Physics Lectures PH123

Fundamentals of Physics II

Wave Motion, Thermal Physics, and Optics

R. Todd Lines

Created in Scientific Workplace ™

Preface

Preface

This set of notes is intended to be an aid to the student. It is what I intend to present in class. There are likely errors and mistakes, so use these notes, but don't expect perfection. If there are things that are confusing, please talk to me or ask questions in class.

Forward

Acknowledgments

I consulted many authors and colleagues in creating these notes. Principle among the authors was Serway and Jewett and Haliday and Resnic. Colleagues of special note are Kevin Kelly, Brian Pyper, Steve Tercotte, and Ryan Nielson. I especially appreciate the PH123 students of the Fall of 2006. They were the first to test run this sets of notes, and I appreciate their feedback (and patience).

BYU-I *R. Todd Lines.*

Contents

1 Introduction to the Course, Simple Harmonic Motion

Fundamental Concepts

- Simple Harmonic Motion
- Frequency and Period
- Mathematical description of Simple Harmonic Motion
- Relationship between Simple Harmonic Motion and circular motion.

Oscillation and Simple Harmonic Motion

Last semester, in PH121 (or Dynamics) you studied how things move. We identified the object we were studying as the *mover mass* and the objects that exerted forces on the mover as the *environmental objects.* You learned about forces and torques which get mover objects moving. You also learned and practiced a lot of math. This semester, we will start with a very special type of motion. It is the motion that results from oscillation. We call this very special type of motion, *simple harmonic motion*.

Simple harmonic motion (SHM) means a motion that repeats in a special, simple way. Some characteristics of this type of motion are as follows:

1. The motion repeats in a regular way, like a grandfather clock pendulum swings back and forth in a set amount of time. This set amount of time is called a period.

2. The mover object moves about a center position and is symmetric about that position. This center position is the bottom of the swing for our grandfather clock. The idea of symmetry just means that the pendulum reaches the same height on both sides as it swings

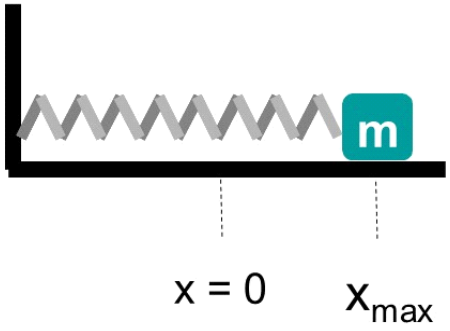
3. The mover object's motion can be traced out in a position vs. time graph. For

SHM, the shape of the position vs. time graph is a sinusoid.

4. The velocity of the mover object constantly changes. It is zero at the extreme points (the points farthest from the center, or the largest positive and the largest negative displacements). That is where it turns around and goes back the other way. It stops there, but just for a split second. The velocity is largest at the equilibrium position (the center position). If we plotted a velocity vs. time graph, the velocity would also be a sinusoid or snake-like shape.

5. The acceleration also constantly changes. And it is also a sinusoid.

The center position is called the Equilibrium position, and for convenience we often define it as the origin of our coordinate system.

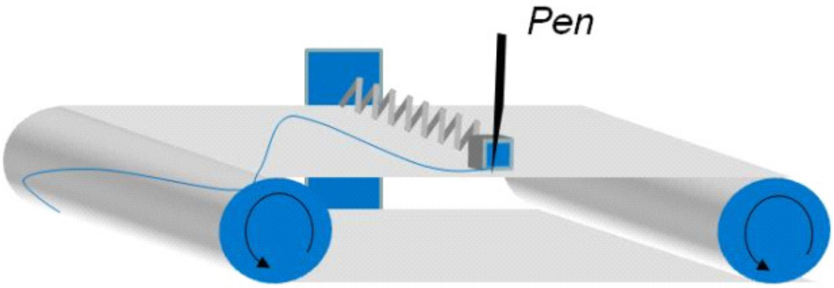


Definition 1.1 *Equilibrium Position: The position of the mover mass (not the spring, which is an environmental object acting on the mover mass) when the spring is neither stretched nor compressed.*

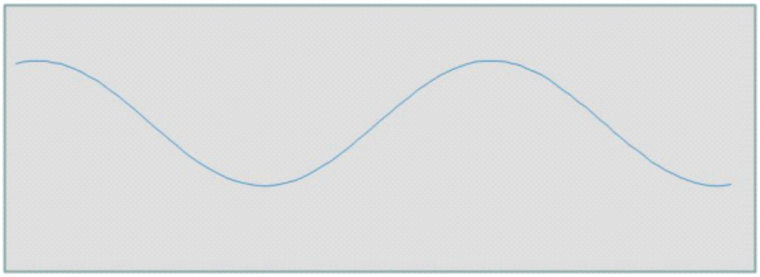
Let's draw a picture of simple harmonic motion. First we need a device to do this. Suppose you go to the supermarket. But instead of putting ramen on the belt at the checkout counter, you strap on a device like this[1]

---

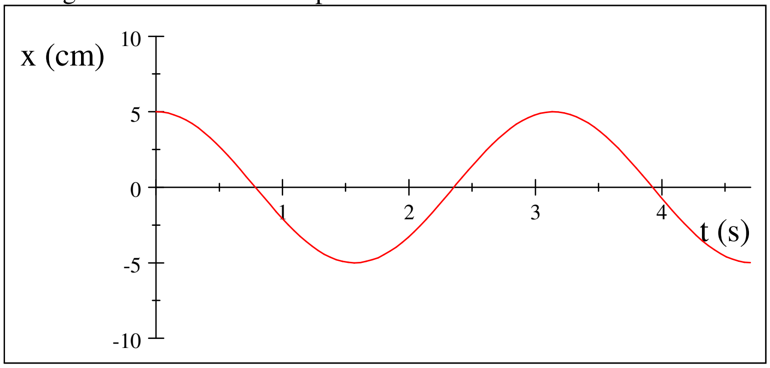
[1] Please don't really try this at the local supermarkets. They don't seem to have any sense of humor about such things at all.



You can see the mover mass on the spring. But we have placed a pen on the mass and that pen is tracing out a pattern as the belt moves. You will recognize this pattern as a trig function. The result might look something like this if you removed the belt.



Of course, what we have made is a position vs. time graph. We remember these from PH121. This gives us a record of the past motion of the mass. $\cos x$



where in this graph, $x_{\max} = 5\,\mathrm{cm}$. Having the power of mathematics, we know we can write an equation that would describe this curve. From your Trigonometry (trig) experience, we can guess that it might look something like

$$x(t) = x_{\max}\cos(\theta)$$

Notice that the instantaneous position, $x(t)$ is, indeed, a function of time. Back in trigonometry we would have said a cosine function was a function of an angle, θ. But we know this is a position vs. time graph, so our angle must be different for different times. Let's write

$$\theta\left(t\right)=\omega t$$

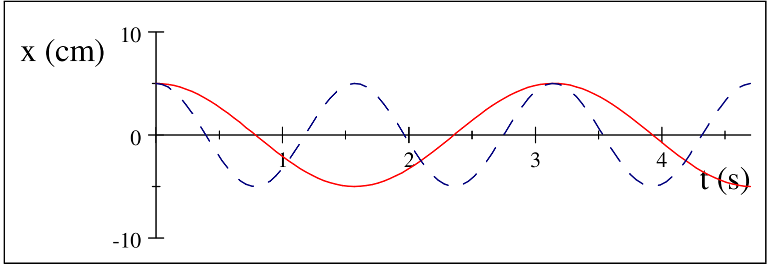
as strange sort of an angle. This "angle" is a function of time, and let's say how fast our "angle" is changing is given by

$$\frac{d\theta}{dt}=\omega$$

This is a sort of speed for how fast our angle is changing. So our equation must be

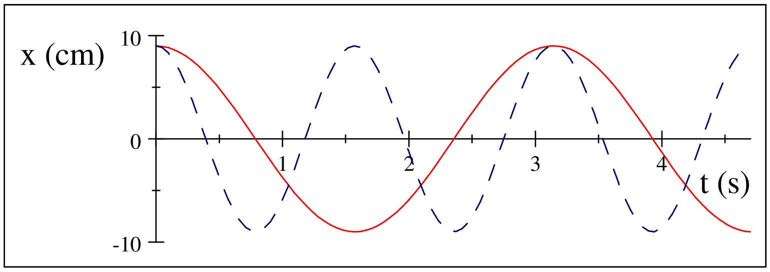
$$x\left(t\right)=x_{\max}\cos\left(\omega t\right)$$

In the next figure $x\left(t\right)=x_{\max}\cos\left(\omega t\right)$ is plotted with two different values for ω.



We can see what ω does for us. It stretches out or compresses our curve. In the blue (dashed) curve our angle changes quickly. In the red (solid) curve the angle changes slowly. Note that we are not plotting position vs. angle. Both plots would be the same if we did! We are plotting position vs. time. And for the blue curve our mass is oscillating much more quickly than it is for the red curve. Let's try this with a coordinate system and the values

I have used $x_{\max}$ where most books use the letter A for *amplitude* to emphasize that the amplitude is the maximum displacement of the mover mass from the mover mass equilibrium position. In our coordinate system, the mass is going back and forth in the x direction. The amplitude means the maximum displacement, so for our coordinate system $x_{\max}$ is a good way to write amplitude. Here is the same graph but with $x_{\max}=9\,\mathrm{cm}$



The two graphs for the two different ω values are not more stretched out in time, but now they are taller along the position axis. This means that the mass is moving farther as it oscillates.

You will remember from your trigonometry class that the *period,* T, tells us the time it takes for the oscillation to go through a complete cycle. You can probably guess that how long it takes to oscillate and how often it oscillates would be related. How often the oscillator completes a cycle is called the frequency. The longer the period, the lower the frequency.

$$f = \frac{1}{T}$$

Think of cars passing you on your way to class. Period is like how long you wait in between cars. Frequency is like how often cars pass. If you wait less time between cars, the cars pass more frequently. And that is just what our equation says! We can see from our graphs that our stretching quantity ω, must be related to the frequency of our oscillation. If the frequency is high, then ω must be large so that we reach different "angles" faster. But for the cosine function to work we need angle units. We will choose radians for our units and we will write our stretching quantity as

$$\omega = 2\pi f$$

$f = \frac{1}{T}$This works, if ω is bigger then our oscillation happens more frequently. The 2π has units of radians. So ω has units of $\mathrm{rad\,Hz}$ or more commonly $\mathrm{rad/\,s}$. That matches our derivative above. Let's give a name to the quantity ω. Since it has radians in it we might guess that it has something to do with circular motion (more on this later) and it has frequency in it. So we will call ω the *angular frequency*.

Velocity and Acceleration

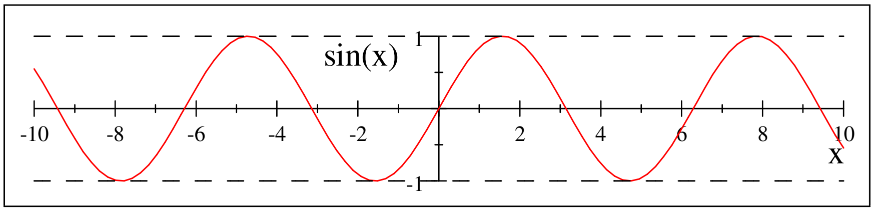
So far we have guessed the descriptive equation for SHM.

$$x(t) = x_{\max}\cos(\omega t) \tag{1.1}$$

This gives us the position of the particle at any given time. Since knowing the position as a function of time is a good description of the motion of our mover mass, we can call this equation the *equation of motion* for our mass. We will see soon that this equation is correct. But let's pretend that we are earlier researchers and we just know the equation makes the right shape. Still, we can learn much from knowing this equation. Since we know how to take derivatives, and we know the derivative of position with respect to time is the velocity, we can see that the velocity of the mass at any given time is given by

$$v(t) = \frac{dx(t)}{dt} = -\omega x_{\max}\sin(\omega t) \tag{1.2}$$

From our fond memories of our trigonometry class we know the maximum of a sine function is always 1.



Notice that the ω and the $x_{\max}$ aren't changing for our oscillation mover mass. They are constant. Then if we want the maximum speed we can simply set the $\sin(\omega t) = 1$. Then the maximum speed will be

$$v_{\max} = \omega x_{\max}(1) \tag{1.3}$$

Notice that this does not tell us when the speed is maximum. Just what the maximum speed is. We will often use this trick of knowing the maximum of sine is one.

We can also find the acceleration. We just take another derivative.

$$\begin{aligned} a(t) &= \frac{dv}{dt} \\ &= \frac{d^2x(t)}{dt^2} \\ &= -\omega^2 x_{\max}\cos(\omega t) \end{aligned}$$

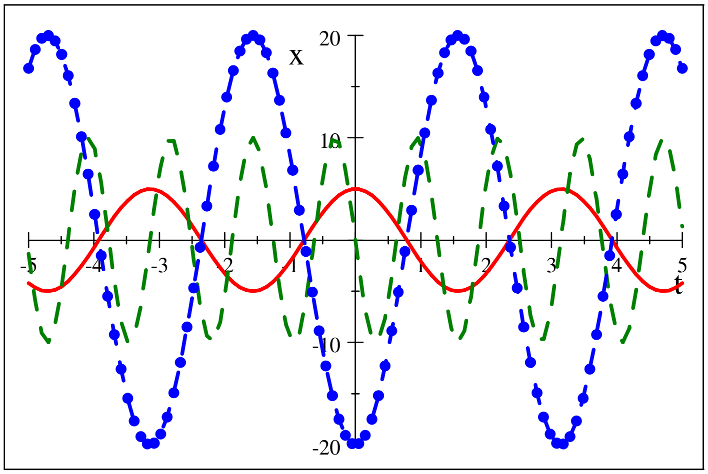
This is the acceleration of the mass attached to the spring. It's also true that the maximum for a cosine function is 1, so the maximum acceleration would be

$$a_{\max} = \omega^2 x_{\max}(1) \tag{1.4}$$

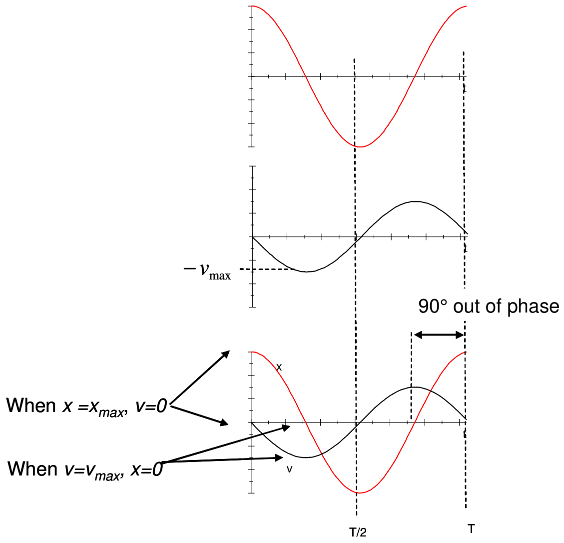
These are significant results, so let's summarize. For a simple harmonic oscillator, the instantaneous position, speed, and acceleration are given by

$$\begin{aligned} x(t) &= x_{\max}\cos(\omega t) \\ v(t) &= -\omega x_{\max}\sin(\omega t) \\ a(t) &= -\omega^2 x_{\max}\cos(\omega t) \end{aligned} \tag{1.5}$$

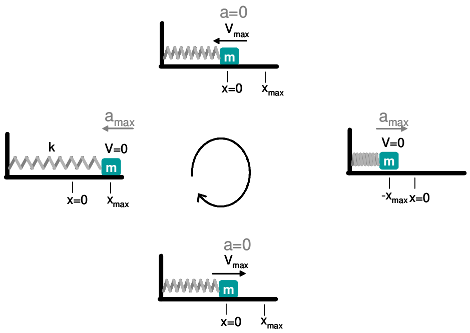
Let's plot $x(t)$, $v(t)$, and $a(t)$ for a specific case



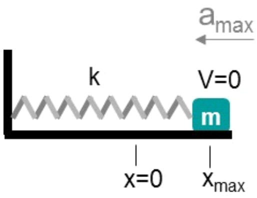
Red (solid) is the displacement, green (dashed) is the velocity, and blue (dot-dashed) is the acceleration. Note that each as a different maximum amplitude. Also note that they don't rise and fall at the same time. We will describe this as being *not in phase*.



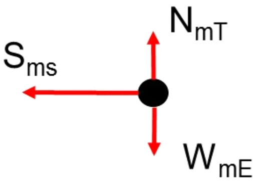
The acceleration is 90° *out of phase* from the velocity. Let's think about why this would be. Suppose we attach a mass to a spring and allow the mass to slide on a frictionless surface.



Let's start by stretching the spring by pulling the mass to the right and releasing. This is the situation in the left hand part of the last figure.



The spring is pulling strongly on the mass. We could draw a free body diagram for this situation. We will need the spring force



Back in PH121 we said that Hooke's Law is not something that is always true, but by "law" we mean a mathematical representation (and equation) that comes from our mental model of how the universe works. In this case, Hooke's law is an equation that comes from Hooke's model of how springs work. It is a good model for most springs as long as we don't stretch them too far. You remember Hooke's law from PH 121. It tells us that the spring force is proportional to how far we stretch or compress the spring (Δx) and how stiff the spring is (k) .

$$\begin{aligned} F_s &= S = -k\Delta x \qquad (1.6) \\ &= -k\left(x - x_o\right) \end{aligned}$$

where x_o is the *equilibrium position of the mass.* If we assume the equilibrium position is at $x = 0$ then we can write our spring force as

$$S = -kx \tag{1.7}$$

If we write out Newton's second law we get

$$\begin{aligned} F_{net_x} &= ma_x = -S_{ms} \\ F_{net_y} &= ma_y = N_{mT} - W_{mE} \end{aligned}$$

We can see that a_y should be zero because the mass won't lift off the table in the y-direction. but in the x-direction

$$a_x = \frac{S_{ms}}{m}$$

and knowing that

$$S_{ms} = -kx$$

from Hooke's law, we can see that

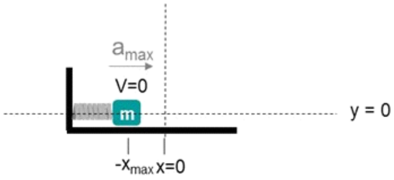
$$a_x = -\frac{kx}{m}$$

and note that since x is large at this point, the acceleration must be big at this point. since the net force is big, the acceleration is big. Of course we realize that way back in our Newton's second law formulation it might have been better to say

$$-F_{net_x} = -ma_x = -S_{ms}$$

but it worked out fine to say we didn't know the direction of a_x. In the end, the minus sign tells us that a_x must be to the left.

But suppose the mass was to the left of $x = 0$



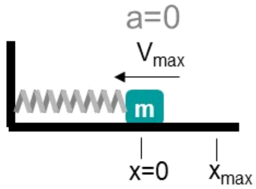
Now $x < 0$ so it is negative. That makes

$$a_x = -\frac{kx}{m}$$

positive. The acceleration always seems to point toward $x = 0$, the equilibrium position for the mass. This is an important part of the definition of simple harmonic motion, having an acceleration that always points toward the equilibrium position. We call a force that makes the acceleration point toward the equilibrium position *restoring force*.

Definition 1.2 *Restoring force: A force that is always directed toward the equilibrium position*

Now lets look at the situation a short time later



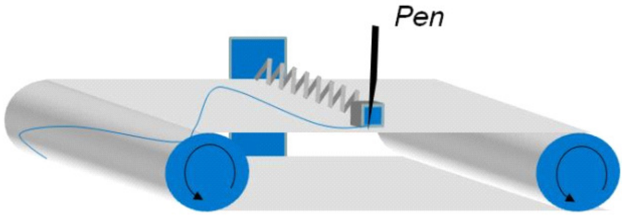
We can see that in this position, $x = 0$ so

$$a_x = \frac{k\,(0)}{m} = 0$$

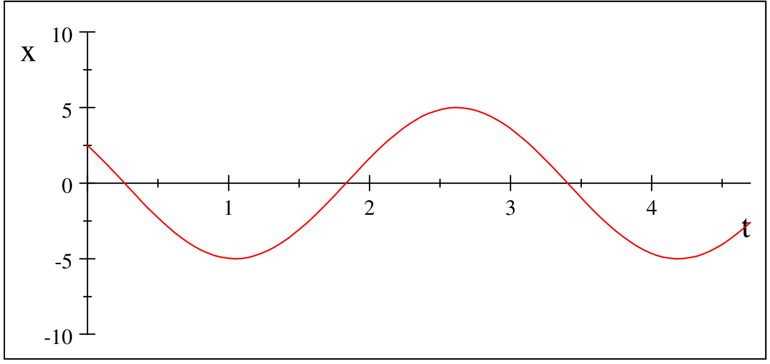
so at this position the acceleration is zero. That is because right at $x = 0$ the spring is neither pushing nor pulling. There is no net force so no acceleration when $x = 0$.

The Idea of Phase

We said before that $x\,(t)$, $v\,(t)$, and $a\,(t)$ are "out of phase." Let's look at the idea of "phase" more carefully. Suppose you return to the grocery store and start your SHM device.



But this time you work with a lab partner, and the lab partner tries to start the stopwatch when you let go of the mass. But, due to having a slow reaction time, your partner starts the clock too late. The resulting graph looks like this

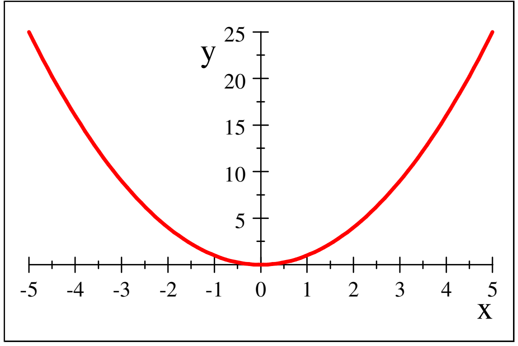


But we know that the only difference between the two situations is that the lab partner was slow, so the graph is shifted on the time axes. We expect we can use the same mathematical model for SHM, but we must need to change something.

We know from our algebra classes what a shift looks like. Take the expression

$$y = x^2$$

We can plot this to get a parabola



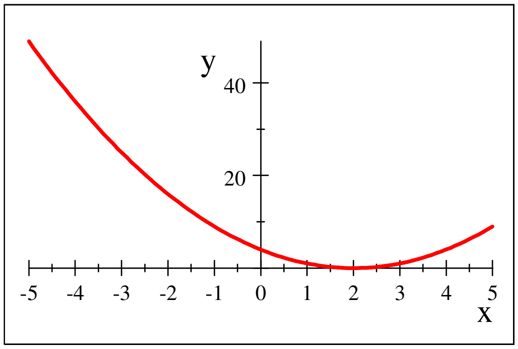
A shifted parabola would be expressed as

$$y = (x - \phi_o)^2$$

where ϕ_o is the amount of the shift. Suppose $\phi_o = 2$, then

$$y = (x - 2)^2$$

and our shifted parabola looks like this



Recall that a shift like

$$y = (x - \phi_o)^2$$

will move the parabola to the right while

$$y = (x + \phi_o)^2$$

will move the parabola to the left.

We can use this idea to write our SHM expression for our situation with the slow lab partner.

$$x(t) = x_{\max} \cos(\omega (t \pm \tau_o))$$

In our case, the lab partner was late by $\tau_o = 4.7124\,\mathrm{s}$. This is not usually how we express the shift, however. We usually distribute the ω so our equation looks like

$$\begin{aligned} x(t) &= x_{\max}\cos(\omega t \pm \omega\tau_o) \\ &= x_{\max}\cos(\omega t \pm \omega\tau_o) \end{aligned}$$

We usually use the symbol $\phi_o = \omega\tau_o$ so we will write our SHM expression for position as a function of time as

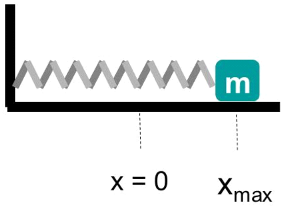
$$x(t) = x_{\max}\cos(\omega t \pm \phi_o)$$

We could call ϕ_o the *slow lab partner constant,* but that is long and not very kind. So let's call ϕ_o by the name *phase constant*. It is also customary to call the entire expression in parenthesis, $(\omega t \pm \phi_o)$, the phase of the cosine function. This is especially used in the fields of Optics and Electrodynamics.

From what we did before, we know that $x(t)$, $v(t)$, and $a(t)$ are out of phase, so there must be a phase constant involved somehow. Let's look for it in what follows.

Initial Conditions

Usually we need to know how we start our oscillator to solve a problem. Let's see how this works.



Suppose we start the motion of a mass attached to a spring (a harmonic oscillator) by pulling the mass to $x = x_{\max}$ and releasing it at $t = 0$. Let's see if we can find the phase. Our initial conditions require

$$\begin{aligned} x(0) &= x_{\max} \\ v(0) &= 0 \end{aligned}$$

Using our formula for $x(t)$ and $v(t)$ we have

$$\begin{aligned} x(0) &= x_{\max} = x_{\max}\cos(0 + \phi_o) \\ v(0) &= 0 = -v_{\max}\sin(0 + \phi_o) \end{aligned}$$

If we choose $\phi_o = 0$, these conditions are met.

Notice that we needed to know the starting time and the position and the velocity at that time. These are what we call *initial conditions*. It is still true that $x(t)$ and $v(t)$ are out of phase. But we fond $\phi_o = 0$. There is another phase term hiding in our expression for $v(t)$ and to find it we need a small trig identity.

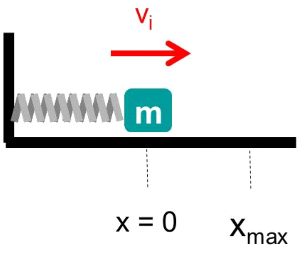
$$\begin{aligned}\sin(\alpha+\phi_o) &= \cos\left(\alpha - \left(\frac{\pi}{2} - \phi_o\right)\right)\\ &= \cos\left(\alpha - \frac{\pi}{2} + \phi_o\right)\end{aligned}$$

so we could write

$$\begin{aligned}v(t) &= -\omega x_{\max}\sin(\omega t+\phi_o)\\ &= -\omega x_{\max}\cos\left(\alpha - \frac{\pi}{2} + \phi_o\right)\end{aligned}$$

and we can see that there was a phase shift of $-\pi/2$ hiding in our sine function. So $v(t)$ really must be out of phase with $x(t)$.

A second example



Using the same equipment, let's start with

$$\begin{aligned}x(0) &= 0\\ v(0) &= v_i\end{aligned}$$

that is, the mover mass is already moving, and we start watching just as it passes the equilibrium point.

$$\begin{aligned}x(0) &= 0 = x_{\max}\cos(0+\phi_o)\\ v(0) &= v_i = -v_{\max}\sin(0+\phi_o)\end{aligned}$$

from the first equation we have

$$0 = x_{\max}\cos(\phi_o)$$

and that gives us

$$\phi_o = \cos^{-1}(0)$$

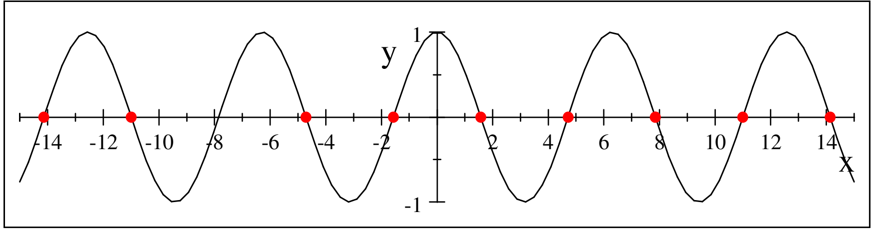
so

$$\phi_o = \pm\frac{\pi}{2}$$

really this is

$$\phi_o = \pm\frac{\pi}{2} \pm n\pi \qquad n = 0, 1, 2, \cdots$$

This gives the red dots in the next plot.



