someone was bound to try our cross and dot product tricks. But we have only shown conservation of energy and angular momentum. We have not reached our goal. So let's return to our basic motion equation that we started with

$$-m_s \overrightarrow{a} - m_s \left(G\frac{M_E}{r^2}\right)\hat{r} = 0$$

and add to it our equation for angular momentum

$$\overrightarrow{L} = \overrightarrow{r} \times m_s \overrightarrow{v}$$

and form the cross product of the two

$$\begin{aligned}
\left(-m_s \overrightarrow{a} - m_s \left(G\frac{M_E}{r^2}\right)\hat{r}\right) \times \overrightarrow{L} &= 0 \times \overrightarrow{L} \\
m_s \overrightarrow{a} \times \overrightarrow{L} + m_s \left(G\frac{M_E}{r^2}\right)\hat{r} \times \overrightarrow{L} &= 0
\end{aligned}$$

Again this may not seem like an obvious thing to do! But we find that

$$m_s \overrightarrow{a} \times \overrightarrow{L} = -\left(G\frac{M_E}{r^2}\right)\hat{r} \times \overrightarrow{L}$$

and it is time for another mathematical trick. Consider the quantity

$$\begin{aligned}\frac{d}{dt}\left(\overrightarrow{v}\times\overrightarrow{L}\right) &= \overrightarrow{v}\times\frac{d\overrightarrow{L}}{dt}+\frac{d\overrightarrow{v}}{dt}\times\overrightarrow{L}\\ &= \overrightarrow{v}\times\frac{d}{dt}\left(\overrightarrow{r}\times m_s\overrightarrow{v}\right)+\overrightarrow{a}\times\overrightarrow{L}\\ &= \overrightarrow{v}\times(0)+\overrightarrow{a}\times\overrightarrow{L}\\ &= \overrightarrow{a}\times\overrightarrow{L}\end{aligned}$$

for our situation because we have already shown that angular momentum is conserved. So we have

$$m_s\frac{d}{dt}\left(\overrightarrow{v}\times\overrightarrow{L}\right)=-\left(G\frac{M_E}{r^2}\right)\hat{r}\times\overrightarrow{L}$$

Now let's look at the right hand side. Writing out the angular momentum gives

$$\hat{r}\times\overrightarrow{L}=\hat{r}\times\left(\overrightarrow{r}\times m_s\overrightarrow{v}\right)$$

and I will use a vector product identity that I will let the math department teach you

$$\overrightarrow{A}\times\left(\overrightarrow{B}\times\overrightarrow{C}\right)=\overrightarrow{B}\left(\overrightarrow{A}\cdot\overrightarrow{C}\right)-\overrightarrow{C}\left(\overrightarrow{A}\cdot\overrightarrow{B}\right)$$

so for us

$$\begin{aligned}\overrightarrow{r}\times\overrightarrow{L} &= m_s\overrightarrow{r}\times\left(\overrightarrow{r}\times\overrightarrow{v}\right)\\ &= m_s\left(\overrightarrow{r}\left(\overrightarrow{r}\cdot\overrightarrow{v}\right)-\overrightarrow{v}\left(\overrightarrow{r}\cdot\overrightarrow{r}\right)\right)\\ &= m_s\left(\overrightarrow{r}\left(\overrightarrow{r}\cdot\overrightarrow{v}\right)-\overrightarrow{v}r^2\right)\end{aligned}$$

We already know that $\hat{r}\cdot\overrightarrow{v}=rdr/dv$ so

$$\hat{r}\times\overrightarrow{L}=m_s\left(r\hat{r}\left(r\frac{dr}{dv}\right)-\overrightarrow{v}r^2\right)$$

then finally

$$\begin{aligned}m_s\frac{d}{dt}\left(\overrightarrow{v}\times\overrightarrow{L}\right) &= -\left(G\frac{M_Em_s}{r^2}\right)\left(r^2\left(\frac{dr}{dv}\right)\hat{r}-\overrightarrow{v}r^2\right)\\ &= -\left(GM_Em_s\right)\left[\left(\frac{dr}{dv}\right)\frac{\hat{r}}{r}-\frac{\overrightarrow{v}}{r}\right]\end{aligned}$$

Let's employ one more mathematical trick

$$\begin{aligned}\frac{d}{dt}\left(\frac{\overrightarrow{r}}{r}\right) &= -\overrightarrow{r}\frac{1}{r^2}\frac{dr}{dt}+\frac{1}{r}\frac{d\overrightarrow{r}}{dt}\\ &= -\overrightarrow{r}\frac{1}{r^2}\frac{dr}{dt}+\frac{1}{r}\overrightarrow{v}\\ &= -\left(\hat{r}\frac{1}{r}\frac{dr}{dt}-\frac{1}{r}\overrightarrow{v}\right)\end{aligned}$$

and this is the part of our equation that I wrote in square brackets, so with a substitution our equation becomes

$$m_s \frac{d}{dt}\left(\overrightarrow{v} \times \overrightarrow{L}\right) = (GM_E m_s)\left(\frac{d}{dt}\left(\frac{\overrightarrow{r}}{r}\right)\right)$$

or, canceling the dt factors from both sides

$$m_s d\left(\overrightarrow{v} \times \overrightarrow{L}\right) = (GM_E m_s)\left(d\left(\frac{\overrightarrow{r}}{r}\right)\right)$$

and we can integrate both sides

$$m_s \int d\left(\overrightarrow{v} \times \overrightarrow{L}\right) = -(GM_E m_s)\int\left(d\left(\frac{\overrightarrow{r}}{r}\right)\right)$$

to find

$$m_s \overrightarrow{v} \times \overrightarrow{L} = (GM_E m_s)\left(\frac{\overrightarrow{r}}{r}\right) + \overrightarrow{B}$$

where $\overrightarrow{B}$ is a vector constant of integration.

Once again for no apparent reason let's take the dot product of this equation with $\overrightarrow{r}$

$$\overrightarrow{r} \cdot \left(m_s \overrightarrow{v} \times \overrightarrow{L}\right) = -(GM_E m_s)\,\overrightarrow{r} \cdot \left(\frac{\overrightarrow{r}}{r}\right) + \overrightarrow{r} \cdot \overrightarrow{B}$$

and use another vector product identity

$$\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C} = \overrightarrow{A} \times \overrightarrow{B} \cdot \overrightarrow{C}$$

We can write this as to write our dot product equation as

$$(m_s \overrightarrow{r} \times \overrightarrow{v}) \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB\cos\theta_{rB}$$

or

$$\frac{1}{m_s}(\overrightarrow{r} \times m_s \overrightarrow{v}) \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB\cos\theta_{rB}$$
$$(\overrightarrow{r} \times m_s \overrightarrow{v}) \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB\cos\theta_{rB}$$
$$\overrightarrow{L} \cdot \overrightarrow{L} = (GM_E m_s)\, r + rB\cos\theta_{rB}$$
$$L^2 = (GM_E m_s)\, r + rB\cos\theta_{rB}$$

and now we can solve for r

$$L^2 = r\left((GM_E m_s) + B\cos\theta_{rB}\right)$$

then

$$r = \frac{L^2}{((GM_E m_s) + B\cos\theta_{rB})}$$

or, rearranging slightly,

$$r = \frac{L^2/(GM_E m_s)}{(1 + (B/GM_E m_s)\cos\theta_{rB})}$$

If we compare this to the parametric equation for a conic section (straight out of your calculus text book),

$$r = \frac{p}{1 + e\cos\nu}$$

we can see that our orbit must be a conic section with a semi-latus rectum, $p = L^2/(GM_E m_s)$ and an eccentricity, $e = B/GM_E m_s$ and an angle $\nu = \theta_{rB}$. This means our orbit could be any conic section, circle, ellipse, parabola, or hyperbola.

For satellites we most often choose ellipses. But the other conic sections are possible.

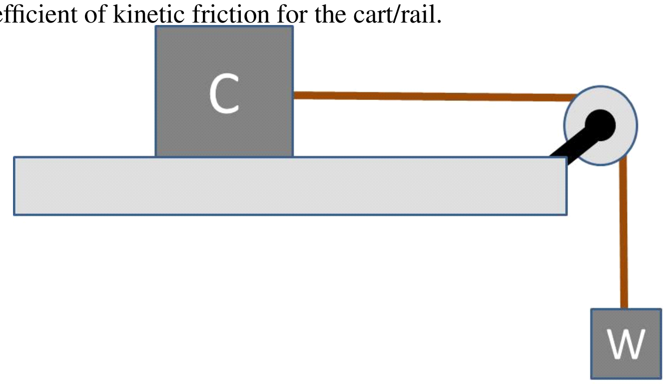
So Kepler was partially right. An ellipse is a general form for an orbit, but it might even be better to write Kepler's law to say that orbits are conic sections.