Notice that at each of these locations $\cos(\phi)$ is zero. but let's make an agreement that we will choose the smallest value for ϕ_o that makes $\cos(\phi_o) = 0$. That is our

$$\phi_o = \pm\frac{\pi}{2}$$

but positive and negative $\pi/2$ are the same "smallness." We don't know the sign. Using our initial velocity condition will let us determine the sign. Let's try it

$$\begin{aligned} v_i &= -v_{\max}\sin\left(\frac{\pi}{2}\right) \\ v_i &= \pm v_{\max} \\ v_i &= \pm\omega x_{\max} \end{aligned}$$

From this we can see

$$x_{\max} = \pm\frac{v_i}{\omega}$$

We defined the initial velocity as positive, and we insist on having positive amplitudes, we choose

$$\phi_o = -\frac{\pi}{2}$$

Our solutions are

$$\begin{aligned} x(t) &= \frac{v_i}{\omega}\cos\left(\omega t - \frac{\pi}{2}\right) \\ v(t) &= v_i \sin\left(\omega t - \frac{\pi}{2}\right) \end{aligned}$$

Remark 1.1 *Generally to have a complete solution, you must find all the constants based on the initial conditions. This would mean we need $x_{\max}$, ω, and ϕ_o to have a complete solution.*

Example

A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is $2.00\,\mathrm{cm}$, and the frequency is $1.50\,\mathrm{Hz}$.

a) show that the position of the particle is given by

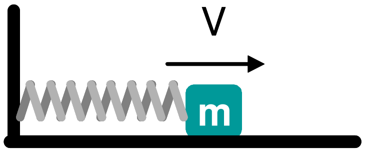
$$x = (2.00\,\mathrm{cm})\sin(3.00\pi t)$$

determine

b) the maximum speed and the earliest time $(t > 0)$ at which the particle has this speed,

c) the maximum acceleration and the earliest time $(t > 0)$ at which the particle has this acceleration, and

d) the total distance traveled between $t = 0$ and $t = 1.00\,\mathrm{s}$



Basic Equations

$$\begin{aligned} x(t) &= x_{\max}\cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max}\sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

Variables

t	time, initial time =0	$t_o = 0$
x	Position, Initial position =0	$x(0) = 0$
v		
a		
x_m	x amplitude	$x_m = 2.00\,\mathrm{cm}$
v_m	v amplitude	
a_m	a amplitude	
ω	angular frequency	
ϕ_o	phase constant	
f	frequency	$f = 1.50\,\mathrm{Hz}$

Symbolic Solution

Part (a)

We can start by recognizing that we know ω because we know the frequency.

$$\omega = 2\pi f$$

We also know the amplitude $A = x_{\max}$ which is given. Knowing that at $t = 0$

$$x(0) = 0 = x_{\max}\cos(0 + \phi_o)$$

we can guess that

$$\phi_o = \pm\frac{\pi}{2}$$

Using

$$v(0) = -\omega x_{\max}\sin\left(0 \pm \frac{\pi}{2}\right)$$

and demanding that amplitudes be positive values, and noting that at $t = 0$ the velocity is positive from the initial conditions:

$$\phi = -\frac{\pi}{2}$$

We also note from out trig identity that we used above

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$$

that we have

$$\begin{aligned} x(t) &= x_{\max}\cos\left(2\pi f t - \frac{\pi}{2}\right) \\ &= x_{\max}\sin(2\pi f t) \end{aligned}$$

which is what we were to show.

Part (b)

We have a formula for

$$\begin{aligned} v_{\max} &= \omega x_{\max} \\ &= 2\pi f x_{\max} \end{aligned}$$

to find when this happens, take

$$v\left(t\right) = v_{\max} = -\omega x_{\max} \sin\left(2\pi f t - \frac{\pi}{2}\right)$$

and recognize that $\sin\left(\theta\right) = 1$ is at a maximum when $\theta = \pi/2$ so

$$\frac{\pi}{2} = 2\pi f t - \frac{\pi}{2}$$
$$\pi = 2\pi f t$$
$$\frac{1}{2f} = t$$
$$\frac{1}{2\left(1.50\,\mathrm{Hz}\right)} = t$$

Part (c) Like with the velocity we must the formula

$$a\left(t\right) = -\omega^2 A \cos\left(\omega t + \phi_o\right)$$

but recognize that the maximum is achieved when $\cos\left(\omega t + \phi_o\right)$ or when $\omega t + \phi_o = 0$

$$\begin{aligned} t &= \frac{\phi_o}{\omega} \\ &= \frac{-\frac{\pi}{2}}{2\pi f} \\ &= \frac{-1}{4f} \end{aligned}$$

The formula for $a_{\max}$ is

$$\begin{aligned} a_{\max} &= -\omega^2 x_{\max} \\ &= -(2\pi f)^2 x_m \end{aligned}$$

Part (d)

We know the period is

$$T = \frac{1}{f}$$

We should find the number of periods in $t = 1.00\,\mathrm{s}$ and find the distance traveled in one period,

$$\frac{t}{T}$$

and multiply them together. In one period the distance traveled is

$$d = 4x_m$$

$$d_{tot} = d * \frac{t}{T} = 4 f x_m t$$

Numerical Solutions

Part (a)

$$\begin{aligned} x(t) &= x_{\max} \sin(2\pi f t) \\ &= (2.00\,\mathrm{cm}) \sin(3.00\pi t) \end{aligned}$$

Part (b)

$$\begin{aligned} v_m &= 2\pi (1.50\,\mathrm{Hz}) (2.00\,\mathrm{cm}) \\ &= 0.188\,50 \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

$$\frac{1}{2f} = t$$

$$\begin{aligned} \frac{1}{2(1.50\,\mathrm{Hz})} &= t \\ &= 0.333 \text{ s} \end{aligned}$$

Part (c)

$$\begin{aligned} t &= \frac{-1}{4f} \\ &= -0.166\,67\,\mathrm{s} \end{aligned}$$

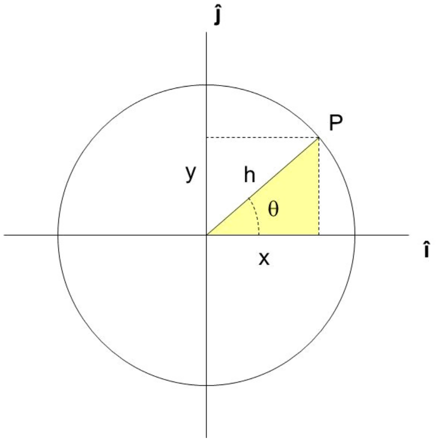
$$\begin{aligned} a_{\max} &= (2\pi f)^2 x_m \\ &= (2\pi 1.5\,\mathrm{Hz})^2 (2.00\,\mathrm{cm}) \\ &= 1.\,776\,5 \frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

Part (d)

$$
\begin{aligned}
d_{tot} &= 4fx_m t \\
&= 8.00\,\mathrm{cm} * 1.50\,\mathrm{Hz} * 1.00\,\mathrm{s} \\
&= 0.12\,\mathrm{m}
\end{aligned}
$$

Comparing Simple Harmonic Motion with Uniform Circular Motion

That circular motion and SHM are related should not be a surprise once we found the solutions to the equations of motion were trig functions. Recall that the trig functions are defined on a unit circle

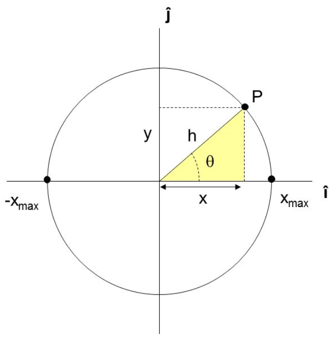


$$\tan\theta = \frac{x}{y} \tag{1.8}$$

$$\cos\theta = \frac{x}{h} \tag{1.9}$$

$$\sin\theta = \frac{y}{h} \tag{1.10}$$

Let's relate this to our equations of motion.



Look at the projection x of the point P on the x axis. This is just the x-component of the position! Lets follow this projection as P travels around the circle. We find it ranges from $-x_{\text{max}}$ to x_{max}. If we watch closely we find it's velocity is zero at the extreme points and is a maximum in the middle. This projection is given as the $\cos$ of the vector from the origin to P. It is just taking the x-component! This model, indeed fits our SHO solution.

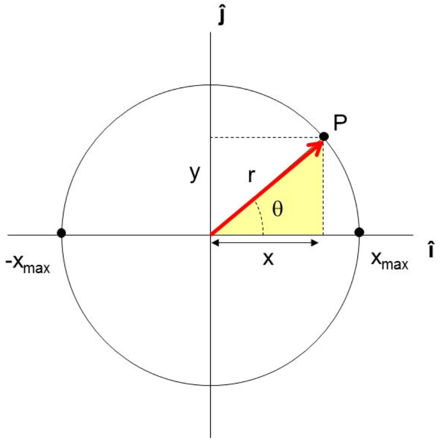
Now lets define a projection of P onto the y axis. Again we have SHM, but this time the projection is a sine function. Because

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\left(\theta\right) \tag{1.11}$$

we can see that this is just a SHO that is $90\,^\circ = \frac{\pi}{2}\,\text{rad}$ out of phase. It is probably worth recalling that a projection of one vector on another can be expressed by a dot product. We could express our length x as

$$x = \overrightarrow{\mathbf{r}} \cdot \hat{\imath} = r\cos\theta$$

where r is the radius of the circle.



We can see that when $\theta = 0$ we have $x = r$ and this will be the largest x value, so $r = x_{\text{max}}$.

So by projecting circular motion onto the x-axis

$$x\left(t\right) = x_{\max} \cos\left(\theta\right)$$

But θ changes in time. We can recall from our PH121 or Dynamics experience that the angular speed

$$\omega = \frac{\Delta\theta}{\Delta t}$$

or, if we agree to start from $\theta_i = 0$ and $t_i = 0$,

$$\omega = \frac{\theta}{t}$$

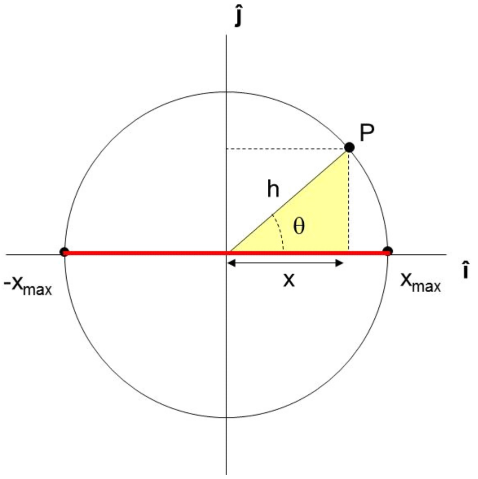
so

$$\theta = \omega t$$

then we

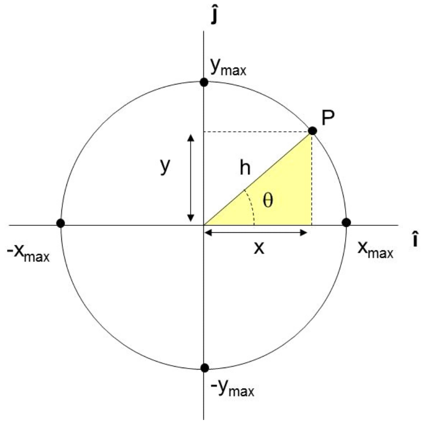
$$x\left(t\right) = x_{\max} \cos\left(\omega t\right)$$

have found SHM. Now we can see why we used "ω" for both angular speed and angular frequency. Really they are very related. Both tell us something about how fast a cyclic event happens. One is how fast the point P goes around the circle, and the other is how often the projection goes back and forth. It makes sense that these have to be the same rate.



The projection of circular motion onto the x-axis gives simple harmonic motion.

It is also true that we can project onto the y-axis.



and we would find that we can describe this projection as

$$y\left(t\right)=y_{\max}\sin\left(\omega t\right)$$

We will choose the cosine function, but from our trig experience it should be clear that these projections are equivalent, just $90\,^{\circ}$ out of phase

$$\cos\left(\theta\pm\frac{\pi}{2}\right)=\mp\sin\theta$$

So

$$x\left(t\right)=x_{\max}\cos\left(\omega t+\phi_{o}\right)$$

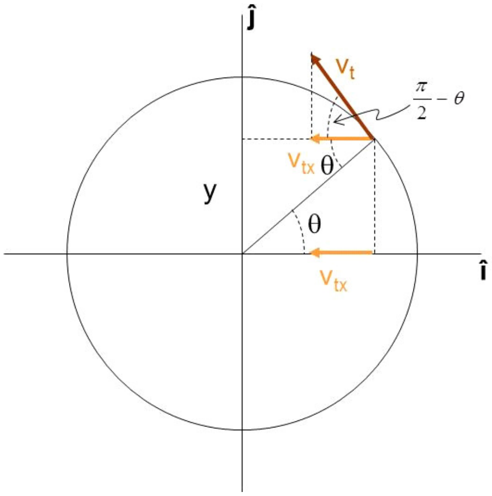
could be a sine function if $\phi_o = \pm\pi/2$. I this way we have incorporated both possibilities into one function.

Note that this is just the function we guessed from our observation!

Remark 1.2 *We see that uniform circular motion can be thought of as the combination of two SHOs, with a phase difference of* $\pi/2\,\mathrm{rad}$.

The angular velocity is given by

$$\omega=\frac{v}{r} \tag{1.12}$$



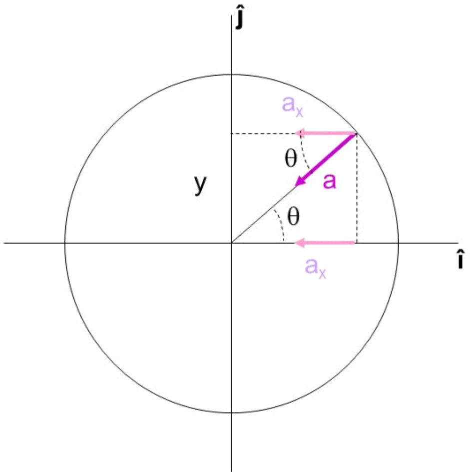
A particle traveling on the x-axis in SHM will travel from $x_{\max}$ to $-x_{\max}$ and from $-x_{\max}$ to $x_{\max}$ (one complete period, T) while the particle traveling with P makes one complete revolution. Thus, the angular frequency ω of the SHO and the angular velocity of the particle at P are the same. (Now we know why we used the same symbol). The magnitude of the tangential velocity is then

$$v_t = \omega r = \omega x_{\max} \tag{1.13}$$

and the projection of this velocity onto the x-axis is

$$v_{tx} = -\omega x_{\max} \sin\left(\omega t + \phi_o\right) \tag{1.14}$$

Which is just what we expected from our observation!



The centripetal acceleration of a particle at P is given by

$$a_c = \frac{v_t^2}{r} = \frac{v_t^2}{x_{\max}} = \frac{\omega^2 x_{\max}^2}{x_{\max}} = \omega^2 x_{\max} \tag{1.15}$$

The direction of the acceleration is inward toward the origin. Of course, we just want the x-component of this, so again we make a projection. If we project this onto the x-axis we have

$$a_x = -\omega^2 x_{\max} \cos(\omega t + \phi) \tag{1.16}$$

Again this is just want we expected from our observation.

So now we have shown that our set of equations

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t) \\ v(t) &= -\omega x_{\max} \sin(\omega t) \\ a(t) &= -\omega^2 x_{\max} \cos(\omega t) \end{aligned} \tag{1.17}$$

is correct for our harmonic oscillator.

Our first problem type: Simple Harmonic Motion

You have probably thought by now that we have a new problem type. We can call it the *simple harmonic motion* problem type or SHM. The equations we have so far for this problem type are

$$\begin{aligned} x(t) &= x_{\max} \cos(\omega t + \phi_o) \\ v(t) &= -\omega x_{\max} \sin(\omega t + \phi_o) \\ a(t) &= -\omega^2 x_{\max} \cos(\omega t + \phi_o) \end{aligned} \tag{1.18}$$

$$\omega = 2\pi f$$

$$v_m = \omega x_m$$

$$a_m = \omega^2 x_m$$

$$T = \frac{1}{f}$$

Next lecture, we will study the energy in a harmonic oscillator.

2 Energy and Dynamics of SHM

Back in PH121 we started with position, velocity, and acceleration to describe motion. We have done that again for a simple harmonic oscillator. In PH121, after basic motion, we found that we could use the idea of a force and Newton's second law to find the acceleration for our description of motion. Then we changed view points and used the idea of energy to find motion. The energy picture was easier in some ways. Let's try to develop and energy picture for simple harmonic oscillators. Of course the goal in physics is to describe the motion of the object, so we will say that we are looking for the *dynamics* of simple harmonic oscillators when we look for the position, velocity, and acceleration as a function of time.

Fundamental Concepts

- Energy and SHM
- Equation of motion
- Vertical oscillations

Oscillators and Energy

You might have noticed that we are calling mover objects that experience simple harmonic motion (SHM) by the name *simple harmonic oscillators (SHO).* Let's consider such a SHO. Because our SHO is moving, we know there must be energy associated with it. To understand the energy involved, let's start with kinetic energy. Recall from PH121 that

$$K = \frac{1}{2}mv^2 \tag{2.1}$$

and we recall that for a spring, we have the spring potential energy given by

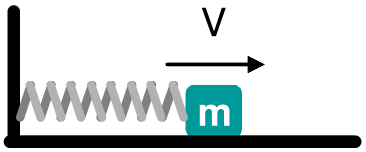
$$U_s = \frac{1}{2}kx^2 \tag{2.2}$$

Let's try a problem.

Find the maximum velocity of the mass in terms of the Amplitude and the angular frequency. We want to use our problem solving steps:

Type of problem: This is an energy and a SHO problem:

Drawing:



Variables:

E	energy
K	Kinetic energy
x	Position
v	velocity or speed
m	mass of the object
k	spring stiffness constant

Basic Equations:

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ U_s &= \frac{1}{2}kx^2 \end{aligned}$$

Symbolic Answer

You might say, this is an easy problem, we know from last time that

$$v_{\mathrm{max}} = x_{\mathrm{max}}\omega$$

but let's find this again using the ideas of energy. In PH121 we often found that using energy made problems easier, so this might be worth a little more work now. The total energy is

$$E = K + U_s + U_g$$

Let's say that our oscillator is moving horizontally, and that we define the center of mass of our object so that it's y-position is right at $y = 0$ so our object has zero gravitational potential energy.

$$E = K + U_s + 0$$

And let's say we have no friction, so that

$$\begin{aligned} E_i &= E_f \\ K_i + U_{si} &= K_f + U_{sf} \\ \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 \end{aligned}$$

and let's start with our initial condition being where the spring is stretched to $x_{\max}$. Then at that moment, $v_i = 0$. And let's take our final position when the mass is moving as fast as possible (because that is what we are looking for, $v_{\max}$). We know from our PH121 experience that this will be right when $U_{sf} = 0$ (so all the energy is kinetic). Then

$$0 + \frac{1}{2}kx_{\max}^2 = \frac{1}{2}mv_{\max}^2 + 0$$

or

$$\frac{1}{2}mv_{\max}^2 = \frac{1}{2}kx_{\max}^2$$

We can solve for $v_{\max}$

$$\begin{aligned} v_{\max}^2 &= \frac{kx_{\max}^2}{m} \\ v_{\max} &= x_{\max}\sqrt{\frac{k}{m}} \end{aligned}$$

But is this what we wanted? We expected that

$$v_{\max} = x_{\max}\omega$$

And this is what we got so long as

$$\omega = \sqrt{\frac{k}{m}}$$

And this is always true for a mass on a spring. We don't need a numeric answer, This is reasonable (just what we expect) and the units check.

Let's look at kinetic and potential energy as a function of time. For our Simple Harmonic Oscillator (SHO) we know the velocity as a function of time,

$$v\left(t\right) = -\omega x_{\max}\sin\left(\omega t + \phi\right)$$

so the kinetic energy as a function of time must be

$$\begin{aligned} K &= \frac{1}{2}m\left(-\omega x_{\max}\sin\left(\omega t + \phi\right)\right)^2 \\ &= \frac{1}{2}m\omega^2 x_{\max}^2\sin^2\left(\omega t + \phi\right) \end{aligned}$$

and now we know that $\omega = \sqrt{k/m}$, so we can write this as

$$\begin{aligned} K &= \\ &= \frac{1}{2}m\left(\sqrt{\frac{k}{m}}\right)^2 x_{\max}^2 \sin^2\left(\omega t + \phi\right) \\ &= \frac{1}{2}m\frac{k}{m}x_{\max}^2 \sin^2\left(\omega t + \phi\right) \\ &= \frac{1}{2}kx_{\max}^2 \sin^2\left(\omega t + \phi\right) \end{aligned}$$

As the spring is stretched or compressed we store energy as spring potential energy. The potential energy due to a spring acting on our SHO (mover mass) is given by (from your PH121 class)

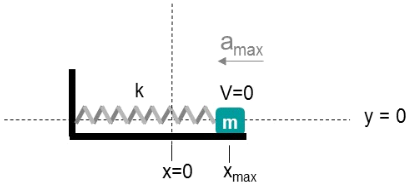
$$U_s = \frac{1}{2}kx^2 \tag{2.3}$$

For our SHO we also know the position as a function of time

$$x\left(t\right) = x_{\max}\cos\left(\omega t + \phi_o\right)$$

so the potential energy as a function of time must be

$$U_s = \frac{1}{2}kx_{\max}^2 \cos^2\left(\omega t + \phi\right)$$



Let’s say that our oscillator is moving horizontally, and that we define the center of mass of our object so that it’s y-position is right at $y = 0$ so our object has zero gravitational potential energy

$$U_g = mgy = 0$$

so the total mechanical energy is given by

$$E = K + U_s$$

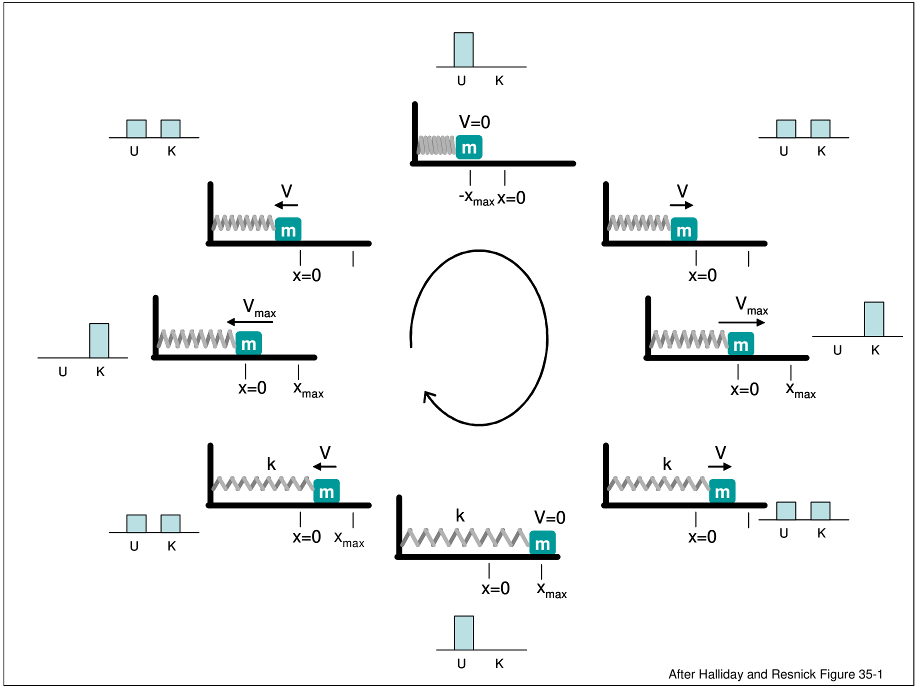
which we can write out as

$$\begin{aligned} E &= K + U_s \\ &= \frac{1}{2}kx_{\max}^2 \sin^2\left(\omega t + \phi\right) + \frac{1}{2}kx_{\max}^2 \cos^2\left(\omega t + \phi\right) \\ &= \frac{1}{2}kx_{\max}^2 \left(\sin^2\left(\omega t + \phi\right) + \cos^2\left(\omega t + \phi\right)\right) \\ &= \frac{1}{2}kx_{\max}^2 \end{aligned}$$

It tells us that if there are no loss mechanisms (e.g. no friction) then the energy in a harmonic oscillator never changes. And we remember that we would say that energy is conserved for such a situation, which is not surprising because we have already used conservation of energy for springs in PH121 and in the previous example. But let's give a new name to a quantity that does not change. Let's call it a *constant of motion*. So we can make a statement about the total energy for our SHO.

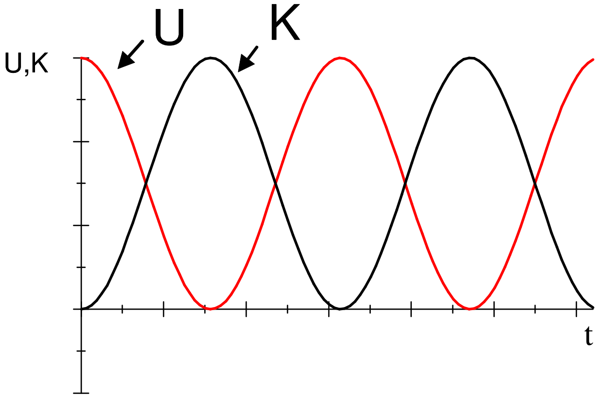
Remark 2.1 *The total mechanical energy of an ideal SHO is a constant of motion*

If we plot the amount of kinetic and potential energy for an oscillator we might find something like this:



After Halliday and Resnick Figure 35-1

Note that the kinetic and potential energy are out of phase with each other. If we plot them on the same scale (for the case $\phi = 0$) we have



Let’s try another problem using energy of a SHO.

Mathematical Representation of Simple Harmonic Motion

We have a mathematical representation of simple harmonic motion from looking at our graph of position vs. time. But as a problem, let’s use math and what we know from PH121 to show that our equation

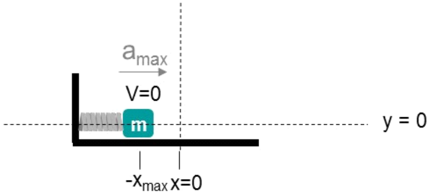
$$x\left(t\right)=x_{\max}\cos\left(\omega t\right)$$

must be right.

Recall from PH 121

$$a=\frac{dv}{dt}=\frac{d^{2}x}{dt^{2}} \tag{2.4}$$

and let’s assume we have SHO moving only in the x-direction.



Further assume the surface the mass rests on is frictionless. Also let’s say that the equilibrium position $x_{em}=0$. Then we can write Newton’s second law as

$$\begin{aligned} F_{net_{x}} &= ma_{x}=-kx \\ m\frac{d^{2}x}{dt^{2}} &= -kx \end{aligned} \tag{2.5}$$

We have a new kind of equation. If you are taking this freshman class as a... well...

freshman, you may not have seen this kind of equation before. It is called a differential equation. The solution of this equation is a function or functions that will describe the motion of our mass-spring system as a function of time. It says that the way the object moves is an equation where the second derivative is almost the same as the original function. The only difference is some constants that are multiplied.

It is this function that we want, so let's see how we can find it.

Start by getting all the constants on the same side of the equation.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

It would be tidier if we defined a quantity ω as

$$\omega^2 = \frac{k}{m} \tag{2.6}$$

why define ω^2? Think of our previous example. We found that $\omega = \sqrt{k/m}$. This is just the square of what we found in our example. Then we can write our differential equation as

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{2.7}$$

We need a function who's second derivative is the negative of itself with just a constant out front. From Math 112 we know a few of these

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \\ x(t) &= A\sin(\omega t + \phi_o) \end{aligned}$$

where A, ω, and ϕ_o are constants that we must find. Let's choose the cosine function and explicitly take it's derivatives to see if this function does solve our differential equation

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_o) \qquad (2.8) \\ \frac{dx(t)}{dt} &= -\omega A\sin(\omega t + \phi_o) \\ \frac{d^2x(t)}{dt^2} &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

Let's substitute these expressions into our differential equation for the motion

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\omega^2 x \\ -\omega^2 A\cos(\omega t + \phi_o) &= -\omega^2 A\cos(\omega t + \phi_o) \end{aligned}$$

As long as the constant ω^2 is our $\omega^2 = k/m$ we have a solution. We could have found that

$$\omega^2 = \frac{k}{m}$$

by solving this differential equation, but it might not have been as meaningful that way.

We can identify ω as the angular frequency.

$$\omega = 2\pi f$$

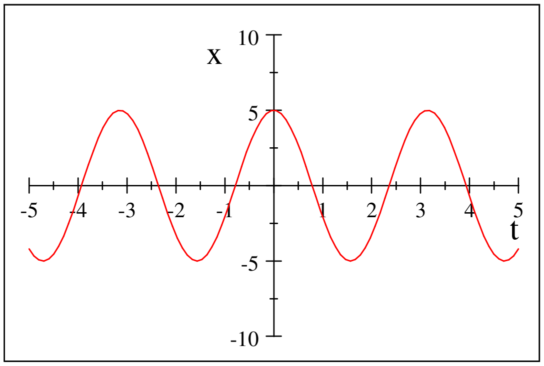
Thus

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f \tag{2.9}$$

This says that if the spring is stiffer, we get a higher frequency, or if the mass is larger, we get a lower frequency.

We still don't have a complete solution, because we don't know A and ϕ_o. We recognize ϕ_o as the initial phase angle. We will have to find this by knowing the initial conditions of the motion. A is the amplitude. That must be the maximum displacement $x_{\max}$. Let's look at a specific case

$$\boxed{\begin{array}{|l|}\hline x_{\max} = 5 \\ \hline \phi_o = 0 \\ \hline \omega = 2 \\ \hline \end{array}} \tag{2.10}$$



We can easily see that the amplitude A corresponds to the maximum displacement $x_{\max}$. (how would you prove this?). We know from trigonometry that a cosine function has a period T.