If you are a normal PH121 student, your reaction to this problem might be "Agh, maybe I should change my major to horticulture!" But don't worry, This was really a junior level problem, and for us physics majors we have many classes (both physics and math classes) to take before we would be expected to do a problem like this. Still it is fun to see that we *can* do a problem like this with the math we learned in lowly PH121 if we are very persistent!

B Class Experiment

An experiment is really just a problem where you have to measure the input values. We will do a problem in class today where we will have to use equipment to measure the mass and acceleration of a cart and weight.[15]

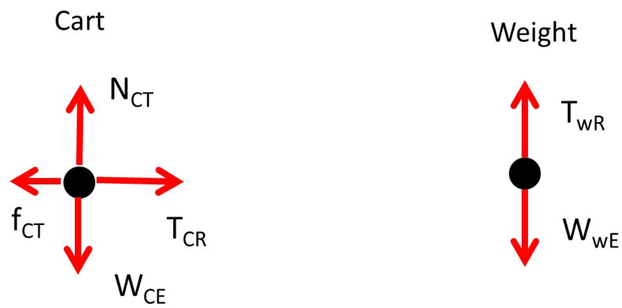
I will bring in some equipment to measure the acceleration of a little cart that rides on a low friction rail. The setup is kind of like what is shown in the figure below. We will let the masses move and let the system measure the acceleration of the cart. We want to find the coefficient of kinetic friction for the cart/rail.



I know this has to be a friction problem, since we want to know the coefficient of kinetic friction for the cart. I suspect that I will have to find the friction force and the normal force to find the coefficient of kinetic friction. So since I am looking for forces, it is probably a Newton's second law problem as well.

Since I think this might be a Newton's second law problem, here are free-body diagrams for both the cart and the hanging weight.

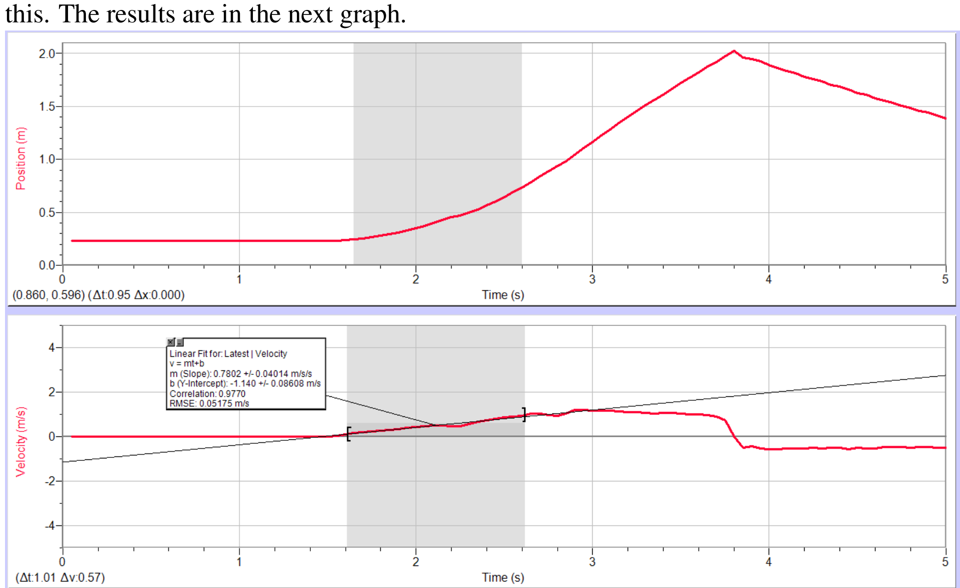
[15] If you are reading this, but not in PH121 at BYU-I, this is still an interesting problem to consider, so read on!



For our class system, we can measure the masses with a triple beam balance. I got

$$\begin{aligned} m_c &= 209.4\,\mathrm{g} \\ m_w &= 20\,\mathrm{g} \end{aligned}$$

and we can measure the acceleration by performing the experiment, letting the cart move, and measuring the position vs. time. We have a sonic position sensor that can do this. The results are in the next graph.



Our sensor also gives us the speed of the object. We can use the velocity vs. time graph to find our acceleration, by finding the slope of the v vs. t graph. I found it to be

$$a = 0.78 \pm 0.04 \frac{\mathrm{m}}{\mathrm{s}^2}$$

We also know

$$g = 9.8004 \frac{\mathrm{m}}{\mathrm{s}^2}$$

(which is true here in Rexburg, where I performed the experiment).

Our basic equations are

$$\overrightarrow{\mathbf{a}}=\frac{\overrightarrow{\mathbf{F}}_{net}}{m}$$

$$a_x = \frac{F_{net_x}}{m}$$

$$a_y = \frac{F_{net_y}}{m}$$

$$F = mg$$

$$f_{BFs} \le \mu_s N_{BF}$$

$$f_{BF_k} = \mu_k N_{BF}$$

From the start, I don't know exactly how to find the answer, but since it is a Newton's second law problem, I will write out Newton's second law in the x- and y-directions for both objects. Then look at the result and see if I can find a way to get μ. So here are the x- and y-equations for the cart

$$F_{net_x} = ma_{cx} = T_{CR} - f_{CT}$$

$$F_{net_y} = ma_{cy} = N_{CT} - W_{CE}$$

we know the cart is accelerating in the x-direction, but it is not accelerating in the y-direction so we can write these as

$$m_c a_c = T_{CR} - f_{CT}$$

$$0 = N_{CT} - W_{CE}$$

and let's add in the friction equation for the cart

$$f_{CT} = \mu_k N_{CT}$$

Now let's do the hanging weight

$$F_{net_x} = ma_{wx} = 0$$

$$F_{net_y} = ma_{wy} = T_{wR} - W_{wE}$$

Notice that the weight *is* accelerating downward. We can't skip the step of writing out Newton's second law and asking if the object is accelerating. It might be tempting to say that the tension and weight are equal, but they really are not! The x-equation just gives $0 = 0$ since there are no x-forces for the weight.

$$0 = \frac{F_{Wnet_x}}{m_w} = \frac{0-0}{m_w} = 0$$

but the y-equation gives us an expression for the acceleration of the weight.

$$-a_{wy} m_w = T_{wR} - W_{wE}$$

Notice that the net force is negative! Since in our Newton's second law we expressly put in the signs for the direction and the rest of the symbols are magnitudes, we will write the net force as $-a_{wy}m_w$ with the minus sign written. The weight is accelerating downward. We can solve for the tension in the rope.

$$T_{wR} = W_{WE} - m_w a_{wy}$$

Note that the fact that the weight is accelerating makes a difference. We know that $T_{wR} = T_{CR}$ so long as our rope is not too massive/ Then let's substitute in T_{wR} into our equation for $m_C a_C$

$$m_c a_c = T_{CR} - f_{CT}$$

so

$$m_c a_c = (W_{WE} - m_w a_{wy}) - f_{CT}$$

or

$$m_c a_c = (W_{WE} - m_w a_{wy}) - \mu_k N_{CT}$$

then from our y-equation from Newton's second law for the cart we know

$$0 = N_{CT} - W_{CE}$$

we have

$$N_{CT} = W_{CE}$$

so

$$(W_{WE} - m_w a_w) - \mu_k N_{CT} = m_c a_c$$

becomes

$$(W_{WE} - m_w a_w) - \mu_k W_{CE} = m_c a_c$$

and we solve for μ_k

$$W_{WE} - m_w a_w - \mu_k W_{CE} = m_c a_c$$
$$W_{WE} - m_w a_w - m_c a_c = \mu_k W_{CE}$$
$$\frac{W_{WE} - m_w a_w - m_c a_c}{W_{CE}} = \mu_k$$

We can substitute in $W_{WE} = m_w g$ and $W_{CE} = m_C g$

$$\frac{m_w g - m_w a_w - m_c a_c}{m_C g} = \mu_k$$
$$\frac{m_w g - m_w a_w - m_c a_c}{m_C g} = \mu_k$$

we also need to realize that $a_w = a_c$. This comes from the fact that the two masses are constrained by the rope that connects them. We call such a relationship a "constraint."

$$\frac{m_w g - m_w a_c - m_c a_c}{m_C g} = \mu_k$$
$$\frac{m_w g - (m_w + m_c)\, a_c}{m_C g} = \mu_k$$
$$\frac{m_w g - (m_w + m_c)\, a_c}{m_C g} = \mu_k$$

$$\begin{aligned}\mu_k &= \frac{(20\,\mathrm{g})\left(9.80004\frac{\mathrm{m}}{\mathrm{s}^2}\right) - (209.4\,\mathrm{g} + 20\,\mathrm{g})\left(0.78\frac{\mathrm{m}}{\mathrm{s}^2}\right)}{(209.4\,\mathrm{g})\left(9.80004\frac{\mathrm{m}}{\mathrm{s}^2}\right)} \\ &= 8.3176 \times 10^{-3}\end{aligned}$$

This seems small, but after all the rail is supposed to be frictionless. So that may be OK.

The units do check since μ_k is unitless.