The period is related to the frequency

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{2.11}$$

We can write the period and frequency in terms of our mass and spring constant

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{2.12}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.13}$$

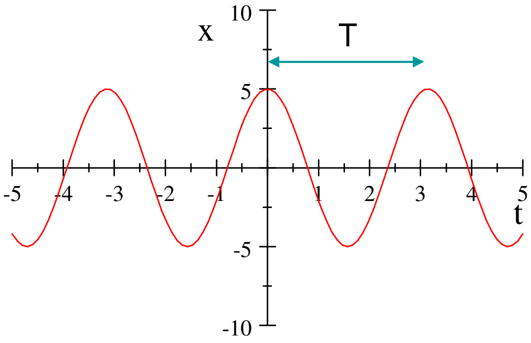
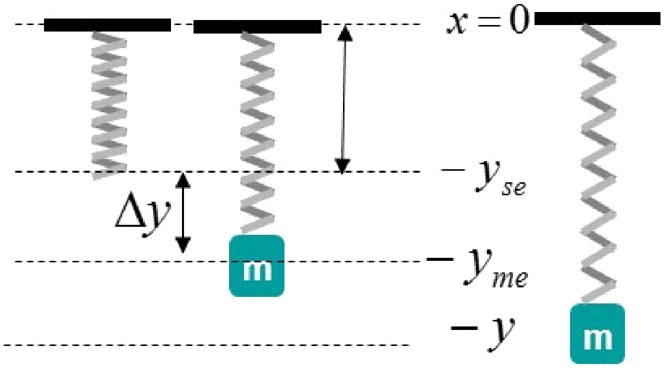


Figure 2.1.

Hanging Springs

Let's do a problem. We now know we have forces involved with our SHOs. So let's do a force problem. In class so far, I have always hung our masses and springs, but we have used horizontal systems for calculations. Let's find the equation of motion for a mass hanging from a spring.



From observation, we would guess that we can just choose the new equilibrium point to be $y = 0$ and use the same equation

$$y(t) = y_{\max}\cos(\omega t - \phi_o)$$

let's see if that is true. Start with a free body diagram for the mass (the hanging mass is our mover, the spring is part of the environment). There are two forces acting on the mass. The force due to gravity, and the force due to the spring.

$$\Sigma F_y = ma_y = S_{ms} - W_{mE}$$

We know the form of these forces

$$\begin{aligned} S_{ms} &= -k\Delta y \\ W_{mE} &= -mg \end{aligned}$$

but we need to carefully choose our origin. Let's try the top of the spring where it attaches to the stand. If the mass just hangs there we would expect the spring to stretch to an equilibrium length y_{eq} and any other motion would either shorten or lengthen the spring. We can write our force equation as

$$\begin{aligned} ma_y &= S_{ms} - W_{mE} \\ &= k\left(y_{em} - y\right) - mg \end{aligned}$$

where y can be positive or negative. If $y = 0$ and the mass is just sitting there, not oscillating, there is no acceleration. Then the stretched length is just y_{em}. In this case

$$\begin{aligned} m\left(0\right) &= k\left(y_{em} - 0\right) - mg \\ ky_{em} &= mg \end{aligned}$$

But now let's let our mass move again. We can substitute our stationary mass answer this into the previous equation for a moving mass

$$\begin{aligned} ma &= k\left(y_{em} - y\right) - mg \\ &= ky_{em} - ky - mg \\ &= mg - ky - mg \\ &= -ky \end{aligned}$$

This gives a net force for the system of

$$F_{net} = -ky$$

It is as though the system were horizontal with no gravitational force and only a spring force. We can see that we are justified in claiming that we could simply choose the origin at the distance y_{em} from the top of the spring, and we can use the equation

$$y\left(t\right) = y_{\max}\cos(\omega t - \phi_o)$$

as our equation of motion.

We have used energy to describe simple harmonic motion, and we have found our equation of motion using a differential equation for mass-spring systems. And we have simple harmonic motion in the y-direction including the weight force due to gravity for mass-spring systems. In our next lecture, we will study simple harmonic motion for different systems, pendula, and other things. It turns out that SHM is a good model for many different systems.

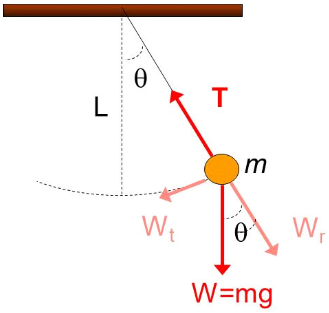
3 More Oscillators, Forces and Friction

You have probably wondered if anyone actually uses mass-spring systems. And we do. They were more common in the past where springs were used to store energy (U_s) to run clocks and toys and machines. But simple harmonic motion can describe other systems as well. You have probably seen an old fashioned clock with a pendulum. A pendulum almost experiences simple harmonic motion. Let's see how this works.

Fundamental Concepts

- Pendula
- Damping
- Driving
- Resonance

The Simple Pendulum



A simple pendulum is a mass on a string. The mass is called a "bob." Usually we study the motion of the pendulum bob, so let's consider the pendulum the mover object. A simple pendulum bob exhibits periodic motion, but not exactly simple harmonic motion. The forces on the bob are $\overrightarrow{\mathbf{W}}$, and $\overrightarrow{\mathbf{T}}$ the tension on the string. The tangential component of W is always directed toward $\theta = 0$. This is a restoring force!

Let's call the path the bob takes "s." Then from Jr. High geometry we recall[2]

$$s = L\theta \tag{3.1}$$

We will use a the cylindrical or rtz coordinate system. The radial axis is directed along the string. The tangential direction is along the circular path the bob takes and is always tangent to the path. In this coordinate system, we can solve for the part of the force directed along the path. This is the restoring part of the net force. Remember from Newton's second law

$$\begin{aligned} F_t &= ma_t \\ F_r &= ma_r \end{aligned}$$

and

$$a_t = \frac{d^2s}{dt^2}$$

Let's write out Newton's second with the sum of the forces part.

$$\begin{aligned} ma_t &= -W\cos\left(90-\theta\right) \\ -ma_r &= -T + W\sin\left(90-\theta\right) \end{aligned}$$

then, using a trig identity (but only a small one)

$$\begin{aligned} a_t &= -\frac{W}{m}\sin\left(\theta\right) \\ &= -g\sin\theta \end{aligned}$$

We have two expressions for $a_{t.}$ We can set them equal

$$\frac{d^2s}{dt^2} = -g\sin\theta \tag{3.2}$$

Remember that $s = L\theta$ so we could write the left hand side as

$$\frac{d^2s}{dt^2} = \frac{d^2}{dt^2}\left(L\theta\right) = L\frac{d^2}{dt^2}\left(\theta\right)$$

then equation (3.2) becomes

$$L\frac{d^2}{dt^2}\left(\theta\right) = -g\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

This is a differential equation much like our differential equation for a harmonic oscillator,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

---

[2] Really I did not recall this, I have to look it up every time, but s is called the *arclength*.

except it has a sine function in it. But, if we take θ as a very small angle, then

$$\sin(\theta) \approx \theta \tag{3.3}$$

This approximation has a name, it is called the "small angle approximation."

In this approximation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

and we have a differential equation we recognize! If we compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

we see that it is a match if

$$\omega^2 = \frac{g}{L} \tag{3.4}$$

we have all the same solutions for θ that we found last time for x. Since ω changed, the frequency and period will now be in terms of g and L.

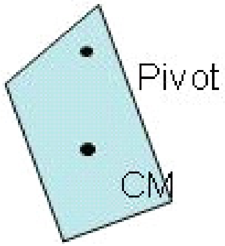
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{3.5}$$

Remark 3.1 ***For a pendulum that oscillates only over small angles, the period and frequency depend only on*** L ***and*** g***!***

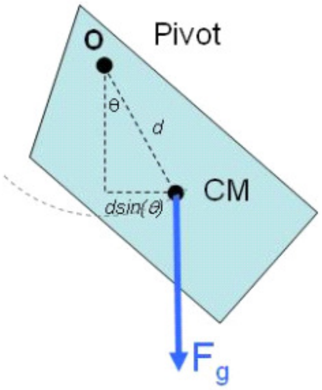
The analysis we just did for the pendulum we can do for other simple harmonic oscillators (or near simple harmonic oscillators). Let's try a few.

Physical Pendulum

Usually when we build a pendulum, we assume the string is so small compared to the bob, that we can ignore it's mass. What if this is not true? Suppose we build a pendulum by making a large solid object swing from one point. Can we describe it's motion?



Let's pull it to the right



then we consider that because of F_g we will have a torque about an axis through O.

$$\begin{aligned}\tau &= \mathbf{r}\times\mathbf{F}\\ &= \mathbf{d}\times\mathbf{F}_g\\ &= -F_g d\sin\theta\\ &= -mgd\sin\theta\end{aligned}$$

Remember that an extended body has a moment of inertia, $\mathbb{I}$. Remember also that angular acceleration is given by

$$\alpha = \frac{dw}{dt} = \frac{d^2\theta}{dt^2}$$

From the rotational form of Newton's second law.

$$\Sigma\tau = \mathbb{I}\alpha$$

we can write

$$-mgd\sin\theta = \mathbb{I}\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{\mathbb{I}}\sin(\theta)$$

and again if we let θ be small so that $\sin(\theta) \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{\mathbb{I}}\theta$$

which we can compare to

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

and we can see that we have the same differential equation if

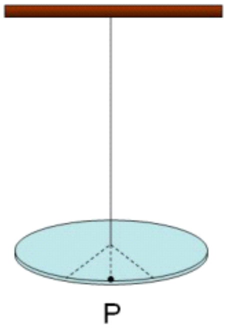
$$\omega^2 = \frac{mgd}{\mathbb{I}} \tag{3.6}$$

In this case

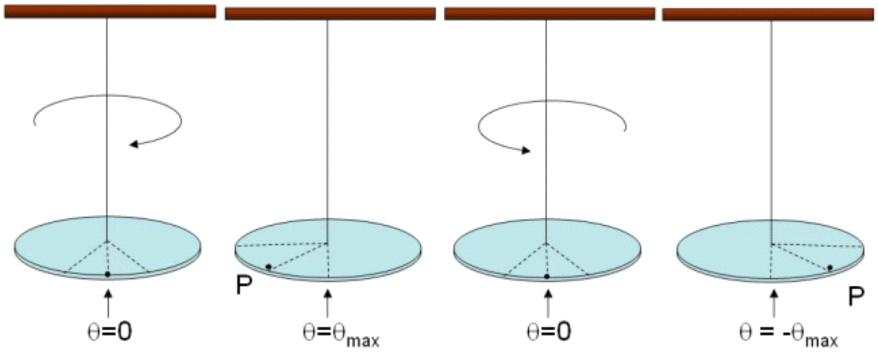
$$T = 2\pi\sqrt{\frac{\mathbb{I}}{mgd}} \tag{3.7}$$

Torsional Pendulum

Physics majors and those taking PH220 in the future, you will see a torsional pendulum.



A torsional pendulum is made by suspending a rigid body from a wire. The body rotates.



The twisted wire exerts a restoring torque on the body that is proportional to the angular position (sound familiar)

$$\boldsymbol{\tau} = -\kappa\theta$$

This looks like a greek version of $F = -kx$! Again

$$\begin{aligned} \Sigma\tau &= \mathbb{I}\alpha \\ &= \mathbb{I}\frac{d^2\theta}{dt^2} \end{aligned}$$

so

$$\begin{aligned}\mathbb{I}\frac{d^2\theta}{dt^2} &= -\kappa\theta \\ \frac{d^2\theta}{dt^2} &= -\frac{\kappa}{\mathbb{I}}\theta\end{aligned}$$

Once again we have our favorite differential equation so long as

$$\omega^2 = \frac{\kappa}{\mathbb{I}} \tag{3.8}$$

which makes the period of the oscillation

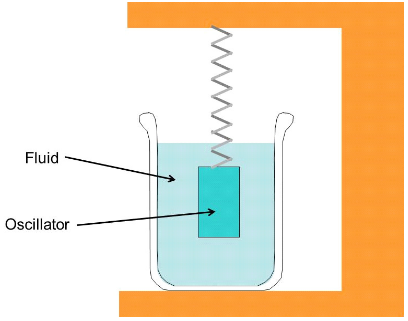
$$T = 2\pi\sqrt{\frac{\mathbb{I}}{\kappa}} \tag{3.9}$$

Damped Oscillations

You remember friction from PH121. So far we have only allowed frictionless oscillators to make the math easy. But what if we do have friction? To investigate this,suppose we add in another force

$$\mathbf{F}_d = -b\mathbf{v} \tag{3.10}$$

This force is proportional to the velocity. This a dissipative (friction-like) force typical of what we find when we moves objects through viscus fluids.



We call b the damping coefficient. Now, from Newton's second law,

$$\Sigma F = -kx - bv_x = ma$$

We can write the acceleration and velocity as derivatives of the position just like we have done before

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

This is another differential equation! But it is harder to guess its solution, and finding that solution is a subject for a differential equations class like M316 or PH332, so we won't learn how to find the solution here, but we can use the results from our trusted colleagues in the math department[3].

$$x(t) = x_{\max} e^{-\frac{b}{2m}t} \cos(\omega t + \phi_o) \tag{3.11}$$

which looks simple enough, but now we have the added complication that ω is more complex

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \tag{3.12}$$

so we get quite a mess if we write equation (3.11) with this new ω.

$$x(t) = x_{\max} e^{-\frac{b}{2m}t} \cos\left(\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right) t + \phi_o\right) \tag{3.13}$$

To see what this solution means, we should study three cases:

1. the damping force is small: $(bv_{\max} < kx_{\max})$ The system oscillates, but the amplitude is smaller as as time goes on. We call this "underdamped."

2. the damping force is large: $(bv_{\max} > kx_{\max})$The system does not oscillate. we call this "overdamped." We can also say that $\frac{b}{2m} > \omega_o$ (after we define ω_o below)

3. The system is "critically damped" (see below).

Let's look at an example. Suppose we have an oscillator with the following characteristics:

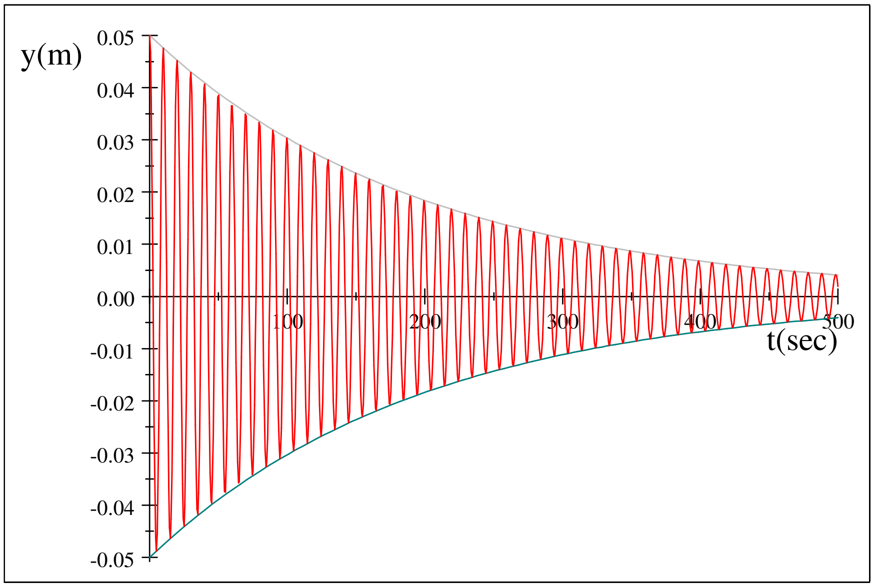
1.

$x_{\max} = 5\,\mathrm{cm}$
$b = 0.005\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.2\frac{\mathrm{N}}{\mathrm{m}}$
$m = .5\,\mathrm{kg}$
$\phi_o = 0$

Graphing the equation of motion $x(t)$, we we get a graph that looks like this

---

[3] That is, until you finish M316, then you will know how to do this kind of problem yourself!



The gray lines are given by

$$\pm x_{\max} e^{-\frac{b}{2m}t}$$

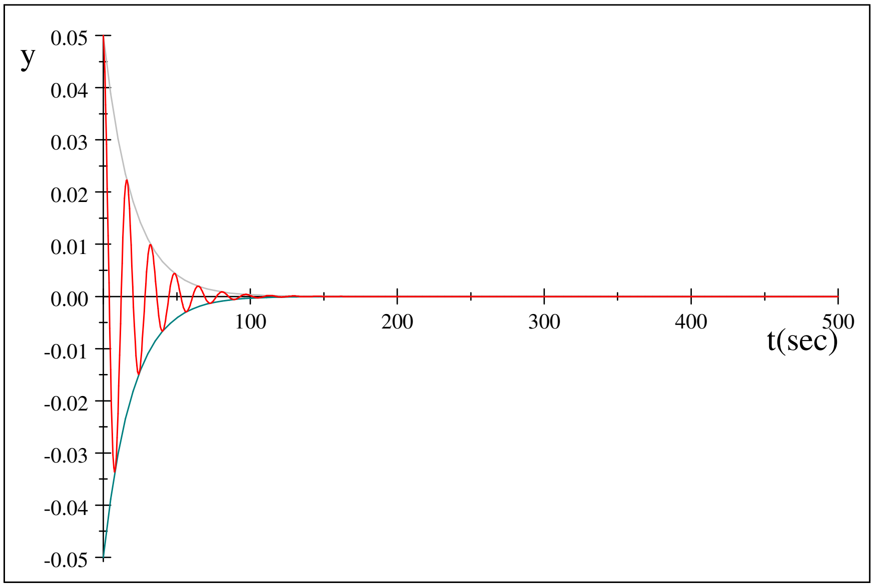
Notice that this quantity is as a collection of terms multiplies the cosine part. It is marked in curly braces blow

$$x(t) = \left\{ x_{\max} e^{-\frac{b}{2m}t} \right\} \cos\left(\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \right) t + \phi_o \right)$$

We know that the part of the equation that multiplies the cosine part is the amplitude. But now the amplitude is more than just $x_{\max}$. And notice that the amplitude $\left\{ x_{\max} e^{-\frac{b}{2m}t} \right\}$ changes with time. The gray lines in the figure show how the amplitude changes. We call this the *envelope* of the curve. The oscillation fits within the gray lines (like an old fashioned letter fits inside a paper envelope).

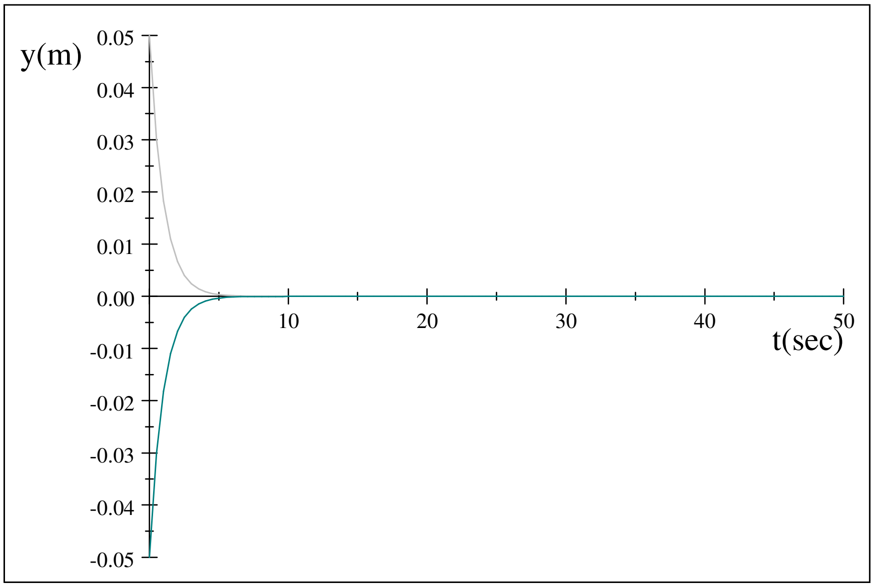
Now let's change b to a larger value

$x_{\max} = 5\,\mathrm{cm}$
$b = 0.05\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.2\frac{\mathrm{N}}{\mathrm{m}}$
$m = .5\,\mathrm{kg}$
$\phi_o = 0$



we see we have less oscillation. The envelope has become more restrictive, making the oscillation die out more quickly. This is a little bit like going over a bump in your car. The car may go up and down a few times, but not many. Now let's increase b even more.

$$\begin{array}{|l|}\hline x_{\max}=5\,\mathrm{cm}\\ \hline b=0.5\frac{\mathrm{kg}}{\mathrm{s}}\\ \hline k=0.2\frac{\mathrm{N}}{\mathrm{m}}\\ \hline m=.5\,\mathrm{kg}\\ \hline \phi_o=0\\ \hline\end{array} \tag{3.14}$$



What happened?

When the damping force gets bigger, the oscillation eventually stops. Only the exponential decay is observed. This happens when

$$\frac{b}{2m} = \sqrt{\frac{k}{m}}$$

When that is true,

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = 0$$

We call this situation *critically damped*. We are just on the edge of oscillation. We define

$$\omega_o = \sqrt{\frac{k}{m}}$$

as the *natural frequency* of the system. Then the value of b that gives us critically damped behavior is

$$b_c = 2m\omega_o$$

When $\frac{b}{2\pi} \geq \omega_o$ the solution in equation (3.11) is not valid! You will find out more about this situation in your advanced mechanics classes.

Forced Oscillations

We found in the last section that if we added a force like

$$\mathbf{F}_d = -b\mathbf{v}$$

our oscillation died out. An example would be a small child on a swing. You give them a push, but eventually they stop swinging.

1. Suppose we want to keep the child going? Let's apply a periodic force like

   $$F(t) = F_o \sin(\omega_f t)$$

   where ω_f is the angular frequency of this new driving force and where F_o is a constant.

   $$\Sigma F = F_o \sin(\omega_f t) - kx - bv_x = ma$$

   When this system starts out, the solution is very messy. It is so messy that we will not give it in this class! But after a while, a steady-state is reached. In this state, the energy added by our driving force $F_o \sin(\omega_f t)$ is equal to the energy lost by the drag force, and we have

   $$x(t) = A\cos(\omega_f t + \phi)$$

   our old friend! BUT NOW the amplitude is given by

   $$A = \frac{\frac{F_o}{m}}{\sqrt{\left(\omega_f^2 - \omega_o^2\right)^2 + \left(\frac{b\omega_f}{m}\right)^2}}$$

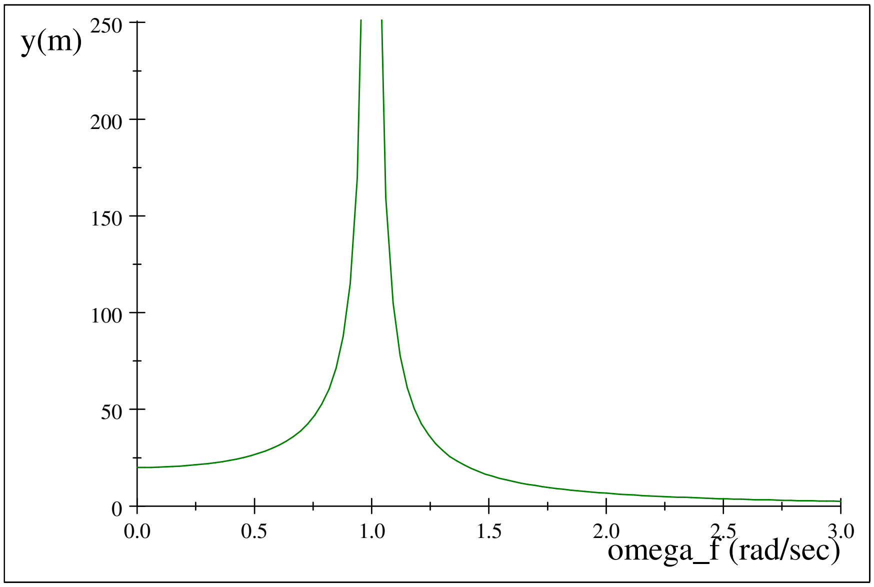
   where

   $$\omega_o = \sqrt{\frac{k}{m}} \tag{3.15}$$

   is the natural frequency as before. So now our solution looks more like our original SHM solution (except for the wild formula for A). In fact, it operates very like our SHM solution. But what does the new version of A mean?

   Lets look at A for some values of ω_f. I will pick some nice numbers for the other values.

$F_o = 2\,\mathrm{N}$
$b = 0.5\frac{\mathrm{kg}}{\mathrm{s}}$
$k = 0.5\frac{\mathrm{N}}{\mathrm{m}}$
$m = 0.5\,\mathrm{kg}$
$\phi_o = 0$



now let's calculate ω_o

$$\begin{aligned}\omega_o &= \sqrt{\frac{0.5\,\frac{\mathrm{N}}{\mathrm{m}}}{0.5\,\mathrm{kg}}} \\ &= \frac{1.0}{\mathrm{s}}\end{aligned}$$

Notice that right at $\omega_f = \omega_o$ our solution gets very big. This is called *resonance*. To see why this happens, think of the velocity for a simple harmonic oscillator

$$\frac{dx\,(t)}{dt} = -\omega A \sin(\omega t + \phi_o)$$

note that our driving force is

$$F\,(t) = F_o \sin(\omega t)$$

Remember that work is given by

$$w = \int \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{x}}$$

or just

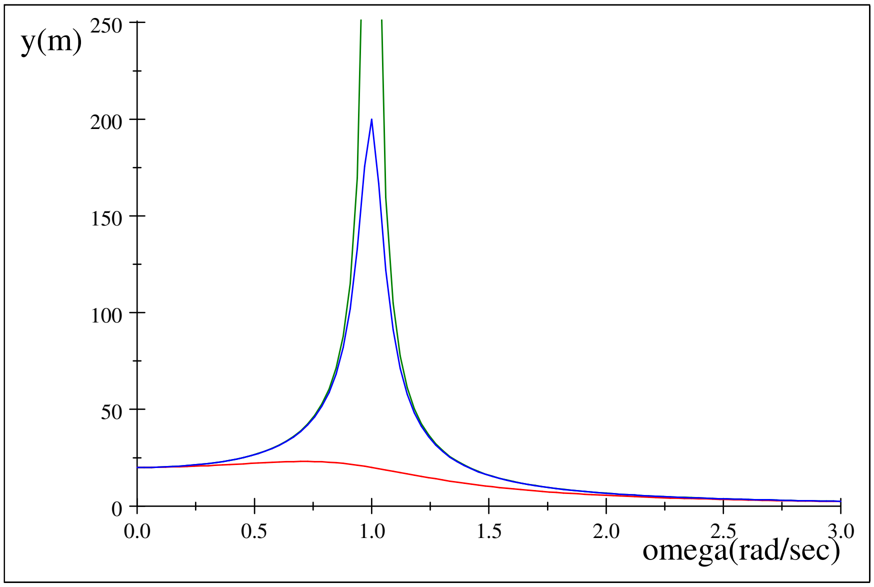
$$w = \overrightarrow{\mathbf{F}} \cdot \Delta\overrightarrow{\mathbf{x}}$$

if our force is constant. The rate at which work is done (power) is

$$\begin{aligned}\mathcal{P} &= \frac{\overrightarrow{\mathbf{F}} \cdot \Delta\overrightarrow{\mathbf{x}}}{\Delta t} = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{v}} && (3.16) \\ &= -F_o \omega A \sin(\omega t) \sin(\omega t + \phi_o) && (3.17)\end{aligned}$$

if F and v are in phase ($\phi_o = 0$), the power will be at a maximum! Think of pushing the child on the swing. You push in phase with the oscillation of the child. When you do this the amplitude of the swing get's bigger.

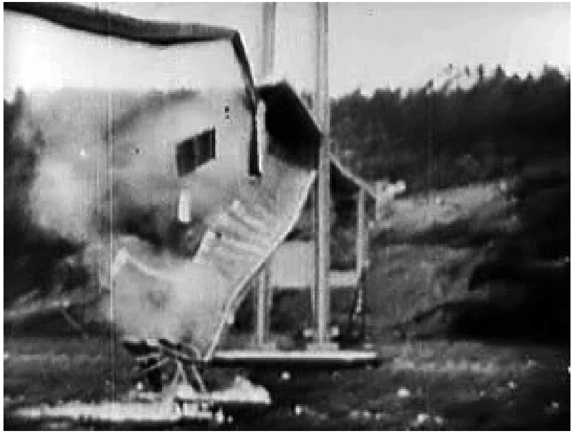
We can plot A for several values of b



Green: b=0.005kg/s; Blue: b=0.05kg/s; Red b=0.01 kg/s

As $b \rightarrow 0$ we see that our resonance peak gets larger. In real systems b can never be zero, but sometimes it can get small. As b →large, the resonance dies down and our A gets small.

Resonance can be great, it can make a musical instrument sound louder (more on this later). But it can also be bad. Here is a picture of the Tacoma Narrows Bridge.



Fall of the Tacoma Narrows Bridge (Image in the Public Domain)

The wind gusts formed a periodic driving force that allowed a driving harmonic oscillation to form. Since the bridge was resonant with the gust frequency, the amplitude grew until the bridge materials broke. Resonance can be a bad thing for structures.

4 What is a Wave?

We studied oscillation with mass-spring systems and pendula. Oscillation is a motion of a mass, but with another object (a spring or string) participating in the motion. The air in our classroom is an example of this type of motion in a way, but instead of one object on a spring, there are millions of objects, the air molecules. The molecules move randomly, but with a specific distribution of speeds. But what would happen if all the molecules moved together in the same direction (more or less) and at the same speed (more or less)? This is what we call wind! Of course we can have bulk motion of millions of objects, like wind. We are going to study an even more specific motion of millions of object that is not random like thermal motion. This specific motion of the objects we will call a *wave.*

Fundamental Concepts

1. A wave requires a disturbance, and a medium that can transfer energy
2. Waves are categorized as longitudinal or transverse (or a combination of the two).