C Table of constants and fundamental values

Charge and mass of elementary particles

Proton Mass	$m_p = 1.6726231 \times 10^{-27}\,\mathrm{kg}$
Neutron Mass	$m_n = 1.6749286 \times 10^{-27}\,\mathrm{kg}$
Electron Mass	$m_e = 9.1093897 \times 10^{-31}\,\mathrm{kg}$
Electron Charge	$q_e = -1.60217733 \times 10^{-19}\,\mathrm{C}$
Proton Charge	$q_p = 1.60217733 \times 10^{-19}\,\mathrm{C}$

α-particle mass[16]	$m_\alpha = 6.64465675(29) \times 10^{-27}\,\mathrm{kg}$
α-particle charge	$q_\alpha = 2q_e$

Fundamental constants

Permittivity of free space	$\epsilon_o = 8.854187817 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\,m}^2}$
Permeability of free space	$\mu_o = 4\pi \times 10^{-7}\,\frac{\mathrm{T\,m}}{\mathrm{A}}$
Coulomb Constant	$K = \frac{1}{4\pi\epsilon_o} = 8.98755 \times 10^{9}\,\mathrm{N\,m^2\,C^{-2}}$
Gravitational Constant	$G = 6.67259 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Speed of light	$c = 2.99792458 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Avogadro's Number	$6.0221367 \times 10^{23}\,\mathrm{mol^{-1}}$
Fundamental unit of charge	$q_f = 1.60217733 \times 10^{-19}\,\mathrm{C}$

Astronomical numbers

[16] http://physics.nist.gov/cgi-bin/cuu/Value?mal

Mass of the Earth[17]	$5.9726 \times 10^{24}\,\mathrm{kg}$
Mass of the Moon[18]	$0.07342 \times 10^{24}\,\mathrm{kg}$
Earth-Moon distance (mean)[19]	$384400\,\mathrm{km}$
Mass of the Sun[20]	$1,988,500 \times 10^{24}\,\mathrm{kg}$
Earth-Sun distance[21]	$149.6 \times 10^{6}\,\mathrm{kg}$

Conductivity and resistivity of various metals

Material	Conductivity $\left(\Omega^{-1}\,\mathrm{m}^{-1}\right)$	Resistivity $(\Omega\,\mathrm{m})$	Temp. Coeff. $\left(\mathrm{K}^{-1}\right)$
Aluminum	3.5×10^{7}	2.8×10^{-8}	3.9×10^{-3}
Copper	6.0×10^{7}	1.7×10^{-8}	3.9×10^{-3}
Gold	4.1×10^{7}	2.4×10^{-8}	3.4×10^{-3}
Iron	1.0×10^{7}	9.7×10^{-8}	5.0×10^{-3}
Silver	6.2×10^{7}	1.6×10^{-8}	3.8×10^{-3}
Tungsten	1.8×10^{7}	5.6×10^{-8}	4.5×10^{-3}
Nichrome	6.7×10^{5}	1.5×10^{-6}	0.4×10^{-3}
Carbon	2.9×10^{4}	3.5×10^{-5}	-0.5×10^{-3}

[17] http://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html
[18] http://nssdc.gsfc.nasa.gov/planetary/factsheet/moonfact.html
[19] http://solarsystem.nasa.gov/planets/profile.cfm?Display=Facts&Object=Moon
[20] http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html
[21] http://nssdc.gsfc.nasa.gov/planetary/factsheet/index.html

D Problem Types and Equation Sets

Throughout the lectures in this book, we refer to problem types and their accompanying equation sets. A set of problem types is compiled here, with a suggested group of equations for the equation set. This may not be an all-inclusive list of problem types, and you may consider grouping the equations differently

Basic definition of displacement

$$\begin{aligned} \Delta y &= y_f - y_i \\ \Delta x &= x_f - x_i \end{aligned}$$

Average velocity and acceleration

$$\overrightarrow{v}_{ave} = \frac{\Delta \overrightarrow{x}}{\Delta t}$$
$$\overrightarrow{a}_{ave} = \frac{\Delta \overrightarrow{v}}{\Delta t}$$

Instantaneous velocity and acceleration

$$\overrightarrow{v} = \frac{d\overrightarrow{x}}{dt}$$
$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$$

$$x_f = x_i + \int_{t_i}^{t_f} v dt$$
$$v_f = v_i + \int_{t_i}^{t_f} a dt$$

Constant Acceleration (Kinematic)

$$\Delta x = v_{ix}\Delta t + \tfrac{1}{2}a_x \Delta t^2$$
$$v_{fx} = v_{ix} + a_x \Delta t$$
$$v_{fx}^2 = v_{ix}^2 + 2a_x \Delta x$$