What is a Wave?

Waves are organized motions in a group of objects. We will give a name to the group of objects, we will call a *medium*.

Spring Demo

Criteria for being a wave

Waves involve energy transfer, but in the case of waves the energy is transferred through space without transfer of matter. Winds transfers both energy and matter. So waves are different than wind. In a wave the objects don't move far from where they start. Think of an oscillation. The mass does move, but never gets too far away from its equilibrium position. Waves are very like this. This is a very specific kind of motion. To be a wave, the motion must have the following characteristics"

Spring Demo-marked part

1. some source of disturbance
2. a medium (group of objects) that can be disturbed
3. some physical mechanism by which the objects of the medium can influence each other

A wave made by a rock thrown into a pond will go out in all directions away from the place where the rock started the wave. This is a normal way that waves are formed. A disturbance starts the wave (the rock disturbs the water) and the energy from the disturbance moves away from the disturbance as a wave. But if we have a wave in a rope or string, the wave can't go in all directions because the string does not go in all directions.

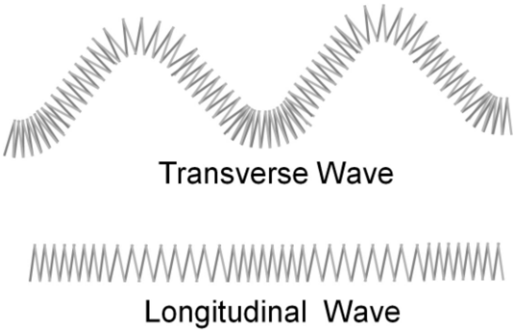
Let's take on this one-dimensional case of a wave on a rope or string first. In the limit that the string mass is negligible we represent a one-dimensional wave mathematically as a function of two variables, position and time, $y(x,t)$.

Longitudinal vs. transverse

We divide the various kinds of waves that occur into two basic types:

Definition 4.1 *transverse wave: a traveling wave or pulse that causes the elements of the disturbed medium to move perpendicular to the direction of propagation*

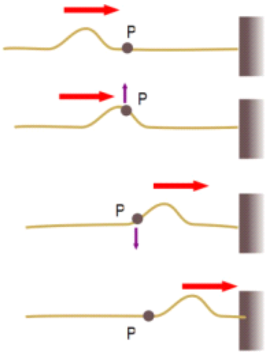
Definition 4.2 *Longitudinal wave: a traveling wave or pulse that causes the elements of the medium to move parallel to the direction of propagation*



Examples of waves:

Let's look at some specific cases of wave motion.

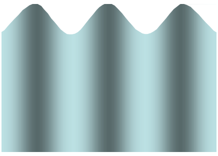
A pulse on a rope:



In the picture above, you see wave that has just one peak traveling to the right. We call such a wave a *pulse*. Notice how the piece of the rope marked P moves up and down, but the wave is moving to the right. This pulse is a transverse wave because the parts of the medium (observe point P) move perpendicular to the direction the wave is moving.

An ocean wave:

Of course, some waves are a combination of these two basic types. You may have noticed that in Physics we tend to define basic types of things, and then use these basic types to define more complex objects. Water waves, for example, are transverse at the surface of the water, but are longitudinal throughout the water.



Earthquake waves:

Earthquakes produce both transverse and longitudinal waves. The two types of waves even travel at different speeds! P waves are longitudinal and travel faster, S waves are transverse and slower.

Wave speed

We can perform an experiment with a rope or a long spring. Make a wave on the rope or spring. Then pull the rope or spring tighter and make another wave. We see that the wave on the tighter spring travels faster.

It is harder to do, but we can also experiment with two different ropes, one light and one heavy. We would find that the heaver the rope, the slower the wave. We can express this as

$$v = \sqrt{\frac{T_s}{\mu}}$$

where T_s is the tension in the rope, and μ is the linear mass density

$$\mu = \frac{m}{L}$$

where m is the mass of the rope, and L is the length.

The term μ might need an analogy to make it seem helpful. So suppose I have an iron bar that has a mass of $200\,\mathrm{kg}$ and is $2\,\mathrm{m}$ long. Further suppose I want to know how much mass there would be in a $20\,\mathrm{cm}$ section cut of the end of the rod. How would I find out?

This is not very hard, We could say that there are $200\,\mathrm{kg}$ spread out over $2\,\mathrm{m}$, so each meter of rod has $100\,\mathrm{kg}$ of mass, that is, there is $100\,\mathrm{kg}/\,\mathrm{m}$ of mass per unit length. Then to find how much mass there is in a $0.20\,\mathrm{m}$ section of the rod I take

$$m = 100\frac{\mathrm{kg}}{\mathrm{m}} \times 0.20\,\mathrm{m} = 20.0\,\mathrm{kg}$$

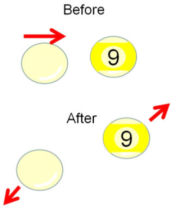
The $100\,\mathrm{kg}/\,\mathrm{m}$ is μ. It is how much mass there is in a unit length segment of something In this example, it is a unit length of iron bar, but for waves on string, we want the mass per unit length of string.

If you are buying stock steel bar, you might be able to buy it with a mass per unit length. If the mass per unit length is higher then the bar is more massive. The same is true with string. The larger μ, the more massive equal string segments will be.

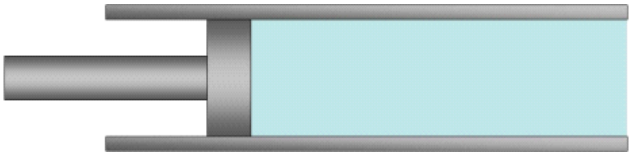
We should note that in forming this relationship, we have used our standard introductory physics assumption that the mass of the rope can be neglected. Let's consider what would happen if this were not true. Say we make a wave in a heavy cable that is suspended. The mass at the lower end of the cable pulls down on the upper part of the cable. The tension will actually change along the length of the cable, and so will the wave speed. Such a situation can't be represented by a single wave speed. But for our class, we will assume that any such changes are small enough to be ignored.

Example: Sound waves

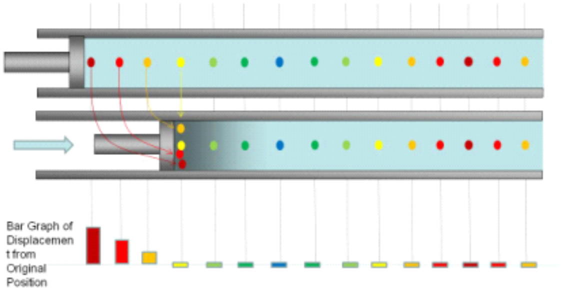
Sound is a wave. The medium is air particles. The transfer of energy is done by collision.



The wave will be a longitudinal wave. Let's see how it forms. We can take a tube with a piston in it.

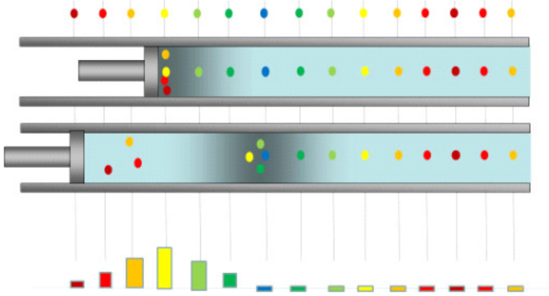


As we exert a force on the piston, the air molecules are compressed into a group. In the next figure, each dot represents a group of air molecules. In the top picture, the air molecules are not displaced. But when the piston moves, the air molecules receive energy by collision. They bunch up. We see this in the second picture.

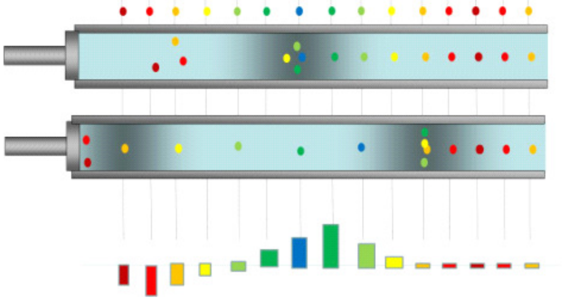


The graph below the two pictures shows how much displacement each molecule group experiences.

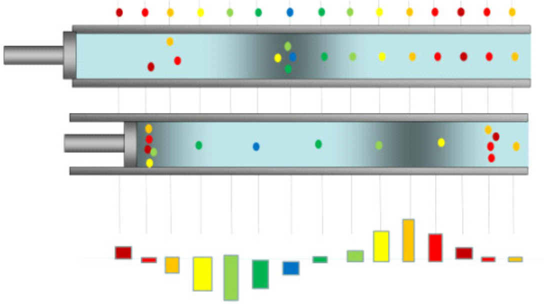
Suppose we now pull the piston back. This would allow the molecules to bounce back to the left, but the molecules that they have collided with will receive some energy and go to the right. This is shown in the next figure. Color coded dots are displayed above the before and after picture so you can see where the molecule groups started.



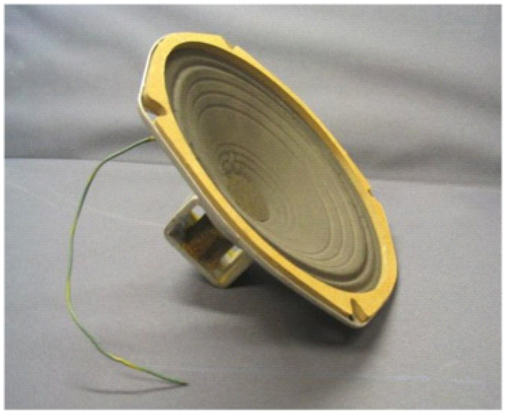
If we pull the piston back further, the molecules can pass their original positions.



Then we can push inward again and compress the gas.



This may seem like a senseless thing to do, but it is really what a speaker does to produce sound. In particular, a speaker is a harmonic oscillator. The simple harmonic motion of the speaker is the disturbance that makes the sound wave.



One dimensional waves

To mathematically describe a wave we will define a function of both time and position.

$$y\left(x,t\right) \tag{4.1}$$

Note, that this is new in our physics experience. Before we usually were concerned about only one variable at a time. For oscillation we had just $y\left(t\right)$ for example. But now we will be concerned about two variables, x, and t.

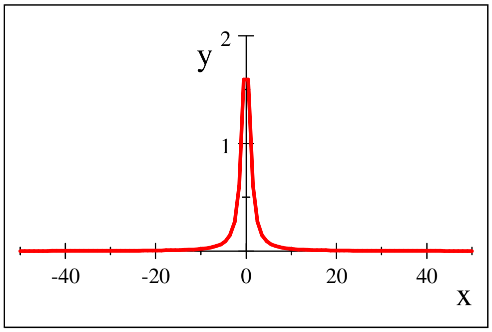
let's take a specific example[4]

$$y=\frac{2}{\left(x-3.0t\right)^{2}+1} \tag{4.2}$$

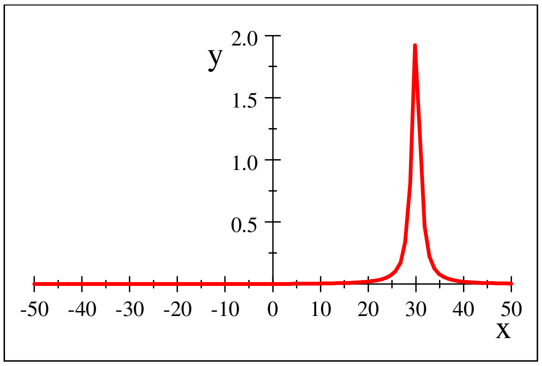
Let's plot this for $t=0$

---

[4] This is not an important wave function, just one I picked because it makes a nice graphic example.



what will this look like for $t = 10$?



The pulse travels along the x-axis as a function of time. Note that there is a value for y for every x position and that these y values change for different times. That is what we meen by saying we have a function of both x and t.

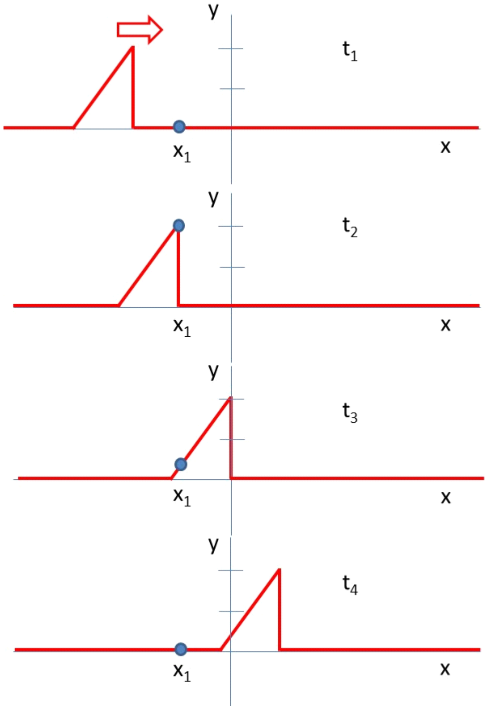
We denote the speed of the pulse as v, then we can define a function

$$y(x,t) = y(x - vt, 0) \tag{4.3}$$

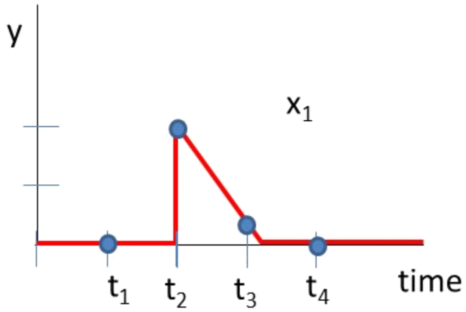
that describes a pulse as it travels. An element of the medium (rope, string, etc.) at position x at some time t, will have the displacement that an element had earlier at $x - vt$ when $t = 0$.

We will give $y(x - vt, 0)$ a special name, the *wave function*. It represents the y position, the transverse position in our example, of any element located at a position x at any time t.

Notice that wave functions depend on two variables, x, and t. It is hard to draw a wave so that this dual dependence is clear. Often we draw two different graphs of the same wave so we can see independently the position and time dependence. So far we have used one of these graphs. A graph of our wave at a specific time, t_o. This gives $y(x, t_o)$. This representation of a wave is very like a photograph of the wave taken with a digital camera. It gives a picture of the entire wave, but only for one time, the time at which the photograph was taken. Of course we could take a series of photographs, but still each would be a picture of the wave at just one time. Here is such a series of graphs at times t_1 through t_4.



The second representation is to observe the wave at just one point in the medium, but for many times. This is very like taking a video camera and using it to record the displacement of just one part of the medium for many times. You could envision marking just one part of a rope, and then using the video recorder to make a movie of the motion of that single part of the rope. We could then go frame by frame through the video, and plot the displacement of our marked part of the rope as a function of time. Such a graph is sometimes called a history graph of the wave. Here is such a graph for the position x_1.



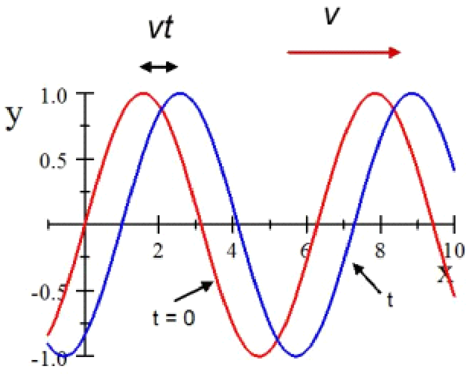
To go from the time graphs to the history graph you observe what happens at the location x_1 for each of the times. Then plot those y positions on the y vs. t graph. Then you must connect the points. This takes some thinking to make sure you connect them right (or a whole lot of points). But with practice, this is not hard and both view points are valuable ways to look at a wave.

5 Sinusoidal Waves in one and Two Dimensions

Fundamental Concepts

- It takes two variables to describe a wave, position and time
- Wave graphs have many named parts, amplitude, period, wavelength crest, trough, etc.
- There is a spatial analog to temporal frequency called spacial frequency and it is represented by the factor k in our equations
- We can also express sound waves in terms of pressure changes
- We don't hear all frequencies equally well
- Waves from point sources are spherical
- Light waves are waves in the electromagnetic field

Sinusoidal Waves



A sinusoidal graph should be familiar from our study of oscillation. Simple harmonic oscillators are described by the function

$$y\left(t\right)=y_{\max}\cos\left(\omega t+\phi\right)\qquad\text{SHM}\tag{5.1}$$

but this only gave us a vertical displacement. Now our sinusoidal function must also be a function of position along the wave.

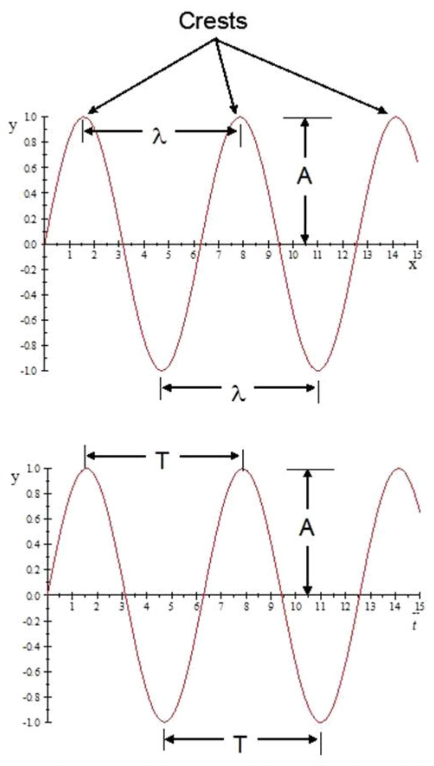
$$y\left(t\right)=y_{\max}\cos\left(ax-\omega t+\phi\right)\qquad\text{waves}\tag{5.2}$$

but before we study the nature of this function, lets see what we can learn from the graph of a sinusoidal wave. We will need both our two views, the camera snapshot and the video of a point. Look at figure **??**. This is two camera snap shots superimposed. The red curve shows the wave (y position for each value of x) at $t=0$. At some some later time t, the wave pattern has moved to the right as shown by the blue curve. The shift is by an amount $\Delta x=v\Delta t$.

Parts of a wave

The peak of a wave is called the crest. For a sine wave we have a series of crests. We define the wavelength as the distance between any to identical points (e.g. crests) on adjacent waves in a snapshot view.

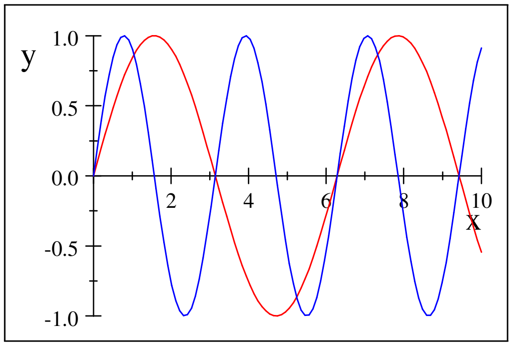
Notice that this is very similar to the definition of the period, T, when we graphed SHM on a y,t set of axis. In fact, this similarity is even more apparent if we plot a sinusoidal wave using our two wave pictures. In the next figure, the snap-shot comes first. We can see that there will be crests. The distance between the crests is given the name *wavelength*. This is not the entire length of the whole wave. But is is a characteristic length of part of the wave that is easy to identify. The next figure shows all this using our snapshot and history graph for a sine wave.



Note that there are crests in the history graph view as well. That is because one marked part of the medium is being displaced as a function of time. Think of a marked piece of the rope going up and down, or think of floating in the ocean at one point, you travel up and down (and a little bit back and forth) as the waves go by. But now the horizontal axis is time. There will be a characteristic time between crests. That time is called the *period*. This is just like oscillatory motion–because it is oscillatory motion! Like the wavelength is not the length of the whole wave, the period is not the time the whole wave exists. It is just the time it takes the part of the medium we are watching to go through one complete cycle. Notice that this video picture is exactly the same as a plot of the motion of a simple harmonic oscillator! For a sinusoidal wave, each part of the medium experiences simple harmonic motion.

We remember frequency from simple harmonic motion. But now we have a wave, and the wave is moving. We can extend our view of frequency by defining it as follows:

The frequency of a periodic wave is the number of crests (or any other point of the wave) that pass a given point in a unit time interval.



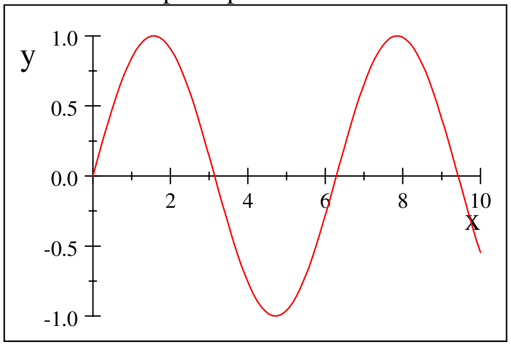
In the previous figure, the blue curve has twice the frequency as the red curve. Notice how it has two crests for every red crest. The maximum displacement of the wave is called the *amplitude* just as it was for simple harmonic oscillators.

Wavenumber and wave speed

Consider again a sinusoidal wave.

$$y\left(x,t\right) = y_{\max}\cos\left(kx - \omega t + \phi_o\right) \tag{5.3}$$

We have drawn the wave in the snapshot picture mode



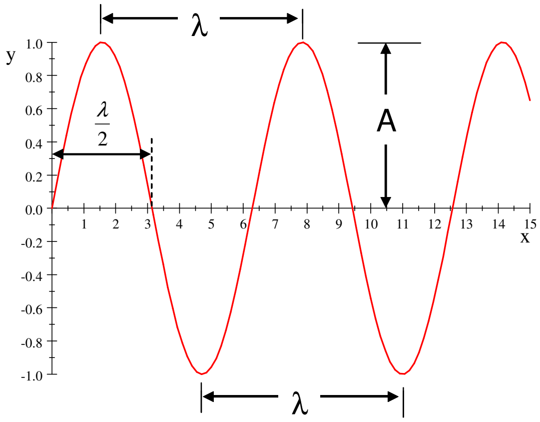
To make this graph, we set $t = 0$ and plot the resulting function

$$y\left(x,0\right) = y_{\max}\sin\left(kx + 0\right) \tag{5.4}$$

$y_{\max}$ is the amplitude. I want to investigate the meaning of the constant k. Lets find k like we did for SHM when we found ω. Consider the point $x = 0$. At this point

$$y\left(0,0\right) = y_{\max}\sin\left(k\left(0\right)\right) = 0 \tag{5.5}$$

The next time $y = 0$ is when $x = \frac{\lambda}{2}$



then

$$y\left(\frac{\lambda}{2},0\right)=y_{\max}\sin\left(k\frac{\lambda}{2}\right)=0 \tag{5.6}$$

From our trigonometry experience, we know that this is true when

$$k\frac{\lambda}{2}=\pi \tag{5.7}$$

solving for k gives

$$k=\frac{2\pi}{\lambda} \tag{5.8}$$

Then we now have a feeling for what k means. It is 2π over the spacing between the crests. The 2π must have units of radians attached. Then

$$y\left(x,0\right)=A\sin\left(\frac{2\pi}{\lambda}x+0\right) \tag{5.9}$$

We have a special name for the quantity k. It is called the *wave number*.

$$k\equiv\frac{2\pi}{\lambda} \tag{5.10}$$

Both the name and the symbol are somewhat unfortunate. Neither gives much insight into the meaning of this quantity. But from what we have done in studying oscillaiton, we can understand this new quantity. For a harmonic oscillator, we know that

$$y\left(t\right)=y_{\max}\sin\left(\omega t\right)$$

where

$$\omega=2\pi f=\frac{2\pi}{T}$$

T is how far, *in time*, the crests are apart, and the inverse of this, $\frac{1}{T}$ is the frequency. The frequency tells us how often we encounter a crest as we march along in time. So $\frac{1}{T}$ must be how many crests we have in a unit amount of time.

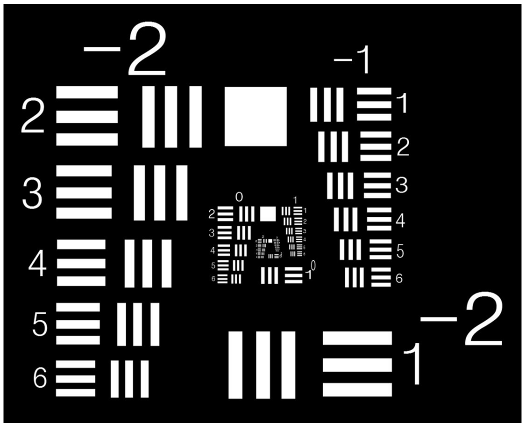
Now think of the relationship between the snapshot and the video representation for a sinusoidal wave. We have a new quantity

$$k = \frac{2\pi}{\lambda}$$

where λ is how far, *in distance*, the crests are apart. This implies that $\frac{1}{\lambda}$ plays the same role in the snap shot graph that f plays in the video graph. It must tell us how many crests we have, but this time it is how many crests in a given amount of distance. We found above that k told us something about how often the zeros (well, every other zero) will occur. But the crests must occur at the same rate. So k tells us how often we encounter a crest in our snapshot graph. This is not too strange. It would be meaningful, for example, to say that a farmer had plowed his or her field to have two furrows per meter.

The frequency in the video graph is how often we encounter a crest, $\frac{1}{\lambda}$ is how often we encounter a crest in the snap shot graph. Thus $\frac{1}{\lambda}$ is playing the same role for a snap shot graph as frequency plays for a history graph. We could call $\frac{1}{\lambda}$a *spatial frequency.* It is how often we encounter a crest as we march along in position, or how many crests we have in a unit amount of distance. And λ could be called a *spatial period*. Both $1/T$ and $1/\lambda$ answer the question "how often something happens in a unit of something" but one asks the question in time and the other in position along the wave.

My mental image for this is the set of groves on the edge of a highway. There is a distance between them, like a wavelength, and how often I encounter one as I move a distance along the road is the spatial frequency. You could say that there are, maybe, $3/\,\mathrm{m}$. That is a spatial frequency! It is how many of something happens in a unit distance. We use this concept in optics to test how well an optical system resolves details in a photograph. The next figure is a test image. A good camera will resolve all spatial frequencies equally well. Notice the test image has sets of bars with different spatial frequencies. By forming an image of this pattern, you can see which spacial frequencies are faithfully represented by the optical system.



Resolution test target based on the USAF 1951 Resolution Test Pattern (not drawn to exact specifications).

In class you will see that our projector does not represent all spatial frequencies equally well! You can also see this now in the copy you are reading. If you are reading on-line or an electronic copy, your screen resolution will limit the representation of some spatial frequencies. Look for the smallest set of three bars where you can still tell for sure that there are three bars without zooming. A printed version that has been printed on a laser printer will usually allow you to see even smaller sets of three bars clearly.

Let's place k in the full equation for the sine wave for any time, t.

$$y\left(t\right)=y_{\max}\cos\left(kx-\omega t+\phi_{o}\right) \tag{5.11}$$

We would like this to look like our wave function equation

$$y\left(x,t\right)=y\left(x-vt,0\right)$$

With a little algebra we can do this

$$\begin{aligned} y\left(t\right) &= y_{\max}\cos\left(kx-\omega t+\phi_{o}\right)\\ &= y_{\max}\cos\left(\frac{2\pi}{\lambda}x-\frac{2\pi}{T}t+\phi_{o}\right)\\ &= y_{\max}\cos\left(\frac{2\pi}{\lambda}\left(x-\frac{\lambda}{T}t\right)+\phi_{o}\right)\end{aligned}$$

This is in the form of a wave function so long as

$$v=\frac{\lambda}{T} \tag{5.12}$$

then

$$y\left(x,t\right)=y_{\max}\sin\left(\frac{2\pi}{\lambda}\left(x-vt\right)\right) \tag{5.13}$$

We can see that the wave travels one wavelength in one period. The simple relationship

$$v=\frac{\lambda}{T} \tag{5.14}$$

is of tremendous importance.

Wave speed forms

We found

$$v = \frac{\lambda}{T} \tag{5.15}$$

but it is easy to see that

$$v = \frac{2\pi\lambda}{2\pi T} = \frac{\omega}{k} \tag{5.16}$$

and

$$v = \lambda f \tag{5.17}$$

This last formula is, perhaps, the most common form encountered in our study of light.

Phase

You may be wondering about the phase constant we learned about in our study of SHM. We have ignored it up to now. But of course we can shift our sine just like we did for our plots of position vs. time for oscillation. Only now with a wave we have two graphs, a history and snapshot graphs, so we could shift along the x in a snapshot graph or along the t axes in a history graph. So the sine wave has the form.

$$y(x,t) = A\sin(kx - \omega t + \phi_o) \tag{5.18}$$

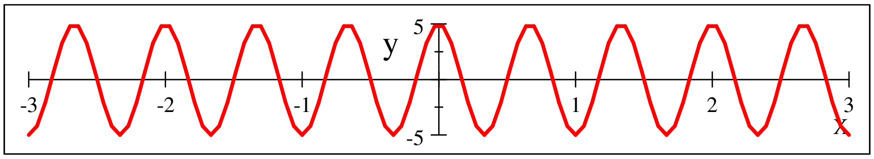
were ϕ_o will need to be determined by initial conditions just like in SHM problems and those initial conditions will include initial positions as well as initial times.