Free Fall

$$\Delta y = v_{iy}\Delta t + \tfrac{1}{2}a_y\Delta t^2$$
$$v_{fy} = v_{iy} + a_y\Delta t$$
$$v_{fy}^2 = v_{iy}^2 + 2a_y\Delta x$$
$$a_y = -g$$

Components of vectors

$$\begin{aligned} v_x &= v\cos\theta \\ v_y &= v\sin\theta \end{aligned}$$
$$v = \sqrt{v_x^2 + v_y^2}$$
$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Projectile motion

$$\begin{array}{ll} \Delta x = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 & \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \\ v_{fx} = v_{ix} + a_x\Delta t & v_{fy} = v_{iy} + a_y\Delta t \\ v_{fx}^2 = v_{ix}^2 + 2a_x\Delta x & v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \end{array}$$
$$\begin{aligned} a_y &= -g \\ a_x &= 0 \end{aligned}$$

Circular motion

$$\begin{array}{ll} \text{Linear} & \text{Circular} \\ \Delta r = r_f - r_i & \Delta\phi = \phi_f - \phi_i \\ \Delta t = t_f - t_i & \Delta t = t_f - t_i \\ v_{ave} = \frac{\Delta r}{\Delta t} & \omega_{ave} = \frac{\Delta\phi}{\Delta t} \\ v = \frac{dr}{dt} & \omega = \frac{d\phi}{dt} \end{array}$$
$$\omega = \tfrac{v_t}{r}$$

Relative Motion

$$\begin{aligned} \vec{\mathbf{v}}_{bA} &= \vec{\mathbf{v}}_{bB} + \vec{\mathbf{V}}_{BA} \\ \vec{\mathbf{v}}_{bB} &= \vec{\mathbf{v}}_{bA} - \vec{\mathbf{V}}_{BA} \end{aligned}$$

Uniform Circular Motion

$$\Delta\phi = \phi_f - \phi_i$$
$$\Delta t = t_f - t_i$$
$$\omega_{ave} = \frac{\Delta\phi}{\Delta t}$$
$$\omega = \frac{d\phi}{dt}$$
$$\omega = \frac{v_t}{r}$$

Non-uniform Circular Motion

$$a_c = \frac{v_t^2}{r}$$
$$\Delta\phi = \phi_f - \phi_i$$
$$\Delta t = t_f - t_i$$
$$\omega_{ave} = \frac{\Delta\phi}{\Delta t}$$
$$\omega = \frac{d\phi}{dt}$$
$$\omega = \frac{v_t}{r}$$
$$\alpha = \frac{d\omega}{dt}$$
$$\alpha = \frac{a_t}{r}$$

rotational kinematics set

$$\omega_f = \omega_i + \alpha\Delta t$$
$$\Delta\theta = \omega_i\Delta t + \frac{1}{2}\alpha\Delta t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\phi$$
$$\omega_{ave} = \frac{\omega_f + \omega_i}{2}$$

Arclength kinematics set

$$v_f = v_i + a_t\Delta t$$

$$s_f = s_i + v_i\Delta t + \frac{1}{2}a_t\Delta t^2$$

$$v_f^2 = v_i^2 + 2a_t\Delta s$$

Forces Basic Set

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\mathbf{F}}_{net}}{m}$$

$$\overrightarrow{\mathbf{F}}_{net} = m\overrightarrow{\mathbf{a}}$$

$$\overrightarrow{F}_{net} = \sum_{n=1}^{6} \overrightarrow{F}_n$$
$$F_{net_x} = \sum_{n=1}^{6} F_{xn}$$

$$F_{net_y} = \sum_{n=1}^{6} F_{yn}$$

$$\overrightarrow{\mathbf{a}} = \frac{\overrightarrow{\mathbf{F}}_{net}}{m}$$

Equilibrium

$$\overrightarrow{\mathbf{F}}_{net} = 0 \qquad \text{equilibrium}$$

Newton's gravitation

$$W = mg$$
$$W = F_g = G\frac{m_m m_E}{r_{mE}^2}$$
$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}$$

$$M_E = 5.89 \times 10^{24}\,\mathrm{kg}$$
$$R_E = 6.37 \times 10^{6}\,\mathrm{m}$$
$$F = G\frac{M_E m_a}{(R_E + h)^2} \qquad \text{orbit with altitude } h$$