Let's consider that we have two views of a wave, the snapshot and history view. Each of these looks like sinusoids for a sinusoidal wave. Let's consider a specific wave,

$$y(x,t) = 5\sin\left(3\pi x - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$

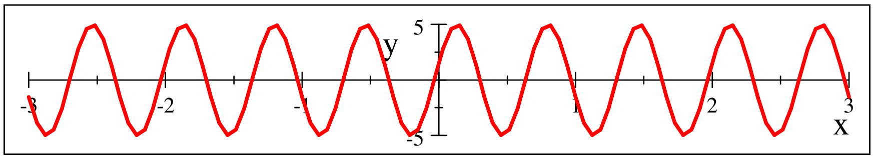
we look at a snapshot graph at $t = 0$

$$y(x,0) = 5\sin\left(3\pi x - \frac{\pi}{5}(0) + \frac{\pi}{2}\right)$$



and another at $t = 2\,\mathrm{s}$

$$y(x,2\,\mathrm{s}) = 5\sin\left(3\pi x - \frac{\pi}{5}(2\,\mathrm{s}) + \frac{\pi}{2}\right)$$



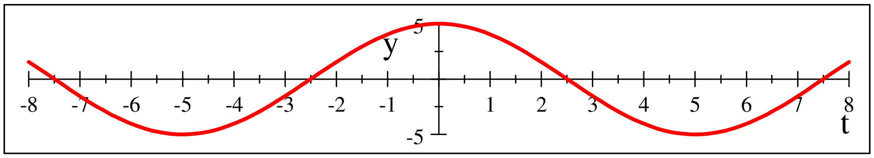
Comparing the two, we could view the latter as having a different phase constant that is the sum of what we have called the phase constant, ϕ_o plus $\omega\Delta t$, which tells us how different the times are between the two graphs. This time difference is what is shifting the graph

$$\phi_{total} = \omega\Delta t + \phi_o = -\frac{\pi}{5}\left(2\,\mathrm{s}\right) + \frac{\pi}{2}$$

that is, within the snapshot view, the time dependent part of the argument of the sign acts like an additional phase constant.

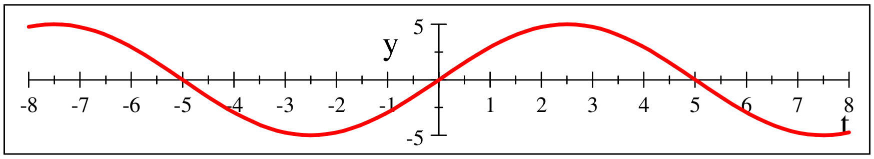
Likewise, in the history view, we can plot our wave at $x = 0$

$$y\left(0, t\right) = 5\sin\left(3\pi\left(0\right) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



and at $x = 1.5\,\mathrm{m}$

$$y\left(1.5\,\mathrm{m}, t\right) = 5\sin\left(3\pi\left(1.5\,\mathrm{m}\right) - \frac{\pi}{5}t + \frac{\pi}{2}\right)$$



Within the history view, the kx part of the argument acts like a phase constant.

$$\phi_{total} = k\Delta x + \phi_o = 3\pi\left(1.5\,\mathrm{m}\right) + \frac{\pi}{2}$$

Of course neither kx nor ωt are constant, But within individual views of the wave we have set them constant to form our snapshot and history representations. We can see that any part of the argument of the sine, $kx - \omega t + \phi_o$ could contribute to a phase shift, depending on the view we are taking.

Because of this, it is customary to call the entire argument of the sine function, $\phi = kx - \omega t + \phi_o$ the *phase of the wave*. Where ϕ_o is the phase constant, ϕ is the phase. Of course then, ϕ must be a function of x and t, so we have a different value for $\phi\left(x, t\right)$ for every point on the wave for every time. This can be a little confusing, ϕ and ϕ_o look a lot the same, but they are different.

The Speed of Waves

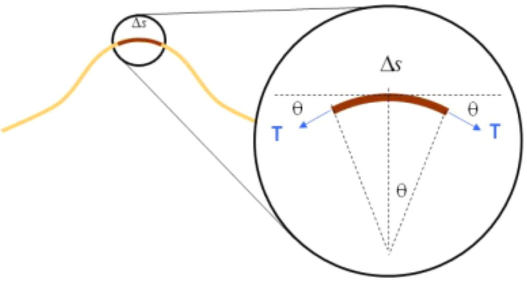
We have already considered the speed of waves on strings

$$v=\sqrt{\frac{T}{\mu}}$$

we should see where this comes from so we know it's limitations. We would expect that there may be things that change the speed of sound waves in air as well (like a tension or a massiveness of the medium). Let's start with a formal derivation of our string speed formula, then take on sound waves in air.

Derivation of the string wave speed formula

Let's work a problem together. Let's find an expression for the speed of the wave as it travels along a string.



What are the forces acting on an element of string?

1. Tension on the RHS of the element from the rest of the string on the right, T_r
2. Tension on the LHS of the element from the rest of the string on the left, T_l
3. F_g

Lets assume that the element of string, Δs, at the crest is approximately an arc of a circle with radius R.

There is a force pulling left on the left end of the element that is tangent to the arc, there is a force pulling right at the right end of the element which is also tangent to the arc. These forces produce centripetal accelerations

$$a=\frac{v^2}{R} \tag{5.19}$$

The horizontal components of the forces cancel $(T\cos\theta)$. The vertical component,

$(T\sin(\theta))$ is directed toward the center of the arc. If the rope is not moving in the x direction, then $T_l = T_r$.

Then, the radial force F_r will have matching components from each side of the element that together are $2T\sin(\theta)$. Notice that to make sure the components are the same we must assume that the string is uniform and that it is not too massive (almost massless string approximation!). Since the element is small,

$$F_r = 2T\sin(\theta) \approx 2T\theta \tag{5.20}$$

The element has a mass m.

$$m = \mu\Delta s \tag{5.21}$$

where μ is the mas per unit length. Using the arc length formula

$$\Delta s = R(2\theta) \tag{5.22}$$

so

$$m = \mu\Delta s = 2\mu R\theta \tag{5.23}$$

and finally we use the formula for the radial acceleration

$$F_r = ma = (2\mu R\theta)\frac{v^2}{R} \tag{5.24}$$

Combining these two expressions for F_r

$$2T\theta = (2\mu R\theta)\frac{v^2}{R} \tag{5.25}$$

$$T = (\mu R)\frac{v^2}{R} \tag{5.26}$$

$$\frac{T}{\mu} = v^2 \tag{5.27}$$

and we find that

$$v = \sqrt{\frac{T}{\mu}} \tag{5.28}$$

which is just what we stated before, only now we see just how much approximation there was in building this formula. We might guess that it would not work so well for large cables supporting bridges or with cables that change size along their length.

Speed of Sound Waves

The speed of sound in air is around $340\,\mathrm{m/s}$. The speed changes when we change media, and even when we are in the same media but the temperature changes. For sound in air, a good approximation is

$$v = v_o\sqrt{1 + \frac{T_c}{T_o}} \tag{5.29}$$

where $v_o = 331\,\frac{\mathrm{m}}{\mathrm{s}}$ and $T_o = 273\,\mathrm{K}$ $(0\,^{\circ}\mathrm{C})$.[5]

Why temperature? We will find out later in this course that the density and pressure of air change with temperature. But for now we can see that changing the air density is a little like changing the linear mass density of a string. It is bound to have an effect on the wave speed in the air.

Waves in two and three dimensions

So far we have written expressions for waves, but our experience tells us that waves don't usually come as one dimensional phenomena. In the next figure, we see the disturbance (a drop) creating a water wave.

Picture of a water drop (Jon Paul Johnson, used by permision)

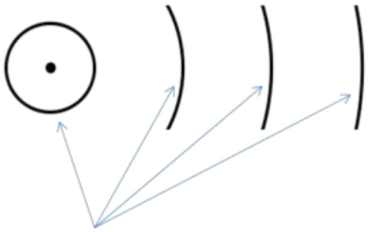
You can see the wave forming, but the wave is clearly not one dimensional. It appears nearly circular. In fact, it is closer to hemispherical, and this limit is only true because the disturbance is at the air-water boundary. Most waves in a uniform medium will be roughly spherical. As such a wave travels away from the source, the energy traveling gets more spread out. This causes the amplitude to decrease. Think of a sound wave, it gets quieter the farther you are from the source. We change our equation to account for this by making the amplitude a function of the distance, r, from the source

$$y\left(r, t\right) = A\left(r\right)\sin\left(kr - \omega t + \phi_o\right) \tag{5.30}$$

Of course, if we look at a very large wave, but we only look at part of the wave, we see that our part looks flatter as the wave expands.

---

[5] $v = v_o\sqrt{1 + \frac{T_C}{T_o}} = v_o\sqrt{\frac{T_o}{T_o} + \frac{T_C}{T_o}} = v_o\sqrt{\frac{T_o + T_C}{T_o}} = v_o\sqrt{\frac{T_K}{T_o}}$



Portion of a Spherical Wave: Wave becomes more flat as it expands

Very far from the source, our wave is flat enough that we can ignore the curvature across it's wave fronts. We call such a wave a *plane wave*. There are no true plane waves in nature, but this idealization makes our mathematical solutions simpler and many waves come close to this approximation.

We have said that sound is a longitudinal wave with a medium of air. Really any solid, liquid, or gas will work as a medium for sound. For our study, we will take sound to be a longitudinal wave and treat liquids and gasses. Solids have additional forces involved due to the tight bonding of the atoms, and therefore are more complicated. Technically in a solid sound can be a transverse wave as well a longitudinal wave, but we usually call transverse waves of this nature *shear waves.*

Periodic Sound Waves, Pressure

Vollyball Demo

Suppose I have a ball and I ask six people from the class to come up and press on the ball from all directions. This is a new force situation, unlike most we dealt with in Dynamics or PH121. Each person exerts a force on the ball. The person uses the area of their hand to exert the force. The motion of the ball, and even it's shape depend on both the force (magnitude and direction) and the area involved in each push.

From our demo, it seems that the force and area of the ball could be related to better describe the situation. Let's look at the ratio

$$\frac{F}{A}$$

What does it represent? This ratio tells us how spread out an applied force may be. The area is important. Think of the sides of a the eraser of a pencil.

It is coinvent to give this concept a name. We will call it *pressure.*

Definition: Pressure is a scalar value that describes how a force acts over an area

$$P \equiv \frac{F}{A} \tag{5.31}$$

But in a fluid, what is the force? Lets consider a ball hitting a wall. Is there a force? During the collision, there is a force of the ball on the wall, and a Newton's third law force of the wall molecules pushing back on the ball. The ball will bounce back.

Ping Pong Ball Demo

This force is small and only lasts during the collision. But now suppose we have many balls, and all the balls impact the wall. Further suppose that every time a ball bounces back is ends up headed back to the wall and bounces again. If the balls keep coming, there will be a force on the wall quite a bit of the time. At least, on average there is a force, anyway. This is the force that causes air pressure. The air molecules in this room are like the small balls. we will find that they have an amount of kinetic energy. They impact the walls (and us) all over our surface area. The result is air pressure.

The water pressure in a swimming pool is caused by moving water particles. You should convince yourself that the reason the water stays in the pool is partly because the air molecules bounce against the water surface exerting a pressure on the waver!

Let's go back to making sounds. Suppose we push our piston as we did before in figure **??**. When we push in the piston, it creates a region of higher pressure next to it.

When we pull back the piston the fluid expands to fill the void. We create a rarefaction next to the piston.

Suppose we drive the piston sinusoidally. Can we describe the motion of the particles and of the wave?

1. Compression: A local region of higher pressure in a fluid
2. Rarefaction: A local region of lower pressure in a fluid

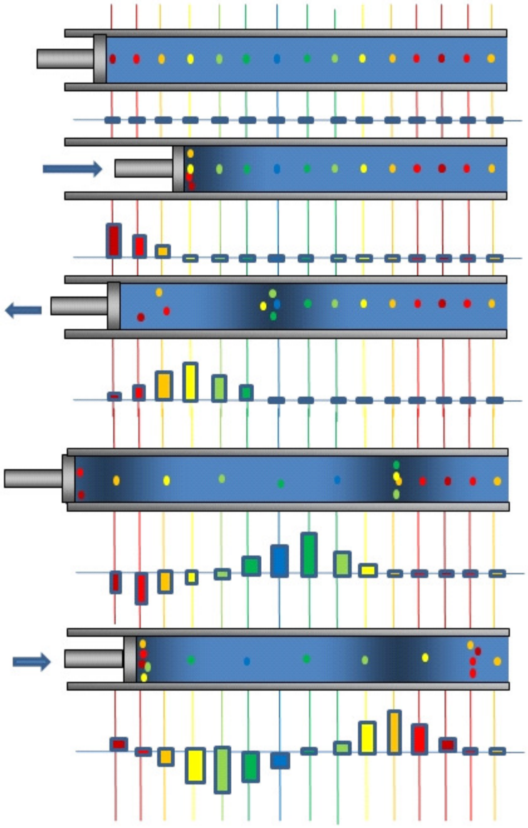
We can identify the distance between two compressions as λ.

We define $s\left(x,t\right)$ (like we defined a wave function, $y\left(x,t\right)$) as the displacement a particle of fluid relative to its equilibrium position.

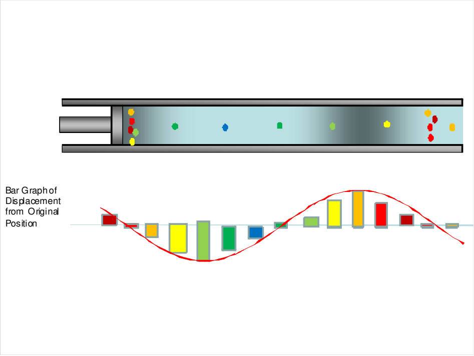
$$s\left(x,t\right)=s_{\max}\cos\left(kx-\omega t\right) \tag{5.32}$$

but what is $s_{\max}$?

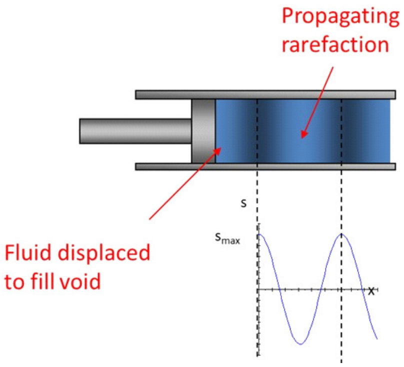
We remember that $s_{\max}$ is the maximum displacement of a particle of fluid from its equilibrium position. We plotted this using a bar graph to show displacement from the equilibrium position for our molecules. As we push the piston in and out we will get something like this.



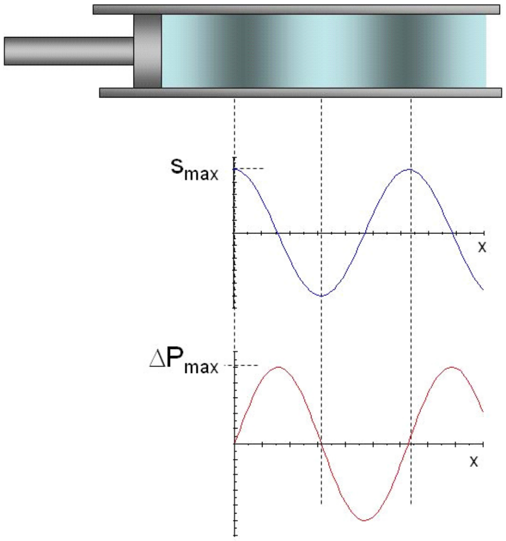
found before that we get something that looks like a sine wave, but remember what the bars represent. They represent the displacement from original position.



We don’t usually draw bar graphs, we usually just draw the sine wave.



The variation of the gas pressure ΔP measured from its equilibrium is also periodic



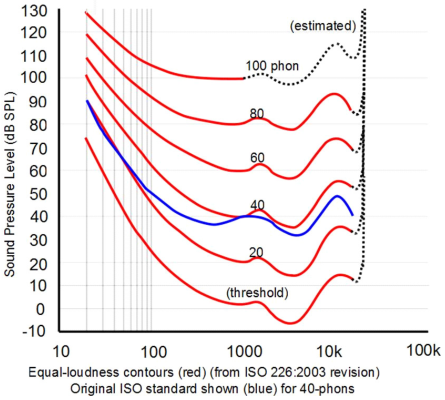
which is why we often refer to a sound wave as a pressure wave. Think of when the wave gets to your ear. the wave consists of a group of particles all headed for your ear drum. When they hit, they exert a force. Pressure is a force spread over an area,

$$P = \frac{F}{A}$$

so in a sense, we hear changes in air pressure!

Loudness and frequency

Frequency Range Demo



Robinson-Dadson equal loudness curves (Image in the Public Domain courtesy Lindosland)

Our ears are not designed to be quantitative scientific instruments (though they are truly amazing in their range and ability). Sounds with the same intensity at different frequencies do not appear to us to have the same loudness. The frequency response graph above shows how this relationship works for test subjects[6].

Boundaries

Suppose two pulses travel in the same medium, say, on a rope, and they approach a different rope with a different linear mass density. If the new rope is heavier, we expect the wave speed to slow down. So as one pulse reaches the boundary, it will go slower. This allows the second pulse to catch-up before it too slows down at the boundary.

Now suppose a sinusoidal wave approaches the boundary. We can envision the crests like pulses, and we expect the first crest to slow down when it reaches the boundary, letting the other crests catch up. Once the wave passes the boundary, the crests will be closer together. The wavelength changes as we move to the slower medium.

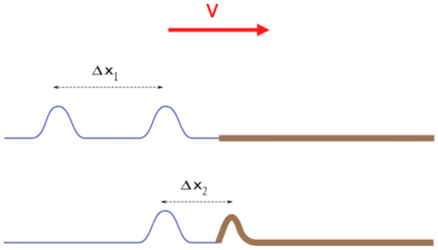
But does the frequency change? We know that

$$v = \lambda f$$

so

$$f = \frac{v}{\lambda}$$

both the speed and the wavelength have changed, but did they change proportionately so f is constant? This must be so. Think that the change in wavelength is due to the relative speed of the wave in the two media. If Δv is small the change in λ will be small because the crests are not delayed too long. If Δv is large, the crests are delayed by a large amount and so the change in λ is large. We won't derive the fact that f is constant, but we can see that is is very believable that it is true.



---

[6] Ones that have not gone to Guns n Roses concerts

This is true for all waves, even light. When a wave crosses a boundary from a fast to a slow or a slow to a fast medium, λ will change and f will remain constant.

Waves in Fields-Light

Sound is a wave in matter, but what is light? It will really take the rest of the course (and then some) to answer this question. But we know that light can travel through a vacuum. Therefore, light can't be a wave in some type of matter. We will find later in this course that there exists something called an electromagnetic field created by charged particles. It turns out that light seems to be a wave in this electromagnetic field. It will take us a while to fully understand this concept–it is done in PH220–but don't worry. Physicists knew that light was a wave for almost 80 years before the electric field was shown to be the medium. We can do a lot just knowing light behaves like a wave.

This makes the light we see just one small part of a whole class of waves that are possible in this electromagnetic field medium. Radio waves, and microwaves, and x-rays are all just different types of electromagnetic waves. The next figure show where all of these electromagnetic waves fit ordered by wavelength (and frequency).

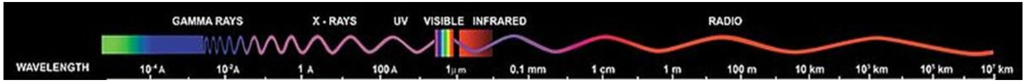


Image Courtesy NASA

There is something very unique about this electromagnetic field medium. The waves in this medium travel at a constant speed no matter what frame of reference we are in. This fact lead to the formation of the Special Theory of Relativity and the famous equation

$$E = mc^2$$

where c is this speed of light

$$c = 299792458\,\frac{\mathrm{m}}{\mathrm{s}}$$

Light does slow down when it enters a material, like glass, or even air. This is not really because it moves slower. what happens is that light is absorbed by the electrons in the atoms of the material substance. The electron temporarily takes up all the energy from a bit of the light wave. But only temporarily. It eventually has to give up the energy a light wave is reformed coming from the atoms. But it has lost some time in the process, so it's average speed is less.[7] How much less depends on how long the electrons in the atoms can hang on to the light. Each substance is different.

We can devise a way to express how much slower light will appear to go in a substance using the ratio

$$\frac{c}{v}$$

the ratio of the speed of light, c, to the average speed in the substance, v. This ratio is so useful that we give it a name, the *index of refraction*.

$$n = \frac{c}{v}$$

In PH121, we used the concept of energy to do problems (specifically to make hard problems easier!). We should expect to be able to use the concept of energy with our waves. After all, the medium must transfer energy to make a wave, so waves must transfer energy! We will take up the topics of energy and waves in our next lecture.

---

[7] There is really a superposition of what is left of the original light wave, and the new light wave generated by the atoms. But we have not gotten to superposition yet. So this explanation will have to do for now.

6 Power, Doppler Effect and Superposition

Fundamental Concepts

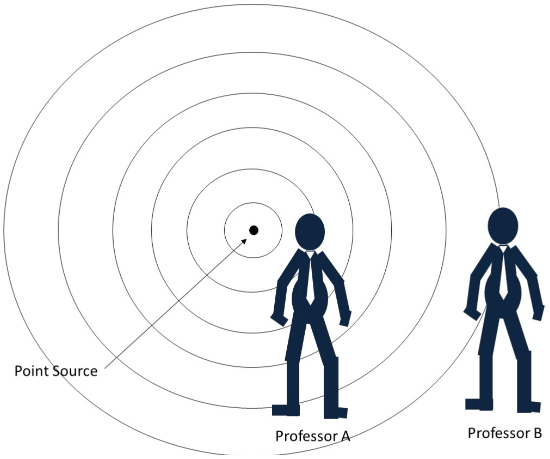
- Because waves are three dimensional, describing the power or energy delivered per time of the wave is not enough, We describe how spatially spread out that power is. We call this spread power *intensity*
- We don't hear a doubling of intensity as twice as loud. To mathematically describe how our ears work we use decibels
- When we observe waves from different reference frames, we observe a apparent frequency shift. This is called the *Doppler effect*
- If we make more than one wave in a medium, the waves "add up" or *superimpose.*

Power and Intensity

We know that energy is being transferred by the wave, whether it is a light or sound wave. We should wonder, how fast is energy transferring. This can mean the difference between a warm ray of sin on a cool spring day and being burned by a laser beam. We will start by considering the rate of energy transfer, *power*. The concept of power should be familiar to us from PH121. We can find the power as the rate at which energy is transferred.

$$\mathcal{P} = \frac{\Delta E}{\Delta t}$$

But if we consider our spherical wave from a point source, we can see that this description isn't good enough. In the next figure we have two professors.



Thinking from our experience we would say that the sound will seem louder for professor A. This is because the energy in the sound wave is being spread over the surface of the wave, and that surface is getting bigger as the wave moves outward. The energy is more spread out by the time it gets to professor B.

Intensity

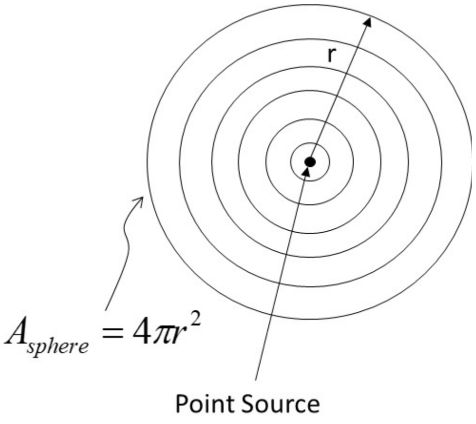
Tuning Fork Demo

To be able to describe how much energy we get from our wave we need define something new.

$$\mathcal{I} \equiv \frac{\mathcal{P}}{A} \tag{6.1}$$

that is, the power divided by an area. But what does it mean?

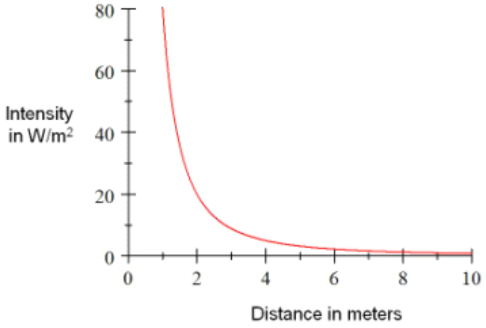
Consider a point source.



it sends out waves in all directions. The wave crests will define a sphere around the points source (the figure shows a cross section but remember it is a wave from a point source, so we are really drawing concentric spheres like balloons inside of balloons.). Then form our point source

$$\mathcal{I} = \frac{\mathcal{P}}{4\pi r^2} \tag{6.2}$$

As the wave travels, its the power per unit area decreases with the square of the distance (think gravity) because the area is getting larger.



This quantity that tells us how spread out our power has become is called the *intensity* of the wave. Professor A would agree with us that the wave he heard was more intense than the wave heard by Professor B. That is because the wave was less spread out for Professor A.

Suppose we cup our hand to our ear. What are we doing? We are increasing the area of our ear. Our ears work by transferring the energy of the sound wave to a mechanical-electro-chemical device that creates a nerve signal.[8] The more energy, the stronger the signal. If we are a distance r away from the source of the sound then the intensity is

$$\mathcal{I} = \frac{\mathcal{P}_{source}}{A_{wave}}$$

But we are collecting the sound wave with another area, the area of our hand. The power received is

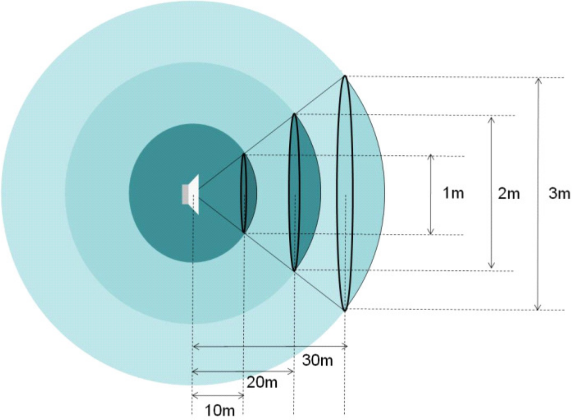
$$\begin{aligned} \mathcal{P}_{received} &= \mathcal{I} A_{hand} \\ &= \frac{A_{hand}}{A_{wave}} \mathcal{P}_{source} \end{aligned}$$

and we can see that, indeed, the larger the hand, the more power, and therefore more energy we collect. This is the idea behind a dish antenna for communications and the idea behind the acoustic dish microphones we see at sporting events. In next figure, we can see that it would take an increasingly larger dish to maintain the same power gath-

---

[8] The inner hair cells in the organ of Corti in the cochlea.

ering capability as we get farther from the source.



Sound Levels in Decibels

Our Design Engineer made an interesting choice in building us. We need to hear very faint sounds, and very loud sounds too. In order to make us able to hear the soft sounds without causing extreme discomfort when we hear the loud, He gave up linearity. That is, we don't hear twice the sound intensity as twice as loud.

The mathematical expression that matches our perception of loudness to the intensity is

$$\beta = 10\log_{10}\left(\frac{I}{I_o}\right) \tag{6.3}$$

where the quantity I_o is a reference intensity.

Sound Meter Demo

We call β the *sound level*. The quantity I_o we choose to be the *threshold of hearing*,

$$I_o = 1\times 10^{-12}\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

the intensity that is just barely audible. Measured this way, we say that intensity is in units of decibels (dB).The decibel, is useful because it can describe a non-linear response in a linear way that is easy to match to our human experience. But it is tricky because, like radians, it is dimensionless because we are comparing two intensities (I/I_o) . We expect the same strange unit behavior with dB that we see with radians.

Suppose we double the intensity by a factor of 2.

$$\begin{aligned} \beta &= 10\log_{10}\left(\frac{2I_o}{I_o}\right) \\ &= 10\log_{10} 2 \\ &= 3.010\,3dB \end{aligned}$$

The sound intensity level is not twice as large, but only $3dB$ larger. It is a tiny increase. This is what we hear. A good rule to remember is that $3dB$ corresponds to a doubling of the intensity.

The tables that follow give some common sounds in units of dB and $\mathrm{W/\,m^2}$. Just for reference, I have measured a Guns n Roses concert at 120 dB outside the stadium.