Friction

$$f_s \leq \mu_s N$$
$$f_k = \mu_k N$$
$$f_r = \mu_r N$$

Material	μ_s	μ_k
Rubber on concrete	1.00	0.80
Steel on steel	0.74	0.57
Wood on wood	0.25 – 0.50	0.20
Waxed wood on snow	0 – 0.14	0.04 – 0.1
Ice on ice	0.10	0.03

Drag

$$D = \frac{1}{2} C \rho A v^2$$

Momentum

$$\begin{aligned}
\overrightarrow{p} &= m\overrightarrow{v} \\
\overrightarrow{F}_{ave} &= \frac{\Delta \overrightarrow{p}}{\Delta t} \\
\Delta \overrightarrow{p} &= \overrightarrow{p}_f - \overrightarrow{p}_i \\
\overrightarrow{p}_f - \overrightarrow{p}_i &= \bar{J} \\
\overrightarrow{p}_f - \overrightarrow{p}_i &= \overrightarrow{F}_{ave} \Delta t \\
\overrightarrow{F}_{net} &= \frac{d\overrightarrow{p}}{dt} \\
\overrightarrow{p}_f - \overrightarrow{p}_i &= \int \overrightarrow{F}_{net}(t)\, dt \\
\overrightarrow{P} &= \sum_{i=1}^{N} \overrightarrow{p}_i \\
\frac{d\overrightarrow{P}}{dt} &= \overrightarrow{F}_{external_{net}} \\
\overrightarrow{P}_f &= \overrightarrow{P}_i
\end{aligned}$$

Energy

$$K = \frac{1}{2} m v^2$$

$$U_g = mgy$$

$$\begin{aligned}
E_i &= E_f \\
K_i + U_i &= K_f - U_f \\
\frac{1}{2} m v_i^2 + mgy_i &= \frac{1}{2} m v_f^2 + mgy_f
\end{aligned}$$

Springs

$$S = F_s = -k\Delta y$$
$$U_s = \frac{1}{2}k\left(x - x_o\right)^2$$
where $x_o = x_e$ is the equilibrium position for the spring.

Work

$$w = \Delta K$$
$$w = \int_{s_i}^{s_f} \overrightarrow{F} \cdot d\overrightarrow{s}$$
$$\Delta U_s = -w$$

Force and potential energy

$$\Delta U = -w = -\int \overrightarrow{F}\left(s\right) \cdot \overrightarrow{ds}$$
$$\Delta U = -\int \overrightarrow{F}\left(s\right) \cdot \overrightarrow{ds}$$

Thermal energy

$$\begin{aligned} E_i &= E_f \\ K_i + U_i + E_{thi} &= K_f + U_f + E_{thf} \\ \frac{1}{2}m_b v_i^2 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f + w_f \end{aligned}$$
$$K_i + U_{gi} + U_{si} + U_{ei} + E_{th_i} = K_f + U_{gf} + U_{sf} + U_{ef} + E_{th_f}$$

Power

$$P = \frac{dw}{dt}$$
$$P = \frac{\Delta U}{\Delta t}$$
$$P = \frac{dE_{any}}{dt}$$
$$P = \overrightarrow{F} \cdot \overrightarrow{v}$$

Dot Product

$$\overrightarrow{A} \cdot \overrightarrow{B} = AB\cos\theta_{AB}$$

Center of Mass

$$x_{cm} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}$$

$$y_{cm} = \frac{\Sigma m_i y_i}{\Sigma m_i} = \frac{m_1 y_1 + m_2 y_2 + \ldots + m_n y_n}{m_1 + m_2 + \ldots + m_n}$$