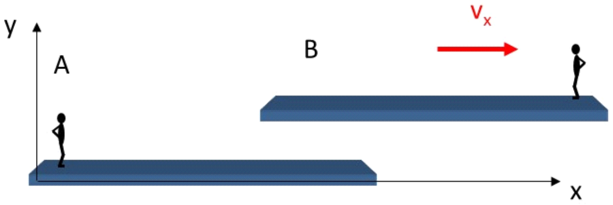
Sound	Sound Level (dB)
Jet Airplane at 30m	140
Rock Concert	120
Siren at 30m	100
Car interior when Traveling 60mi/hr	90
Street Traffic	70
Talk at 30cm	65
Whisper	20
Rustle of Leaves	10
Quietest thing we can hear (Io)	0

Doppler Effect

Doppler Ball Demo

Let's start by considering an inertial reference frame (remember this from Dynamics/PH121?)

Suppose we pick two reference frames, one traveling with a velocity v_r with respect to the other. Let's also place them far away from any other object.

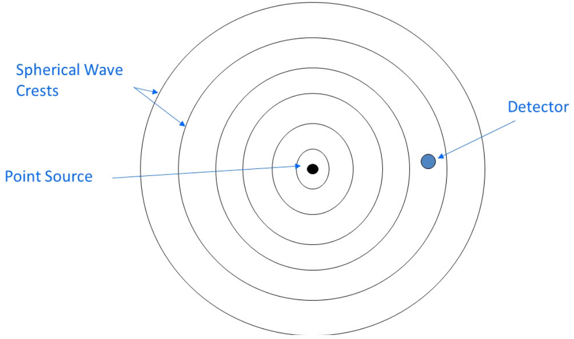


Person A sees himself as stationary and sees person B traveling with velocity v_x. Person B sees himself as stationary, and person A traveling with velocity $-v_x$.

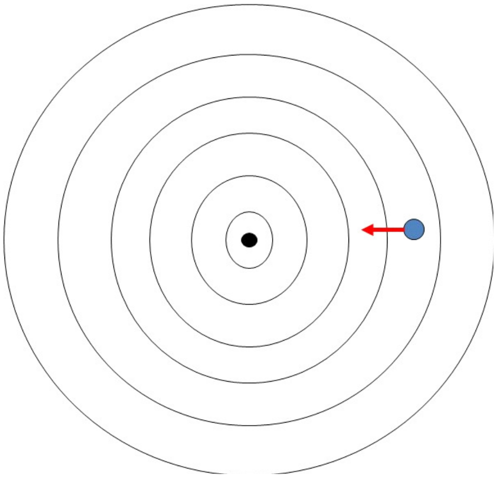
In looking at this situation it is the *relative* speed v_x that we must consider. We recall that we could write the speed of guy b as seen from platform A as

$$v_{bA} = v_{bB} + v_{BA}$$

where v_{bB} is guy b's speed seen from platform B and v_{BA} is the speed of platform B as seen from platform A. In this case $v_{BA} = v_x$. Now suppose we have a wave generator (a point source) creating spherical waves. Let the point source be at rest. We will call this point source an emitter and use a subscript e for it. BYU Demo

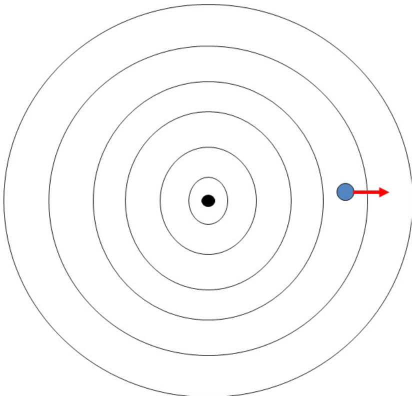


Let's also assume a detector. If the detector is stationary with respect to the emitter so the detector is in the emitter's reference frame, it sees a frequency of the wave in the emitter frame, f_{we}. But lets have the detector move relative to the emitter. Move George



Remember, that the frequency is the number of crests that pass by a given point in a unit time. Does the moving detector see the same number of crests per unit time as when it was stationary?

No, the frequency appears to be higher! How about if we let the detector move the other way?



Again the frequency seen by the detector is different, but this time lower.

We can quantify this change. Take our usual variables f_{we}, λ_{we}, and the velocity of sound $v_{sound} = v_{we}$ for the wave speed of the sound wave because the air itself is not moving with respect to the emitter and the wave travels through the air. When the detector moves toward the source, it sees the wave velocity as

$$v_{wd} = v_{we} + v_{de} \tag{6.4}$$

where v_{wd} is the velocity of the wave in the detector frame, and v_{de} is the velocity of the detector in the emitter frame. The wavelength will not be changed, so

$$v_{we} = \lambda_{we} f_{we}$$

can just be written as

$$v_{we} = \lambda f_{we}$$

which tells us the frequency must change.

$$f_{wd} = \frac{v_{wd}}{\lambda} = \frac{v_{we} + v_{de}}{\lambda}$$

We can eliminate λ from this expression for the change in f by using $v_{we} = \lambda f_{we}$ again, this time solving for λ we get

$$\lambda = \frac{v_{we}}{f_{we}}$$

and substitute this into our f_{wd} equation

$$f_{wd} = \frac{v_{we} + v_{de}}{\frac{v_{we}}{f_{we}}}$$

or, after rearranging

$$f_{wd} = \frac{v_{we} + v_{de}}{v_{we}} f_{we} \qquad \text{detector moving toward the emitter} \tag{6.5}$$

or recognizing that v_{we} is the speed of sound in the stationary frame of the emitter

$$f_{wd} = \frac{v_{sound} + v_{de}}{v_{sound}} f_{we} \qquad \text{detector moving toward the emitter} \tag{6.6}$$

Change the Demo

Now if the detector is going the other way

$$v_{wd} = v_{we} - v_{de}$$

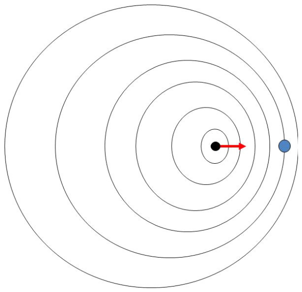
and the same reasoning gives

$$f_{wd} = \frac{v_{we} - v_{de}}{v_{we}} f_{we} \qquad \text{detector moving away from the emitter} \tag{6.7}$$

or

$$f_{wd} = \frac{v_{sound} - v_{de}}{v_{sound}} f_{we} \qquad \text{detector moving away from the emitter} \tag{6.8}$$

From our thinking about the motion of two inertial reference frames, we should expect a similar situation if the detector is stationary and the source moves. In this case the detector will see a different wavelength.



In fact, if we measure the distance between the crests we must account for the fact that the source moved by an amount

$$\Delta x_{ed} = v_{ed} T_{we} = \frac{v_{ed}}{f_{we}}$$

during one period of oscillation of the emitter before making the next crest. We can see that in this case, $v_{ed} = v_{sound}$ because this time the air is not moving with respect to the detector, so they are in the same reference frame. Then the wavelength will be shorter by this amount! That is, if we take the wavelength that we would get if the emitter were at rest, and subtract Δx_{ed} we should have our new wavelength at the detector. Let's start by finding the wavelength we expect if the emitter were at rest.

$$\lambda_{we,rest} = \frac{v_{we}}{f_{we}} = \frac{v_{sound}}{f_{we}}$$

This is because if the emitter were at rest, then $v_{we} = v_{sound.}$ But in our case the emitter is moving, so we must subtract Δx_{ed} from this

$$\lambda_{wd} = \lambda_{we,rest} - \Delta x_{ed}$$

Then the wavelength we would see at the detector would be

$$\lambda_{wd} = \lambda_{we,rest} - \frac{v_{ed}}{f_{we}}$$

Using

$$\lambda_{wd} = \frac{v_{wd}}{f_{wd}}$$

once more, we can write the frequency in the detector frame as

$$\begin{aligned} f_{wd} &= \frac{v_{wd}}{\lambda_{wd}} = \frac{v_{wd}}{\lambda_{we,rest} - \frac{v_{ed}}{f_{we}}} \\ &= \frac{v_{wd}}{\frac{v_{sound}}{f_{we}} - \frac{v_{ed}}{f_{we}}} \\ & \frac{v_{wd}}{v_{sound} - v_{ed}} f_{we} \end{aligned}$$

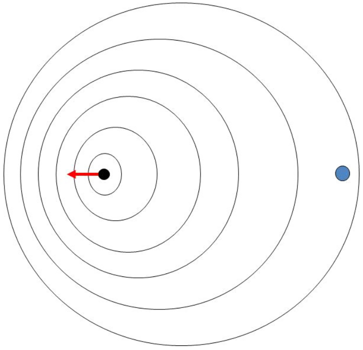
or, again with a little rearranging

$$f_{wd} = \frac{v_{wd}}{v_{sound} - v_{ed}} f_{we} \qquad \text{emitter moving toward detector} \tag{6.9}$$

Now recall that it is the detector that is stationary this time so $v_{sound} = v_{wd}$

$$f_{wd} = \frac{v_{sound}}{v_{sound} - v_{ed}} f_{we} \qquad \text{emitter moving toward detector} \tag{6.10}$$

When the source is moving away from the detector,



we expect the wavelength to be larger. The same reasoning gives

$$f' = \frac{v_{wd}}{v_{wd} + v_{ed}} f_{we} \qquad \textbf{emitter moving away from detector} \tag{6.11}$$

or

$$f' = \frac{v_{sound}}{v_{sound} + v_{ed}} f_{we} \qquad \textbf{emitter moving away from detector} \tag{6.12}$$

Combined Doppler Equation

We can combine these formulae to make one expression, but to do so we need to remember what v_{de} and v_{ed} mean. The first was the speed of the detector when the emitter was not moving. The second was the speed of the emitter when the detector was not moving. But we are experienced with relative motion. We should ask, "not moving with respect to what?" Let's envision a reference frame that is not tied to either the emitter or detector. In this reference frame v_{dR} is the speed of the detector, and v_{eR} is the speed of the emitter. In this R reference frame our first Doppler equation for a moving detector with a stationary emitter might be written as

$$f_{wd} = \frac{v_{sound} \pm v_{dR}}{v_{sound}} f_{we} \qquad \text{detector moving emitter stationary in R frame} \tag{6.13}$$

and our second equation for a moving emitter with a stationary detector might be written as

$$f' = \frac{v_{wd}}{v_{wd} \mp v_{eR}} f_{we} \qquad \text{emitter moving detector stationary in R frame} \tag{6.14}$$

We could, of course have both the detector and emitter moving in the R frame. This would combine both of our previous scenarios

$$f_{wd} = \frac{v_{sound} \pm v_{dR}}{v_{sound} \mp v_{eR}} f_{we} \tag{6.15}$$

where we use the top sign for the speed when the mover is going toward the non-mover.

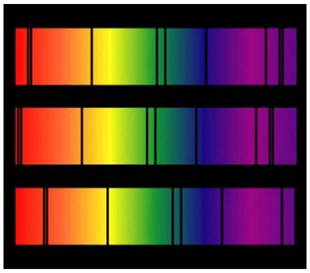
With this view, that we are neither in the e frame nor the d frame but in a separate R frame, we could even drop the R subscript without too much confusion

$$f_{wd} = \frac{v_{sound} \pm v_{d}}{v_{sound} \mp v_{e}} f_{we} \tag{6.16}$$

where now v_d is the speed of the detector in the reference frame of the observer, and v_e is the speed of the emitter in the reference frame of the observer. The quantity f_{we} is still the frequency of the wave as seen by the emitter and f_{wd} is still the frequency of the wave recorded by the detector.

Doppler effect in light

Light is also a wave, and so we would expect a Doppler shift in light. Indeed we do see a Doppler shift when we look at moving objects. Here is an optical spectrum of the Sun on the top and a spectrum of a similar start moving away from us in the middle. The final spectrum is for a star moving toward us.

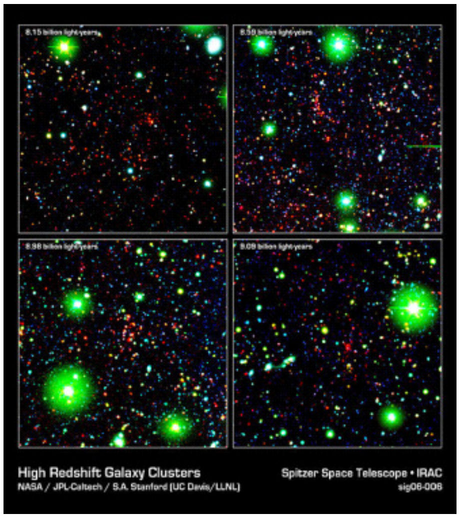


Top: Normal ‘dark’ spectral line positions at rest. Middle: Source moving away from observer. Bottom: Source moving towards observer. (Public domain image courtesy NASA: http://www.jwst.nasa.gov/education/7Page45.pdf)

Note that the wavelength of the lines is shifted toward the red part of the spectrum when the glowing object moves away from us. This is equivalent to lowering of the frequency of a truck engine noise as it goes away from us. The larger wavelengths indicate a lower frequency of light because

$$f = \frac{c}{\lambda}$$

This gives us a way to determine if distant stars and galaxies are moving toward or away from us. We look for the chemical signature pattern of lines, then see whether they are shifted to the red (moving away from us) or blue (moving toward us) compared to the position in their spectrum of the Sun. This photo is of some of the most distant galaxies that are moving very fast away from us. Their redshift is very large.



High Redshift Galaxy Cluster shown here in false color from the Spitzer Space Telescope. (Public domain image courtesy NASA/JPL-Caltech/S.A. Stanford (UC Davis/LLNL)

Deriving the Doppler equation for light is more tricky because the speed of light is constant and the same in every reference frame. We really tackle this in our PH279 class. So I will just quote the result here.

$$\lambda_- = \lambda_o \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \qquad \text{receding source} \tag{6.17}$$

$$\lambda_- = \lambda_o \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \qquad \text{Approaching source} \tag{6.18}$$

Superposition Principle

Wave Machine Demo

What happens if we have more than one wave propagating in a medium? If you remember being a little child in a bath tub, you will probably remember making waves in the water. If you made a wave with each hand, the two waves seemed to "pile up" in the middle and make a big splash. We should expect something like this for any kind of wave. We call the "piling up" of waves *superposition.* The word literally means putting one wave on top of another.

Definition 6.1 *Superposition: If two or more traveling waves are moving through a medium, the resultant wave formed at any point is the algebraic sum of the values of the individual wave forms.*

So if we have

$$y_1(x,t) \tag{6.19}$$

and

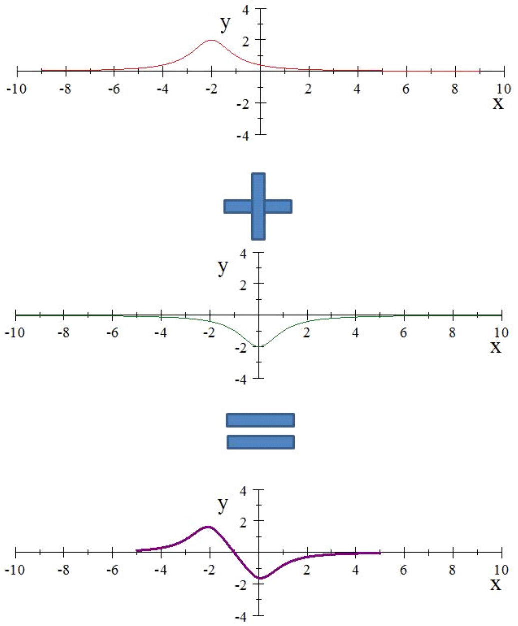
$$y_2(x,t) \tag{6.20}$$

both propagating on a string, then we would see

$$y_r(x,t) = y_1(x,t) + y_2(x,t) \tag{6.21}$$

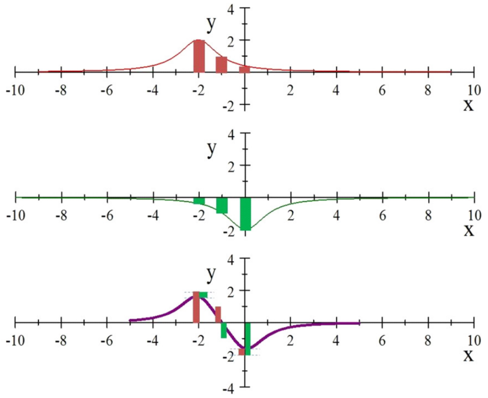
This is a fantastically simple way for the universe to act!

Let's look at an example. let's add the top wave (red) to the middle wave(green). We get the bottom wave (purple)



Of course we are adding these in the snapshot view. So this is all done for just one instant of time.

Let’s see how to do this.



Start at $x = -2$. In the figure, I drew a red bar to show the y value at $x = -2$ for the red curve. Likewise, I have a green bar sowing the value of y at $x = -2$ for the green wave. Note that this is negative. On the bottom graph, the bars have been repeated, and we can see that the red bar minus the green bar brings us to the value for the resulting wave at the point $x = -2$. We need to do this at every point along all the waves for this instant of time.

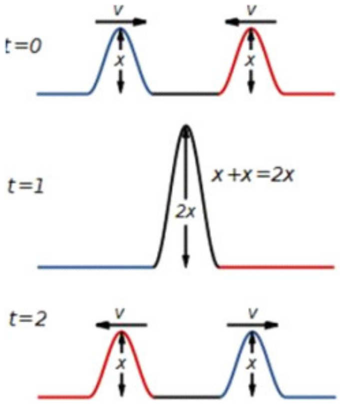
This is tedious by hand, so we won't generally do this calculation by hand. But a computer can do it easily.

Note that this is really only true for *linear* systems. Let's take the example of a slinky. If we form two waves in the slinky, they behave according to the superposition principle most of the time. But suppose we make the amplitude of the individual waves large. They may travel individually OK, but when the amplitudes add we may overstretch the slinky. Then it would never return to it's original shape. The wave form would have to change. Such a wave is not linear. There is a good rule of thumb for when waves are linear.

A wave is generally linear when its amplitude is much smaller than its wavelength.

Consequences of superposition

Linear waves traveling in media can pass through each other without being destroyed or altered!

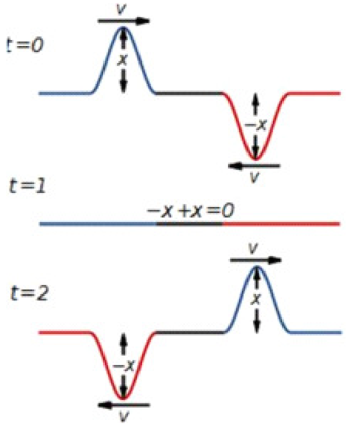


Constructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Constructive_interference.svg)

Our wave on the string makes the string segments move in the y direction. Both waves do this. So putting the two waves together just makes the string segments move more! There is a special name for what we observe

1. *interference*: The combination of separate waves in the same region of space to produce a resultant wave.

What happens if one of the pulses is inverted?

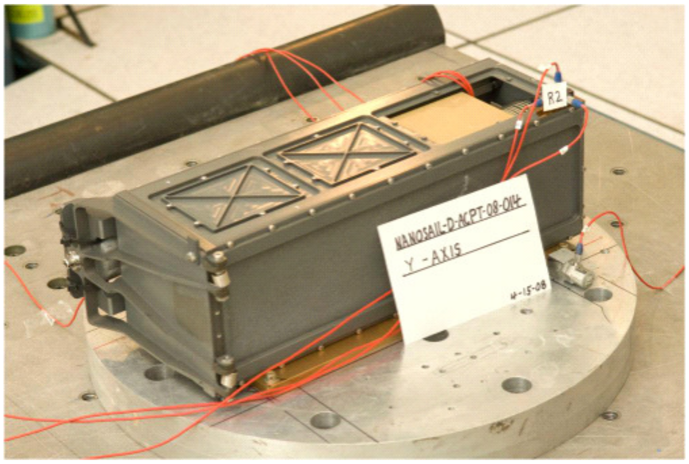


Destructive Interference (Public Domain image by Inductiveload, http://commons.wikimedia.org/wiki/File:Destructive_interference1.svg)

When the two pulses meet, they "cancel each other out." But do they go away? No! the energy is still there, the string segment motions have just summed vectorially to zero, the energy carried by each wave is still there. We have a few more definitions. The type of interference we have just seen is the first

1. *Destructive Interference*: Interference between waves when the displacements caused by the two waves are opposite in direction
2. *Constructive Interference*: interference between waves when the displacements caused by the two waves are in the same direction

The combination of waves is important for both scientists and engineers. In engineering this is the hart of vibrometry.



Marshall and Cal Poly technicians wired the NanoSail-D spacecraft to accelerometers, instruments which measure vibration response during simulated launch conditions. Image couracy NASA, image in the Public Domain.

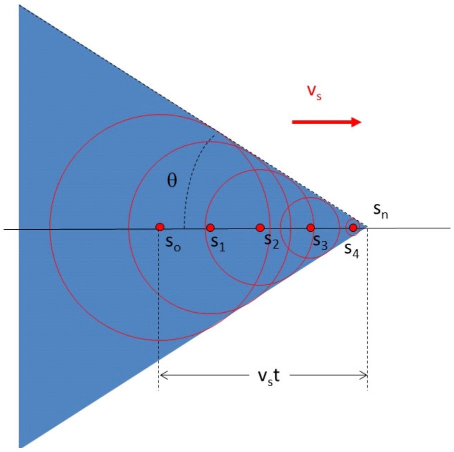
Mechanical systems have moving parts. These moving parts can be the disturbance that creates a wave. If more than one wave crest arrives at a location in the device, the amplitude at that location could become large. The oscillation of this part of the device could rattle apart welds or bolts, destroying the device. Later, as we study spectroscopy, we will see how to diagnose such a problem and hint at how to correct it.

Shock Waves

What happens when the speed of the source is greater than the wave speed?

Remember that the wave speed depends only on the medium. Let's call the crests of a wave the *wave front*. In the picture below, a point source is generating a wave and the red lines are the wave fronts.

When $v_e = v_{sound}$ the waves begin to pile up. If we allow $v_e > v_{sound}$ then the wave fronts are no longer generated within each other.



The leading edge of the wave fronts “build up” to form a cone shape. We recognize this as a superposition of the waves. This is constructive interference. The half angle of this cone is called the *Mach angle*

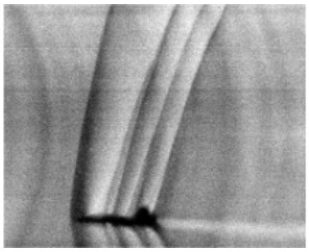
$$\sin\theta = \frac{v_{sound}t}{v_e t} = \frac{v_{sound}}{v_e} \tag{6.22}$$

This ration v_{sound}/v_e is called the Mach number and the conical wave front is called a shock wave. We see them often in water

Boat wakes as a Doppler cone. Image courtacy US Navy. Image is in the Public Domain.

and hear them when jet aircraft go supersonic. In the next figure we can see a picture of a T-38 breaking the sound barrier. You can see the Mach cones, but notice that there are several! Remember that a disturbance creates a wave. There are disturbances created by the nose of the plane, the rudder, and the wings, and perhaps the cockpit in this Schlieren photograph.

Dopler Movie



Dr. Leonard Weinstein's Schlieren photograph of a T-38 Talon at Mach 1.1, altitude 13,700 feet, taken at NASA Langley Research Center, Wallops in 1993. Image Courtacy NASA, image is in the Public Domain.

Superposition is the basis of making music and we will see how this works in our next lecture.

7 Standing Waves

Reading Assignment 21.2, 21.3

Fundamental Concepts

Mathematical Description of Superposition

We know what superposition is, but we don't really want to add values for millions of points in a medium to find out what a combination of waves will look like. At the very least, we want to make a computer do that (and programs like OpenFoam do something very akin to this!). But where we can, we would like to combine wave functions algebraically. Let's see how this can work.

Lets define two wave functions

$$y_1 = y_{\max} \sin(kx - \omega t)$$

and

$$y_2 = y_{\max} \sin(kx - \omega t + \phi_o)$$

These are two waves with the same frequency and wave number traveling the same direction in the medium, but they at $t = 0$ the y values are not the same because of the phase shift. The graph of y_2 is shifted by an amount ϕ_o.

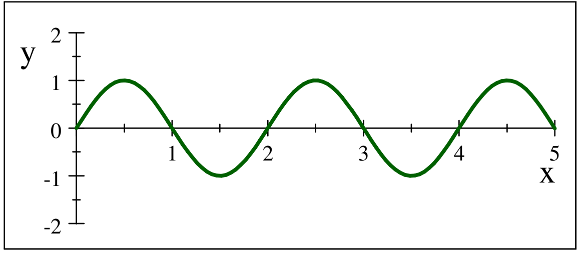
I will pick some values for the constants

$$\begin{aligned} &\lambda = 2 \\ &k = \tfrac{2\pi}{\lambda} \\ &\omega = 1 \\ &\phi_o = \tfrac{\pi}{6} \\ &t = 0 \\ &A = 1 \end{aligned}$$

then for y_1 we have

$$\begin{aligned} y_1 &= (1)\sin\left(\frac{2\pi}{\lambda}x - (1)\,t\right) \\ &= \sin\left(\frac{2\pi}{2}x - (1)\,t\right) \\ &= \sin(\pi x - t) \end{aligned}$$

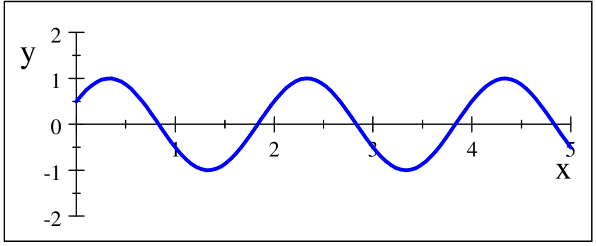
here is a plot of the wave function, y_1



Now let's consider $y_{2.}$ Using the values we chose, y_2 can be written as

$$\begin{aligned} y_2 &= y_{\max}\sin(kx - \omega t + \phi_o) \\ &= \sin\left(\pi x - t + \frac{\pi}{6}\right) \end{aligned}$$

which looks like this



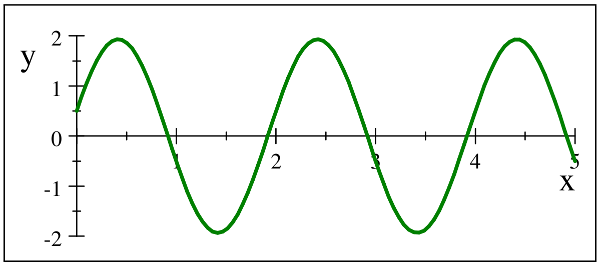
What does it look like if we add these waves using superposition? Symbolically we have

$$y_r = y_{\max}\sin(kx - \omega t) + y_{\max}\sin(kx - \omega t + \phi_o) \tag{7.1}$$

and putting in the numbers gives

$$y_r = \sin(\pi x - t) + \sin\left(\pi x - t + \frac{\pi}{6}\right)$$

which is shown in the next graph.



Notice that the wave form is taller (larger amplitude). Noticed it is shifted along the x axis.

We can find out by how much by rewriting y_r. We need a trig identity

$$\sin a+\sin b=2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

Then let $a=kx-\omega t$ and $b=kx-\omega t+\phi$

$$\begin{aligned}
y_r &= y_{\max}\sin(kx-\omega t)+A\sin(kx-\omega t+\phi_o)\\
&= 2y_{\max}\cos\left(\frac{(kx-\omega t)-(kx-\omega t+\phi_o)}{2}\right)\sin\left(\frac{(kx-\omega t)+(kx-\omega t+\phi_o)}{2}\right)\\
&= 2y_{\max}\cos\left(\frac{-\phi_o}{2}\right)\sin\left(\frac{2kx-2\omega t+\phi_o}{2}\right)\\
&= 2y_{\max}\cos\left(\frac{-\phi_o}{2}\right)\sin\left(kx-\omega t+\frac{\phi_o}{2}\right)\\
&= 2y_{\max}\cos\left(\frac{\phi_o}{2}\right)\sin\left(kx-\omega t+\frac{\phi_o}{2}\right)
\end{aligned}$$

where we use the fact that $\cos(-\theta)=\cos(\theta)$.

Let's look at the parts of this expression. First take the sine part.

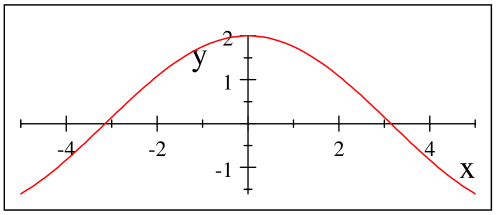
$$\sin\left(kx-\omega t+\frac{\phi_o}{2}\right) \tag{7.2}$$

This part is a traveling wave with the same k and ω as our original waves, but it has a phase of $\phi_o/2$. So our combined wave is shifted by $\phi_o/2$ or half the phase shift of y_2.

Now let's look at other factor

$$2y_{\max}\cos\left(\frac{\phi_o}{2}\right) \tag{7.3}$$

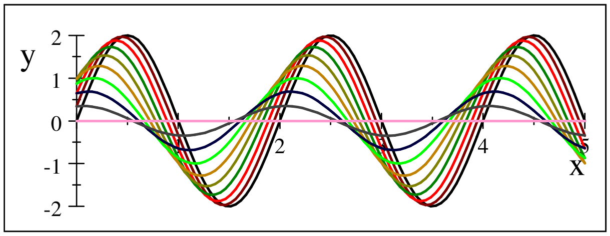
The sine part of our wave equation is multiplied by all of this factor. So all of this part is the new amplitude. It has a maximum value when $\phi_o=0$



When $\phi_o = \pi$, then

$$2y_{\max}\cos\left(\frac{\pi}{2}\right) = 0$$

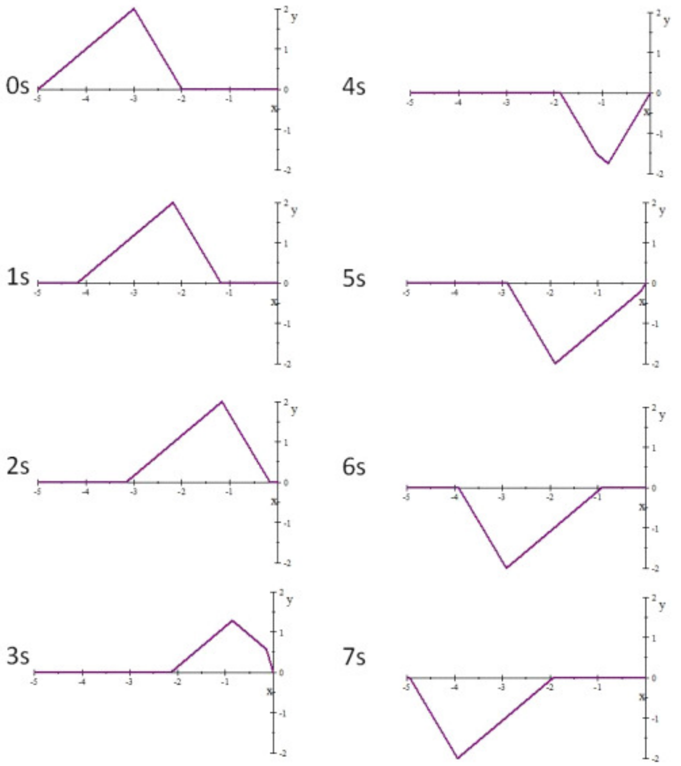
so when $\phi_o = 0$ we have a new maximum amplitude of $2y_{\max}$ and when $\phi_o = \pi$ we have a zero amplitude. Here is our wave for several choices of ϕ_o.



Reflection and Transmission

In our examples so far, we have not explained how we got two waves into a medium. One way is to simply reflect one wave back on top of itself.

In class we will made pulses on a spring with one end of the rope fixed (held by a class member). What happened when the pulse reached the end of the rope?

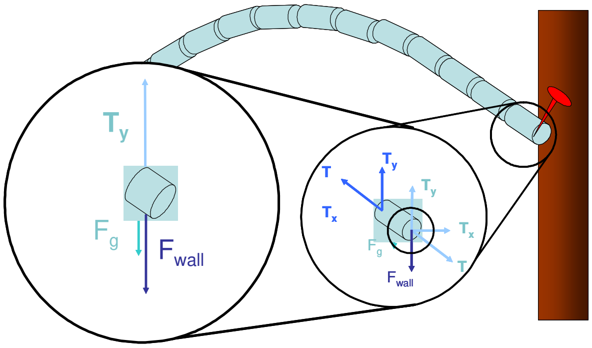


Case I: Fixed rope end.

There is a big change in the medium at the end of the rope. The rope ends. There is a person or (as in the next figure) some thing holding the rope in place. This change in medium causes a reflection.

In the fixed end case, the pulse is inverted. Why?

In the next figure I have envisioned a rope made of small rope segments.



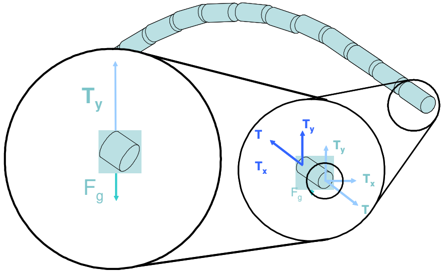
The end of the rope pushes up on the support (person in class, or nail in the figure). By Newton's third law there must be a normal force downward on the rope end. Compressing the molecules in the support stores potential energy in those compressed atoms. They will release that potential energy and create kinetic energy in the rope end. That kinetic energy will have a rope end velocity

$$K = \frac{1}{2}mv^2$$

and that velocity will be downward. This will pull the rope down, inverting the wave.

But what happens if the rope end is not fixed?

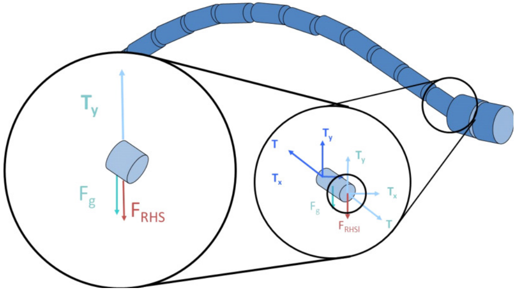
The rope end rises, and therefore there is no force exerted. The pulse (or at least part of the pulse energy) is still reflected, but there is no inversion because there was not downward force or no stored potential energy in the support!



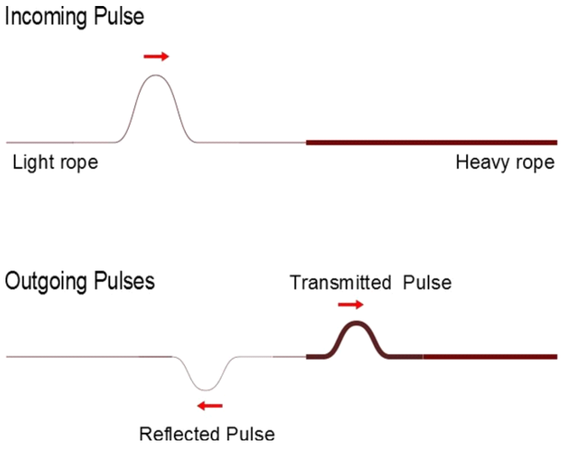
Case III: Partially attached rope end

Now lets tie the rope to another rope that is larger, more dense, than the rope we have been using, what will happen?

The light end of the rope exerts a force on the heavy beginning of the new rope. The atoms in the heavy rope will be pulled and compressed. The heavy rope will move, but the heaviness of the rope will prevent them from going very far. The fibers on the end of the heavy rope will build up potential energy because their bonds will be compressed. They will push downward on the light end of the rope. Once again the potential energy from the compressed heavy rope atoms will transfer to kinetic energy in the light rope end. Once again that light rope end will move downward.



We expect part of the energy to reflect back along the light rope, and this pulse will be inverted. Notice that the heavy rope did move upward a little, so there will also be a pulse on the heavy rope. Since this pulse formed from the light rope pulling up the heavy rope, and the light rope atoms were not compressed, this pulse will not be inverted.



Mathematical description of standing waves

Now that we have a way to make two waves to superimpose, we can study the special case of standing waves.

Standing Wave Demo

A standing wave pattern is the result of the superposition of two traveling waves with the same frequency going in opposite directions

$$\begin{aligned} y_1 &= y_{\max}\sin\left(kx-\omega t\right) \\ y_2 &= y_{\max}\sin\left(kx+\omega t\right) \end{aligned}$$

The sum is

$$y_r = y_1 + y_2 = y_{\max}\sin\left(kx-\omega t\right) + A\sin\left(kx+\omega t\right)$$

To gain insight into what these two waves produce, we use another of our favorite trig identities

$$\sin\left(a\pm b\right) = \sin\left(a\right)\cos\left(b\right) \pm \cos\left(a\right)\sin\left(b\right)$$

to get

$$\begin{aligned} y_r &= y_{\max}\sin\left(kx-\omega t\right) + y_{\max}\sin\left(kx+\omega t\right) \\ &= y_{\max}\sin\left(kx\right)\cos\left(\omega t\right) - y_{\max}\cos\left(kx\right)\sin\left(\omega t\right) + y_{\max}\sin\left(kx\right)\cos\left(\omega t\right) + y_{\max}\cos\left(kx\right)\sin\left(\omega t\right) \\ &= 2y_{\max}\sin\left(kx\right)\cos\left(\omega t\right) \\ &= \left(2y_{\max}\sin\left(kx\right)\right)\cos\left(\omega t\right) \end{aligned}$$

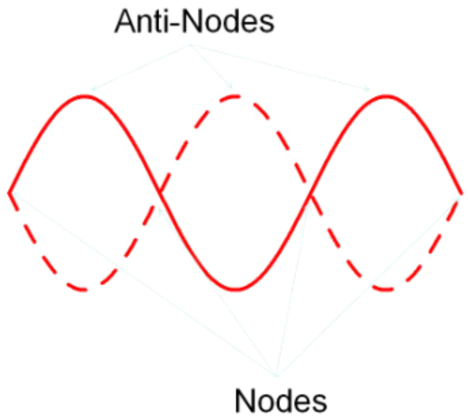
This looks like the harmonic oscillator equation

$$y = y_{\max}\cos\left(\omega t + \phi_o\right)$$

with $\phi_o = 0$ and with another complicated amplitude

$$2y_{\max}\sin\left(kx\right)$$

That is, we have a set of harmonic oscillators who's amplitude is different for each value of x.



We can identify spots along the x axis where the amplitude is always zero! we will call these spots *nodes*. These happen when $\sin(kx) = 0$ or when

$$kx = n\pi$$

By using

$$k = \frac{2\pi}{\lambda}$$

we have

$$\begin{aligned} \frac{2\pi}{\lambda}x &= n\pi \\ \frac{2}{\lambda}x &= n \\ x &= n\frac{\lambda}{2} \end{aligned}$$

We can also find the places along x where the amplitude will be largest. this occurs when $\sin(kx) = 1$ or when

$$kx = n\frac{\pi}{2}$$

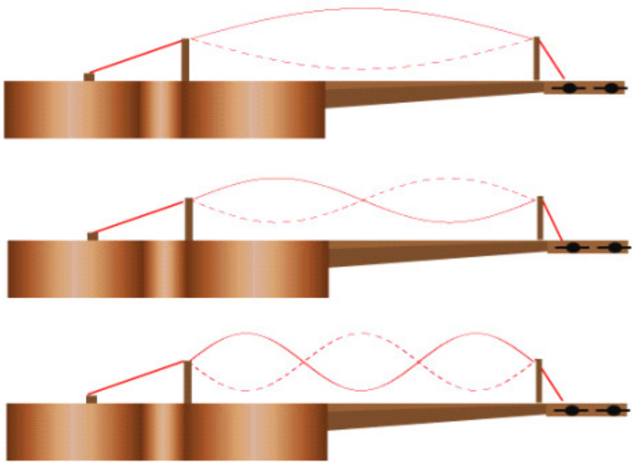
or

$$\begin{aligned} \frac{2\pi}{\lambda}x &= n\frac{\pi}{2} \\ x &= n\frac{\lambda}{4} \end{aligned}$$

these are called *antinodes*.

We can also create standing waves with sound or even light waves!

Standing Waves in a String Fixed at Both Ends



If we attach a string to something on both ends, we find something interesting in the standing wave pattern. Not all possible standing waves can be realized. Some frequencies are preferred, and some never show up. The standing wave pattern is *quantized*. The patterns that are allowed are called *normal modes*. We will see this any time a wave confined by boundary conditions (light in a resonant cavity, radio waves in a wave guide, electrons in an atom, etc.).

The figure shows some normal modes for a string.

We find which modes are allowed by first imposing the boundary condition that each end must be a node. We start with

$$y = 2y_{\max} \sin(kx) \cos(\omega t)$$

and recognize that we have one condition met because

$$y = 0$$

when

$$x = 0$$

we need $y = 0$ when $x = L$. That happens when

$$kL = n\pi$$

I will write this as

$$k_n L = n\pi$$

to indicate there are many values of k that can work. Solving this for λ_n gives

$$\begin{aligned} \frac{2\pi}{\lambda_n} L &= n\pi \\ \frac{2L}{n} &= \lambda_n \end{aligned}$$

Which says there are many wavelengths that will work. Let's see how this works, the first mode will have

$$\lambda_1 = 2L$$

where L is the length of the string. Looking at the figure, this works.

The second mode has three nodes (one on each end and one in the middle). This gives

$$\lambda_2 = L$$

We can keep going, the third mode will have five nodes

$$\lambda_3 = \frac{3L}{2}$$

and so forth to give

$$\lambda_n = \frac{2L}{n}$$

We use our old friend

$$v = f\lambda$$

to find the frequencies of the modes

$$f = \frac{v}{\lambda}$$

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

or, in general

$$\begin{aligned} f_n &= \frac{v}{\lambda_n} = n\frac{v}{2L} \\ &= \frac{n}{2L}v \\ &= \frac{n}{2L}\sqrt{\frac{T}{\mu}} \end{aligned}$$

The lowest frequency has a special name, the *fundamental frequency*. The higher frequencies are integer multiples of the fundamental. When this happens we say that the frequencies form a *harmonic series,* and the modes are called *harmonics*. Since only certain frequencies work, we say that the frequencies of waves on the string that make standing waves are *quantized*! Quantization means that some values are allowed. This idea is the basis behind Quantum mechanics (which views light and even matter as waves).

Musical Strings

So how do we get different notes on a guitar or Piano?

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \tag{7.4}$$

A guitar uses tension to change the frequency or pitch (tuning) and length of string (your fingers pressing on the strings) to change notes. A Piano uses both tension and length of string (and mass per unit length as well!). What do you expect and organ will do?

8

Light and Sound Standing waves

Fundamental Concepts

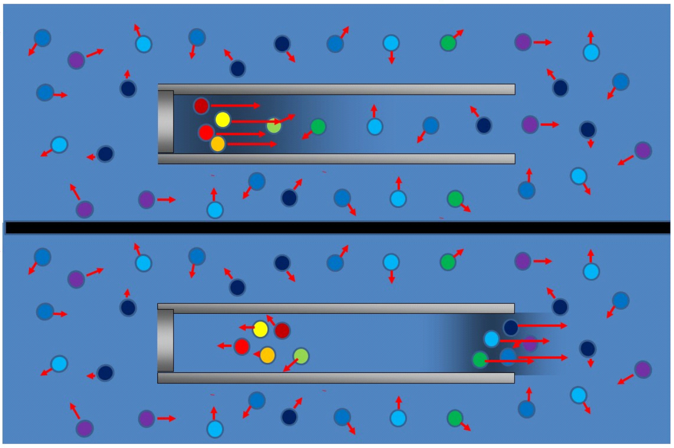
- Sound Standing waves (music)

Sound Standing waves.

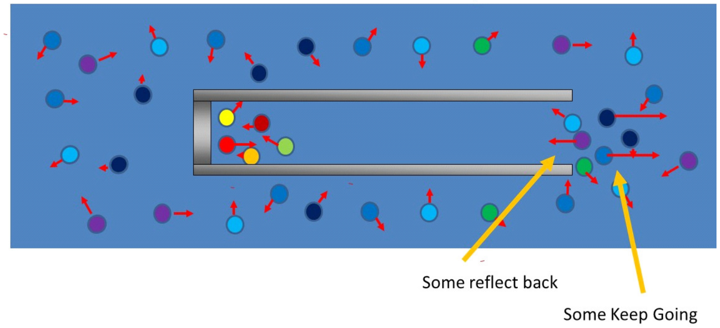
We have spent some time studying standing waves on strings. But we also have studied sound waves, could we make sound standing waves?

For our attempt, let's take a pipe as shown in the next figures. The pipe will control the direction of the sound wave. The sound wave will (mostly) go down the pipe. But will there be a reflection at the end of the pipe like there was for the string?

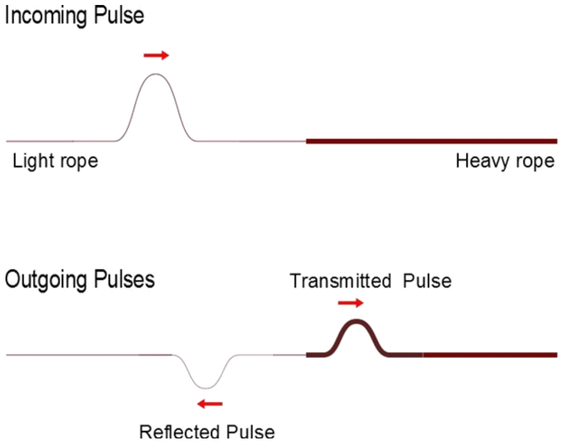
It turns out that there will be a reflection. Let's consider our sound wave as a change in pressure to see why. For the string wave, we had a reflected pulse because the person (or guitar bridge, or whatever is holding the string end) exerted a force on the string. If a sound wave travels down our pipe, the pressure will change as the wave goes down the pipe. Remember that pressure is a force spread over an area. When the sound wave gets to the end of the pipe, the rest of the room's air is waiting. And that air mass pushes back on the air molecules that are trying to leave the pipe. The air mass of the room won't change it's pressure much. That resistance to pressure change (the force due to the rest of the room air molecules colliding with our wave molecules, pushing them back) will send our tube molecules back down the tube. And that starts a reflection. Thin about this for a moment. We need to picture more than we have before. The room is full of air, and it turns out that the air molecules aren't sitting still. So when we make our wave start but hitting molecules with a piston, the molecules at the end of the tube will hit room air molecules that are already moving.



Think of pool balls. If the cue ball and another ball, say, the 6 ball, collide, usually the 6 ball is stationary at the beginning. But if it is not, that will change the outcome of our conservation of momentum problem. if the 6 ball is initially moving toward the cue ball, the cue ball will bounce back the way it came. That will happen with some of the molecules at the end of the pipe. But some will keep going.

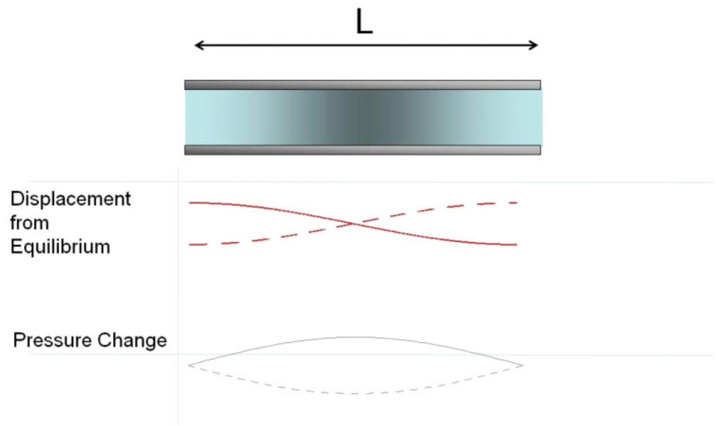


We will split the energy from the wave into to waves, much like we did with waves on strings when we came to an interface.



Now that we have a reflected wave in the tube, we can end up with two waves so long as the piston keeps working at making new waves. And with two waves in the same medium, we have the possibility of having standing waves!

If we have a pipe open at both ends, we can see that air molecules are free to move in and out of the ends of the pipes. If the air molecules can move, the ends must not be nodes. In our string case, the part of the string that experienced destructive interference and did not move was called a node and we always had a node on the end of the string. But the pipe is different than the string case! We can see that air molecules will move out and then bounce back in at the end of the pipe. Still, we expect that there must be a node somewhere. We can reasonably guess that there will be a node in the middle of the pipe. Of course, the pressure on both ends must be atmospheric pressure. So, remembering that pressure and displacement are $90\,^{\circ}$ out of phase for sound waves, we can guess that there are pressure nodes on both ends. But there is a displacement anti-node at the ends of the pipe.



Then for the first harmonic we can draw a displacement node in the middle and we see that

$$\lambda_1 = 2L \tag{8.1}$$

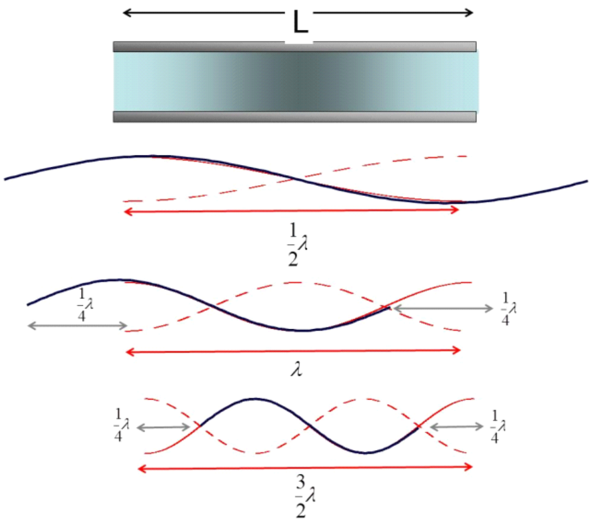
and

$$f_1 = \frac{v}{2L} \tag{8.2}$$

The next mode fits a whole wavelength

$$\lambda_2 = L \tag{8.3}$$

$$f_2 = \frac{v}{L} \tag{8.4}$$



but the next mode fits a wavelength and a half

$$\lambda_3 = \frac{3}{2}L \tag{8.5}$$

$$f_3 = \frac{2v}{3L} \tag{8.6}$$

If we keep going

$$\lambda_n = \frac{2}{n}L \tag{8.7}$$

$$f_n = n\frac{v}{2L} \qquad n = 1, 2, 3, 4 \ldots \tag{8.8}$$

This is the same mathematical form that we achieved for a standing wave on a string!

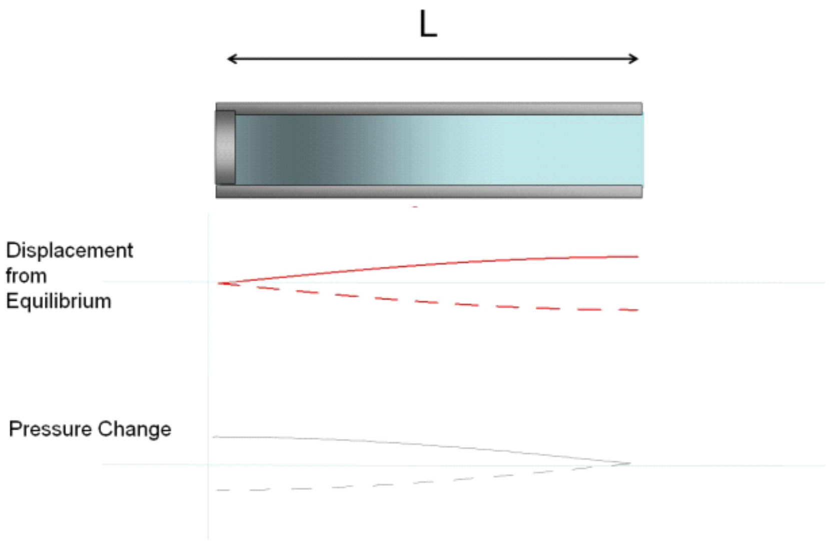
Pipes closed on one end

Boom Whacker Demo

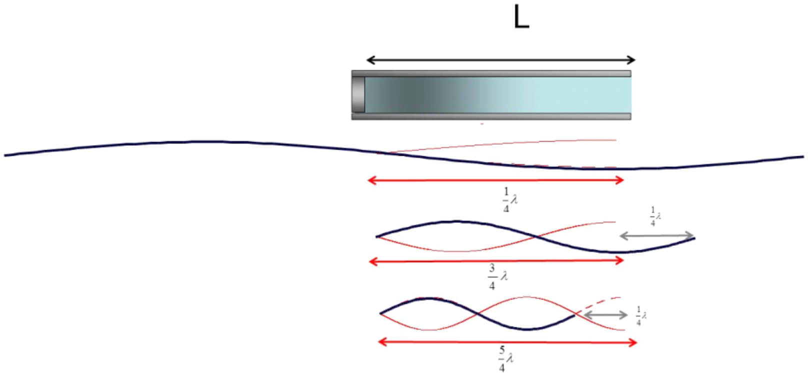
But what happens if we put a cap on one end of the pipe? The air molecules cannot move longitudinally once they hit the cap. And where the molecules don't move, that is a node. So our capped end must be a displacement node.

Question 123.24.2

The open end is still a displacement antinode. Our standing wave will have a node on the capped side, and an anti-node on the open side. It might look like this



In the next figure we draw the first few harmonics for the "closed on one end" case.



The first harmonic for the closed pipe are found by using

$$v = \lambda f \tag{8.9}$$

$$f = \frac{v}{\lambda} \tag{8.10}$$

We know the speed of sound, so we have v. Knowing that the first harmonic has a node at one end and an anti node at the other end gives us the wavelength. If the pipe is L in length, then L must be

$$L = \frac{1}{4}\lambda_1 \tag{8.11}$$

or

$$\lambda_1 = 4L \tag{8.12}$$

then

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{4L} \tag{8.13}$$

The next configuration that will have a node on one end and an antinode on the other will have

$$L = \frac{3}{4}\lambda_2 \tag{8.14}$$

which gives

$$\lambda_2 = \frac{4}{3L} \tag{8.15}$$

and

$$f_2 = \frac{v}{\lambda_2} = \frac{3v}{4L} \tag{8.16}$$

If we continued, we would find

$$\lambda_n = \frac{4}{n}L \tag{8.17}$$

and

$$f_n = n\frac{v}{4L} \qquad n = 1, 3, 5 \ldots \tag{8.18}$$

Example: organ pipe

Organ Pipe Demo

The organ pipe shown is closed at one end so we expect

$$f_n = n\frac{v}{4L} \qquad n = 1, 3, 5 \ldots \tag{8.19}$$

Measuring the pipe, and assuming about $20\,^{\circ}\mathrm{C}$ for the room temperature we have

$$\begin{aligned} L &= 0.41\,\mathrm{m} \\ R &= 0.06\,\mathrm{m} \\ v &= 343\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned} \tag{8.20}$$

There is a detail we have ignored in our analysis, the width of the pipe matters a little. I will include a fudge factor to account for this. With the fudge factor, the wavelength is

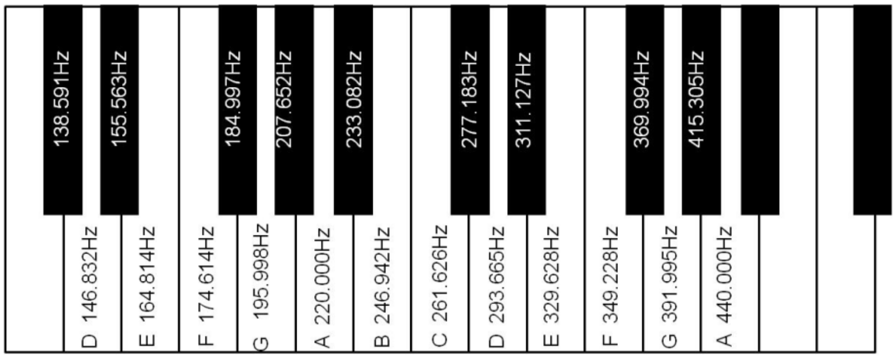
$$\begin{aligned} \lambda_1 &= 4\,(L + 0.6R) \\ &= 178.\,4\,\mathrm{cm} \end{aligned}$$

then our fundamental frequency is

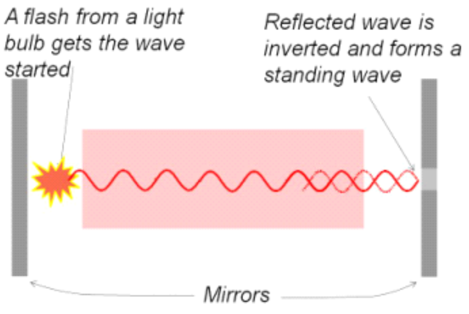
$$f_1 = \frac{v}{\lambda_1} \tag{8.21}$$

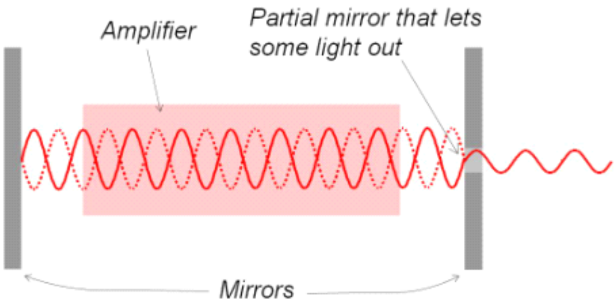
$$= 192.\,26 \tag{8.22}$$

We can identify this note, and compare to a standard, like a tuning fork or a piano to verify our prediction.



Lasers and standing waves



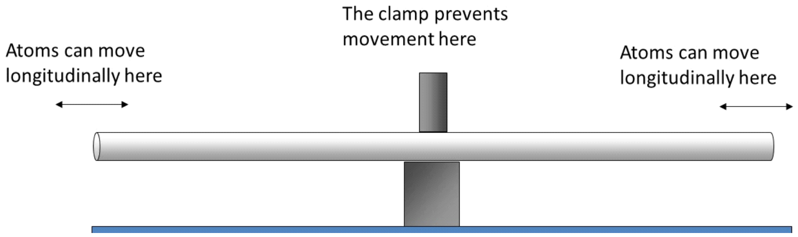


Question 123.24.3

Question 123.24.4

Standing Waves in Rods and Membranes

We have hinted all chapter that the analysis techniques we were building apply to structures. We need more math and computational tools to analyze complex structures like bridges and buildings, but we can tackle a simple structure like a rod that is clamped. The atoms in the rod can vibrate longitudinally

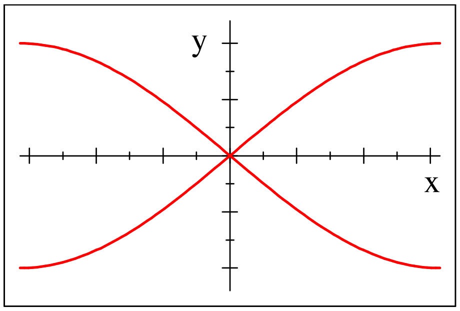


Since we have motion possible on both ends and not in the middle, we surmise that this system will have similar solutions as did the open ended pipe.