$$x_{cm} = \frac{1}{M} \int x dm$$

$$y_{cm} = \frac{1}{M} \int y dm$$

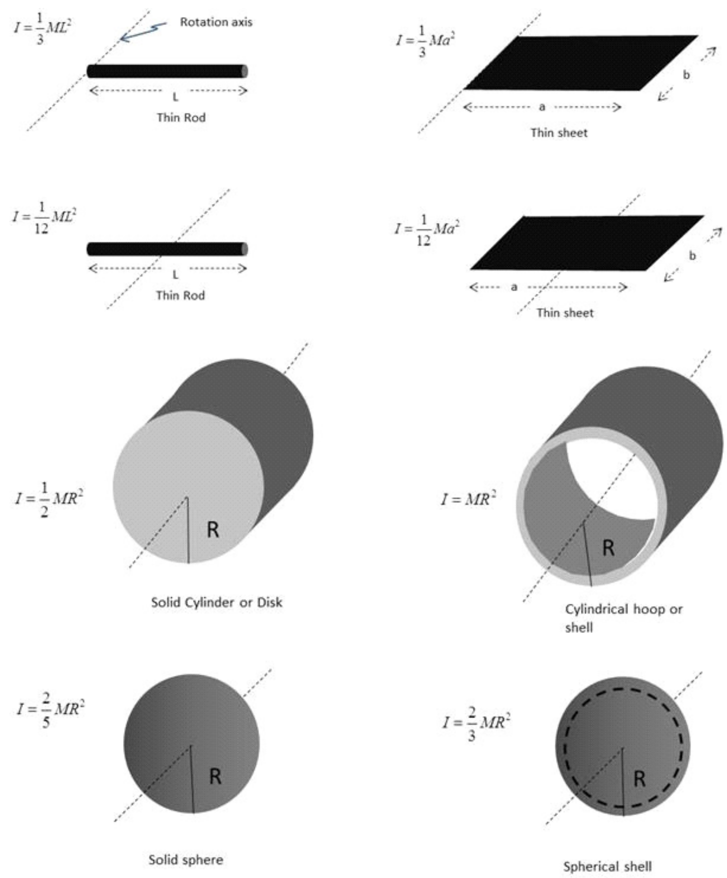
$$\lambda = \frac{M}{L}$$

Moment of Inertia

$$I = \sum_i m_i r_i^2$$

$$I = \int r^2 dm$$

$$I = I_{cm} + d^2 M$$



Rotational Kinetic Energy

$$K_{rot} = \frac{1}{2}\omega^2 I$$

$$E_{mech} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2 + U_g$$

Torque

$$\tau = rF\sin\theta_{rF}$$

Rotational Dynamics

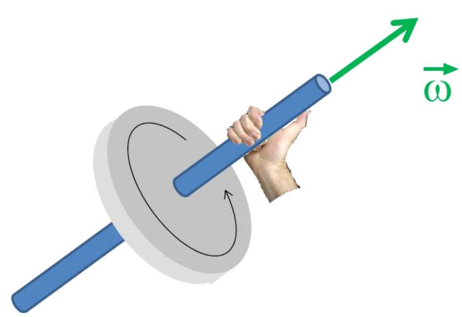
$$\alpha = \frac{\tau_{net}}{I}$$

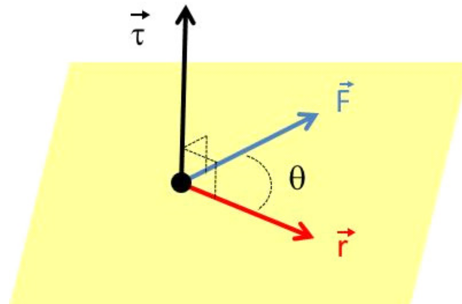
Rotating without slipping

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_t$$

$$\begin{aligned} K_{total} &= K_{rot} + K_t \\ &= \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 \end{aligned}$$

Direction for Angular quantities





Cross Products

$$\overrightarrow{A} \times \overrightarrow{B}$$

$$\left|\overrightarrow{A} \times \overrightarrow{B}\right| = AB \sin\theta_{AB}$$

Torque Again

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$$

Equilibrium

$$\begin{aligned} \overrightarrow{\mathbf{F}}_{net} &= 0 \qquad \text{equilibrium} \\ \overrightarrow{\tau}_{net} &= 0 \qquad \text{equilibrium} \end{aligned}$$

Angular Momentum

$$\overrightarrow{L} = I\overrightarrow{\omega}$$
$$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$$
$$\overrightarrow{L} = \sum_i \overrightarrow{L}_i$$

Newton's gravitation

$$F_g = G\frac{m_m m_E}{r_{mE}^2}$$
$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}$$
$$g\left(r\right) = G\frac{M_E}{r_{mE}^2}$$
$$U_g\left(r\right) = -G\frac{M_E m_m}{r_{mE}}$$

Circular orbits

$$K = -\frac{1}{2}U_g$$
$$v_t = \sqrt{G\frac{M_E}{r_{Es}}}$$
$$E_{mech} = \frac{1}{2}U_g$$

Kepler's laws

$$T^2 = \frac{4\pi^2}{GM_E}r_{Es}^3$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{\Delta A}{\Delta t} = \frac{1}{2m_s}L$$
$$\frac{\Delta A}{\Delta t} = \frac{1}{2m_s}\left(m_s r v \sin\phi_{rp}\right)$$
$$v_t = \sqrt{G\frac{M_E}{r_{Es}}}$$

E References

F Index