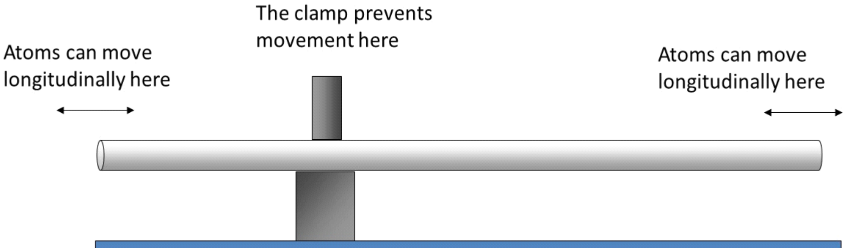
$$f_n = n\frac{v}{2L}$$

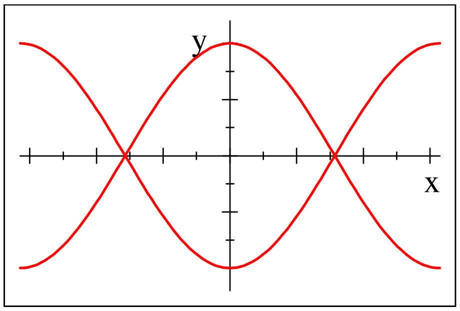
The fundamental looks like

$\sin(x)$



But suppose we move the clamp. The clamp forces a node where it is placed. If we place the clap at $L/4$





We can perform a similar analysis for a drum head, but it is much more complicated. The modes are not points, but lines or curves, and the frequencies of oscillation are not integer multiples of each other. See for example http://physics.usask.ca/~hirose/ep225/animation/drum/anim-drum.htm.

Of course structures can also waggle on the ends. the ends can rotate counter to each other, etc. These are more complex modes than the longitudinal modes we have considered.

Single Frequency Interference in one dimension

Two speaker demo

We have seen interference from superimposing two waves in the same medium. Whether we get constructive or destructive interference really depends on the total phase difference. Let's review this, but let's make our two waves as general as possible.

$$\begin{aligned} y_1 &= y_{\max}\sin\left(k_1x_1-\omega_1t_1+\phi_1\right) \\ y_2 &= y_{\max}\sin\left(k_2x_2-\omega_2t_2+\phi_2\right) \end{aligned}$$

where we have allowed the wavelengths to be different (so then the wave numbers, k_1 and k_2 could be different), the distances traveled by the waves could be different (x_2 might not be equal to x_1), the frequencies might be different, the times observed might be different, and the phase constants might be different. We have not made the amplitudes different (but it is likely to happen in an upcoming homework problem).

Then the resulting wave would be

$$\begin{aligned} y_r &= y_{\max}\sin\left(k_2x_2-\omega_2t_2+\phi_2\right)+A\sin\left(k_1x_1-\omega_1t_1+\phi_1\right) \\ &= 2y_{\max}\cos\left(\frac{(k_2x_2-\omega_2t_2+\phi_2)-(k_1x_1-\omega_1t_1+\phi_1)}{2}\right) \\ &\quad\times\sin\left(\frac{(k_2x_2-\omega_2t_2+\phi_2)+(k_1x_1-\omega_1t_1+\phi_1)}{2}\right) \end{aligned}$$

just as before. We can rewire this as

$$\begin{aligned} y_r &= 2y_{\max}\cos\left(\frac{1}{2}\left[(k_2x_2-\omega_2t_2+\phi_2)-(k_1x_1-\omega_1t_1+\phi_1)\right]\right) \\ &\quad\times\sin\left(\frac{(k_2x_2-\omega_2t_2+\phi_2)+(k_1x_1-\omega_1t_1+\phi_1)}{2}\right) \end{aligned}$$

and recognize that the sine part is still a function of position and time (just a complicated one) so it must be a wave. The cosine part must be part of the amplitude. Let's write out this amplitude

$$A=2y_{\max}\cos\left(\frac{1}{2}\left[(k_2x_2-\omega_2t_2+\phi_2)-(k_1x_1-\omega_1t_1+\phi_1)\right]\right)$$

The part in square brackets is what we have been calling the phase difference

$$\Delta\phi = (k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1)$$

Notice that to make constructive interference we want the amplitude to be as big as possible. This happens when the cosine is ±1. We can find when this happens by recalling that cosine is ±1 when the argument of the cosine is a multiple of π. Notice that the amplitude has the cosine of half the phase difference

$$A = 2y_{\max}\cos\left(\frac{1}{2}\Delta\phi\right)$$

so

$$\frac{1}{2}\Delta\phi = \pi m \qquad m = 0, \pm1, \pm2, \pm3, \cdots$$

or

$$\Delta\phi = 2\pi m \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Constructive}$$

For total destructive interference we want the amplitude to be zero. To achieve this the cosine must be zero and this happens at odd multiples of $\pi/2$.

$$\frac{1}{2}\Delta\phi = \frac{\pi}{2}m \qquad m = \pm1, \pm3, \pm5, \cdots$$

Then we could say that our phase difference

$$\Delta\phi = \pi m \qquad m = \pm1, \pm3, \pm5, \cdots$$

another way to say this is to allow m to be any integer but to write the condition as

$$\Delta\phi = \left(m + \frac{1}{2}\right)2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive}$$

Let's write these out in detail

$$\begin{aligned}
[(k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1)] &= 2\pi m \qquad m = 0, \pm1, \pm2, \pm3, \cdots \\
[(k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1)] &= \left(m + \frac{1}{2}\right)2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots
\end{aligned}$$

We can see that there are at least four sources of phase difference here. We can change the phase difference by changing wavelength ($k_2 \neq k_1$), changing frequency ($\omega_2 \neq \omega_1$), having the waves travel different distances ($x_2 \neq x_1$), having different phase constants ($\phi_2 \neq \phi_1$) or even consider somehow mixing the waves at different times ($t_2 \neq t_1$). This last one is not as interesting, we want our waves mixed together at the same time so let's set

$$t_2 = t_1 = t$$

but let's let any of the other variables be changeable. This gives us many ways to produce constructive or destructive interference. In our next lecture we will use this analysis to solve specific problems. The pattern will be to use our two criteria

$$\begin{aligned}\Delta\phi &= 2\pi m \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Constructive}\\ \Delta\phi &= \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive}\end{aligned}$$

to determine if we have constructive, destructive, or partially constructive or destructive interference.

9 Single Frequency Interference, Multiple Dimensions

Fundamental Concepts

- When two waves interfere we get a new wave with a more complicated amplitude
- Thin films can cause reflections that create interference
- There may be a phase shift when a wave reflects (the wave may invert-this is a review)
- If two waves are out of phase by half a wavelength we have total destructive interference (review)
- If two waves are out of phase by a full wavelength we have total constructive interference (review)
- Other phases provide partial constructive or partial destructive interference
- If two waves superimpose but the waves have different frequencies, we have beating.

Mathematical treatment of single frequency interference

It is time to put our treatment of interference on a general mathematical footing. You know that the phase difference, $\Delta\phi$, is the key factor in determining whether we will have constructive or destructive interference. Here is our equation from last lecture for the amplitude of two mixed waves

$$A = 2y_{\max}\cos\left(\frac{1}{2}\left[(k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1)\right]\right)$$

We defined

$$\Delta\phi = (k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1)$$

and found conditions for constructive and destructive interference.

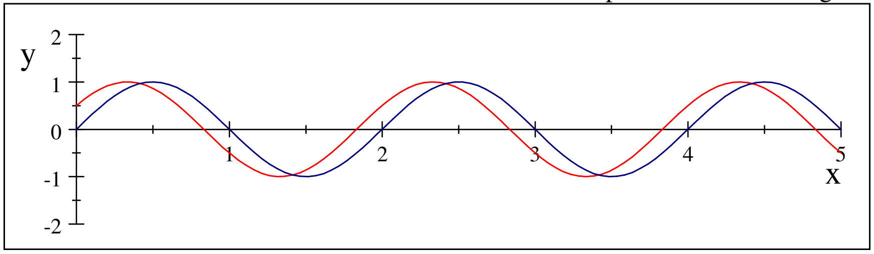
$$\begin{aligned} \Delta\phi &= 2\pi m \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Constructive} \\ \Delta\phi &= \left(m + \frac{1}{2}\right)2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive} \end{aligned}$$

Let’s try a problem using these ideas.

Suppose we have the following two waves.

$$\begin{aligned} y_1 &= \sin\left(\pi x - \omega t + \frac{1}{6}\pi\right) \\ y_2 &= \sin\left(\pi x - \omega t\right) \end{aligned}$$

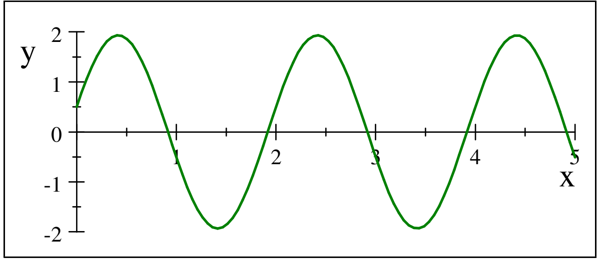
We want to find the equation of the resultant wave and to determine if we have constructive or destructive interference. The two waves are plotted in the next figure.



We can identify that $y_{\max} = 1\,\mathrm{m}$, $k_1 = k_2 = \pi$, $\omega_1 = \omega_2 = \omega$, $t_1 = t_2 = t$, $\phi_1 = \pi/6$ and $\phi_2 = 0$. We have this for all x so we can chose $x_1 = x_2 = x$ for all x. The superposition yields.

$$\begin{aligned} y_r &= y_{\max}\sin\left(k_2x_2 - \omega_2t_2 + \phi_2\right) + A\sin\left(k_1x_1 - \omega_1t_1 + \phi_1\right) \\ &= 2y_{\max}\cos\left(\left(\frac{1}{2}\left[\left(k_2x_2 - \omega_2t_2 + \phi_2\right) - \left(k_1x_1 - \omega_1t_1 + \phi_1\right)\right]\right)\right) \\ &\quad \times\sin\left(\frac{\left(k_1x_1 - \omega t + \phi_2\right) + \left(k_1x_2 - \omega t + \phi_1\right)}{2}\right) \\ &= 2y_{\max}\cos\left(\frac{1}{2}\left[\left(\pi x - \omega t + 0\right) - \left(\pi x - \omega t + \frac{\pi}{6}\right)\right]\right) \\ &\quad \times\sin\left(\pi x - \omega t + \frac{0 - \frac{\pi}{6}}{2}\right) \\ &= 2\cos\left(-\frac{\pi}{12}\right)\sin\left(\pi x - \omega t + \frac{\pi}{12}\right) \end{aligned}$$

Our specific case is graphed in the next figure.



Notice that the wave form is taller (larger amplitude). Noticed it is shifted along the

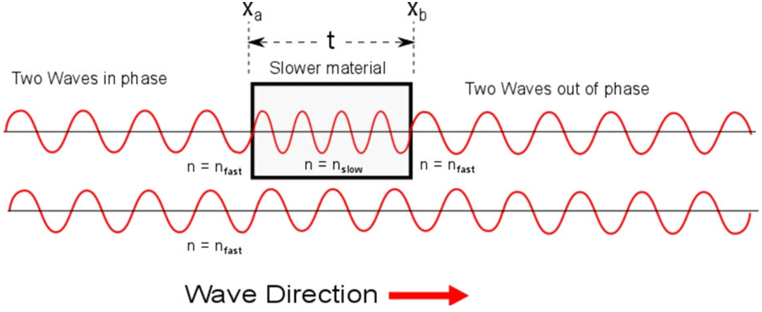
x axis. This graph is not surprising to us now, because we have done a case like this before. The shirt will be $\phi_R = \pi/12$. Let's test this for constructive or destructive interference. We could use our conditions that we developed, but we can also just evaluate the amplitude.

$$2\cos\left(-\frac{\pi}{12}\right) = 1.931\,9$$

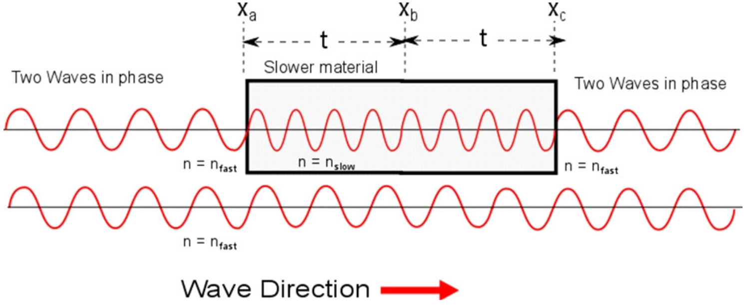
This is bigger than $1\,\mathrm{m}$, which is our individual wave amplitude. It is a little smaller than $2\,\mathrm{m}$ which would be total constructive interference. We would call this partial constructive interference.

Let's do another example problem, a harder one this time.

Consider two waves traveling to the right. One passes through a slower medium. We expect that the frequency won't change ($\omega_2 = \omega_1$), but the wavelength will shorten for the wave that enters the slower material ($k_2 \neq k_1$). The two waves approach point x_a with the same phase, but by point x_b they are out of phase.



If the thickness ($\mathsf{t} = x_b - x_a$) were larger ($\mathsf{t} = x_c - x_a$) we might be back in phase. This picture shows twice the original thickness and we see we have constructive interference after the slower material.



Using this setup, let's find the smallest thickness of material that can cause constructive interference. The thickness of the slower material will be ($\mathsf{t} = x_c - x_a$)

This time we want constructive interference, so we can use our criteria

$$\begin{aligned}\Delta\phi &= 2\pi m \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Constructive}\\ \Delta\phi &= \left(mn + \frac{1}{2}\right)2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive}\end{aligned}$$

choosing the first for the constructive case. We know

$$\Delta\phi = (k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1)$$

and we can put in what we know for this specific problem. We know the $\omega's$ are the same. We want the waves to be at the same place x_b at the same time. They start at point x_a with the same phase constant $\phi_2 = \phi_1 = \phi_o$. Both waves travel the distance $x_b - x_a$ as they go from position x_a to x_b so

$$\begin{aligned}\Delta\phi &= (k_2(x_b - x_a) - \omega t + \phi_0) - (k_1(x_b - x_a) - \omega t + \phi_o)\\ &= k_2(x_b - x_a) - k_1(x_b - x_a)\\ &= (x_b - x_a)\left(\frac{2\pi}{\lambda_2} - \frac{2\pi}{\lambda_1}\right)\end{aligned}$$

We know that λ_2 is different than λ_1, but how different? We can find this by remembering that the frequency does not change as we cross a material boundary. So if f_{out} is the frequency outside the slower material, then

$$f_{in} = f_{out}$$

the frequency did not change as the wave moved into the slower material. Using

$$v = f\lambda$$

or

$$f = \frac{v}{\lambda}$$

we can find an expression for the wavelength inside the slower material for wave 1.

$$\begin{aligned}f_{in} &= f_{out}\\ \frac{v_{in}}{\lambda_{in}} &= \frac{v_{out}}{\lambda_{out}}\end{aligned}$$

Let's rewrite this for our case

$$\lambda_{in} = \frac{v_{in}}{v_{out}}\lambda_{out}$$

and we can multiple top and bottom by c so

$$\begin{aligned}\lambda_{in} &= \frac{c}{c}\frac{v_{in}}{v_{out}}\lambda_{out}\\ &= \frac{\frac{c}{v_{out}}}{\frac{c}{v_{in}}}\lambda_{out}\end{aligned}$$

Remember we defined the index of refraction earlier in our course

$$n = \frac{c}{v}$$

and we can see we have the index of refraction twice in our last equation.

$$\begin{aligned}\lambda_{in} &= \frac{\frac{c}{v_{out}}}{\frac{c}{v_{in}}}\lambda_{out} \\ &= \frac{n_{out}}{n_{in}}\lambda_{out} \\ &= \frac{1}{n}\lambda_{out}\end{aligned}$$

where we used the idea that $n_{out} = n_{air} \approx 1$.

Returning to our problem, wave 1 will have $\lambda_1 = \lambda_{in}$ and wave 2 will have $\lambda_2 = \lambda_{out}$

$$\begin{aligned}\Delta\phi &= (x_b - x_a)\left(\frac{2\pi}{\lambda_{out}} - \frac{2\pi}{\frac{1}{n}\lambda_{out}}\right) \\ &= \frac{2\pi\,(x_b - x_a)}{\lambda_{out}}(1 - n)\end{aligned}$$

Now we use our condition for destructive interference

$$\Delta\phi = 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \qquad \text{Constructive}$$

and place in it our phase difference for this problem

$$\frac{2\pi\,(x_b - x_a)}{\lambda_{out}}(1 - n) = 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots$$

We can see that if the thickness of the slower material is

$$(x_b - x_a) = \mathsf{t} = \frac{\lambda_{out}}{(1 - n)}m$$

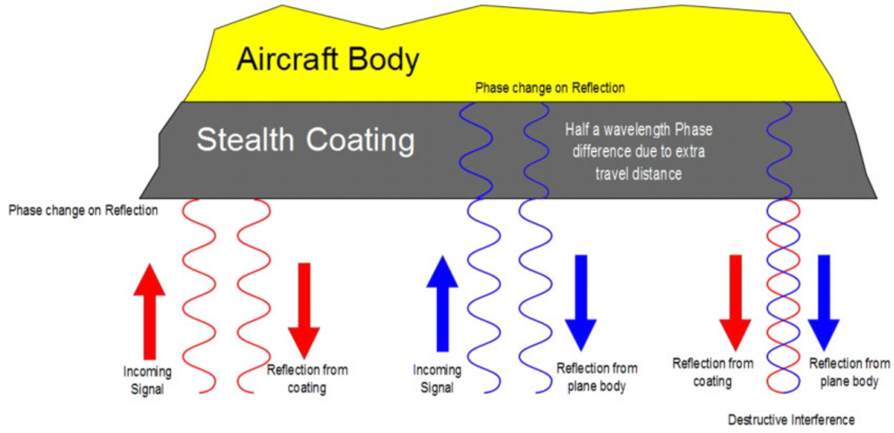
to find the smallest thankless that will work, we set $m = 1$ so

$$\mathsf{t} = \frac{\lambda_{out}}{(1 - n)}$$

we will have constructive interference.

This is the thinnest thickness that will make constructive interference.

Let’s try another problem.



The stealth fighter is coated with an anti-reflective polymer. This is part of it's mechanism for making the plane invisible to radar. Suppose we have a radar system with a wavelength of $3.00\,\mathrm{cm}$. Further suppose that the index of refraction of the anti-reflective polymer is $n = 1.50$, and that the aircraft index of refraction is very large, how thick would you make the coating?

The radar waves all hit the plane in phase. From the figure, we see that the radar wave will reflect off of the coating. Because the index of refraction of the coating is large, this is like a fixed end. There will be an inversion. Let's call this first reflected wave, wave 1 and from what we have just learned $\phi_1 = \pi$ when it leaves the surface of the polymer.

But some of the wave will penetrate the polymer. This will reflect off of the plane body. Let's call this wave 2. The plane body has a very large index of refraction, so once again the wave will experience an inversion. We expect then that $\phi_2 = \pi$. Next think of the distances traveled. Both waves get to the polymer together. Let's set our y_a right at the outside edge of the polymer. But wave 1 just bounces off. It travels no distance before it leaves $(y_1 = 0)$. Wave 2 however traves twice the thickness of the polymer before it leaves $(y_2 = 2\mathrm{t})$. Once again, inside the material, we know that the wavelength will change $(\lambda_2 \neq \lambda_1)$ but we expect the frequency to stay the same $(\omega_2 = \omega_1)$. Once again we can write our phase difference as

$$\Delta\phi = (k_2 y_2 - \omega_2 t_2 + \phi_2) - (k_1 y_1 - \omega_1 t_1 + \phi_1)$$

and put in what we know from our problem

$$\begin{aligned}\Delta\phi &= \left(\frac{2\pi}{\lambda_2}(2\mathrm{t}) - \omega t + \pi\right) - \left(\frac{2\pi}{\lambda_1}(0) - \omega t + \pi\right) \\ &= \frac{2\pi}{\lambda_2}(2\mathrm{t})\end{aligned}$$

We can force destructive interference by using the second of our interference criteria equations.

$$\begin{aligned}\Delta\phi &= 2\pi m \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Constructive}\\ \Delta\phi &= \left(m+\frac{1}{2}\right)2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots \qquad \text{Destructive}\end{aligned}$$

so that

$$\frac{2\pi}{\lambda_2}(2\mathsf{t}) = \left(m+\frac{1}{2}\right)2\pi \qquad m = 0, \pm1, \pm2, \pm3, \cdots$$

or

$$\mathsf{t} = \frac{\lambda_2}{2}\left(m+\frac{1}{2}\right)$$

But we have the same problem as in our last problem. The wave entered into a material. So the wavelength changed. We can use what we found before

$$\lambda_{in} = \frac{1}{n}\lambda_{out}$$

and identify $\lambda_2 = \lambda_{in}$ so

$$\mathsf{t} = \frac{\frac{1}{n}\lambda_{out}}{2}\left(m+\frac{1}{2}\right)$$

then for the smallest thickness that will create destructive interference we set $m = 0$

$$\mathsf{t} = \frac{\lambda_{out}}{4n_{coating}}$$

Putting in the numbers, our thickness would be

$$t = \left(\frac{1}{4}\right)\left(\frac{3.00\,\text{cm}}{1.50}\right) = 0.5\,\text{cm}$$

Half a centimeter of coating doesn't seem too unreasonable!

Of course we could also make a plane that would be more visible to radar. Suppose we are building a search and rescue plane. We could make a plane coating that gave constructive interference. We would start with

$$\Delta\phi = \left(\frac{2\pi}{\lambda_{in}}(2\mathsf{t}) + 0\right)$$

but this time we would make $\Delta\phi = m2\pi$. This would give

$$m2\pi = \left(\frac{2\pi}{\lambda_{in}}(2\mathsf{t})\right)$$

$$m2\pi = \left(\frac{2\pi}{\frac{\lambda_{air}}{n_{coating}}}(2\mathsf{t})\right)$$

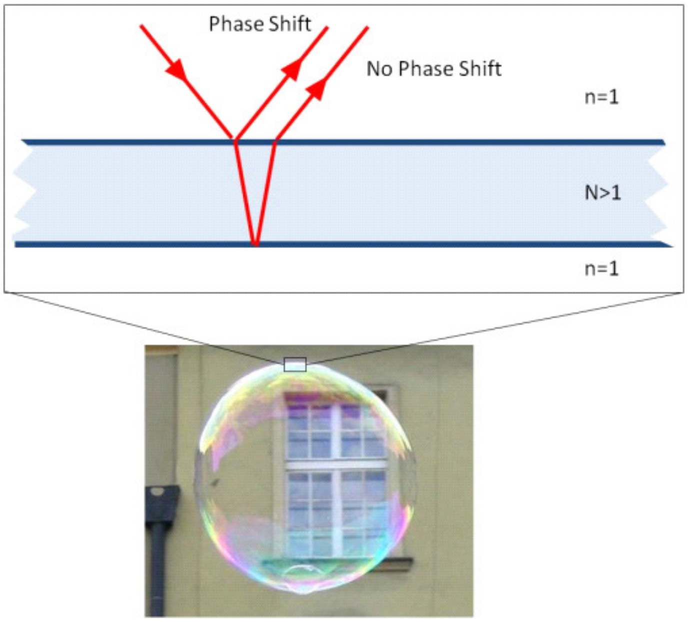
which makes a coating thickness so that

$$\mathsf{t} = \frac{m}{2}\left(\frac{\lambda_{air}}{n_{coating}}\right) \qquad m = 0, 1, 2, \cdots$$

and now the coating will provide constructive interference, making it easier to track on radar from the command center. For the thinnest possibility, set $m = 1$ and

$$\begin{aligned} t &= m\frac{1}{2}\left(\frac{\lambda_{air}}{n_{coating}}\right) \\ &= \frac{1}{2}\left(\frac{3.00\,\mathrm{cm}}{1.50}\right) \\ &= 1\,\mathrm{cm} \end{aligned}$$

Note that we reasoned out these equations for the boundary conditions that we have in our problem. If the boundary conditions change, so do the equations. Take a soap bubble for example.



Interference from a soap bubble. (Bubble image in the Public Domain, courtesy Marcin Deręgowski)

Suppose we see the nice blue color near the bottom, this must be constructive interference for blue if we can see it. So the bubble thickness must be just right to make constructive interference for blue. So $\Delta\phi = m2\pi$. Let's find the thickness. Again start with.

$$\Delta\phi = (k_2 y_2 - \omega_2 t_2 + \phi_2) - (k_1 y_1 - \omega_1 t_1 + \phi_1)$$

and again $\omega_2 = \omega_1$ and $t_2 = t_1$. We can see that one wave (wave 1) bounces off the outside of the bubble film. For this wave $\lambda_1 = \lambda_{air}$ and if we define $y_a = 0$ at this boundary than $y_1 = 0$. For wave 2, the wave travels into the bubble material so it travels a distance $y_2 \approx 2\mathrm{t}$ and there is a bounce, but this bounce will be like a free end. The wave won't invert. So $\phi_2 = 0$. Let's try to put this all into $\Delta\phi$

$$\Delta\phi = (k_2 y_2 - \omega t + 0) - (0 - \omega t + \pi)$$

Wave 2 goes into the bubble film so once again

$$\lambda_2 = \lambda_{in} = \frac{1}{n_{film}}\lambda_{out}$$

$$\begin{aligned} \Delta\phi &= \left(\frac{2\pi}{\lambda_{out}} n_{film}(2\mathrm{t}) - \omega t + 0\right) - (0 - \omega t + \pi) \\ &= \frac{2\pi}{\lambda_{out}} n_{film}(2\mathrm{t}) - \pi \end{aligned}$$

Using our criteria for constructive interference we have

$$\frac{2\pi}{\lambda_{out}} n_{film}(2\mathrm{t}) - \pi = 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots$$

Solving for the thickness

$$\begin{aligned} \frac{1}{\lambda_{out}} n_{film}(2\mathrm{t}) - \frac{1}{2} &= m \\ \frac{1}{\lambda_{out}} n_{film}(2\mathrm{t}) &= m + \frac{1}{2} \\ \mathrm{t} &= \frac{\lambda_{out}}{2n_{film}}\left(m + \frac{1}{2}\right) \end{aligned}$$

So if the thickness is

$$\mathrm{t} = \frac{\lambda_{out}}{2n_{film}}\left(m + \frac{1}{2}\right) \qquad m = 0, 1, 2, \cdots$$

There will be constructive interference. The thinnest film that will show the color due to constructive interference would be the $m = 0$ case

$$\mathrm{t} = \frac{\lambda_{out}}{4n_{film}}$$

In each of these problems we used our criteria for constructive or destructive interference

$$\begin{aligned} \Delta\phi &= 2\pi m \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \qquad \text{Constructive} \\ \Delta\phi &= \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots \qquad \text{Destructive} \end{aligned}$$

and we used what we know from the problem to determine the total phase change using

$$\Delta\phi = (k_2 x_2 - \omega_2 t_2 + \phi_2) - (k_1 x_1 - \omega_1 t_1 + \phi_1)$$

We can view this as three main methods of causing a phase difference. One is having a path difference where either x_2 and x_2 are different, or the wavelengths are different along the path. Another is that the frequency changes so that ω_2 and ω_1 are not the same. The final is a phase shift, where either the waves start differently or there is a change in the phase constant (like when there is a fixed end reflection). This will work for most problems involving mixing of two waves. Let's do a two dimensional problem.

Single frequency interference in more than one dimension

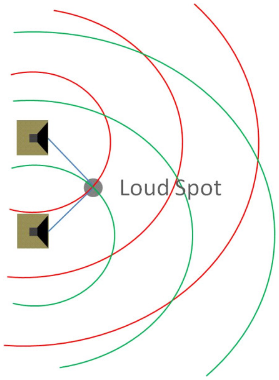
But what happens if our waves don't travel along the same line? Suppose you are at a dance, and there are two speakers. Further suppose that you are testing the system with a constant tone so $\omega_2 = \omega_1$ (either that or you have somewhat boring music with long, sustained tones). Suppose the two speakers make waves in phase ($\phi_1 = \phi_2$). There is no slower medium, so we expect $k_2 = k_1$. If you are equal distance from the two speakers, you would expect constructive interference because both $\Delta\phi_o = 0$ and $\Delta x = 0$ for this case.

$$\begin{aligned}\Delta\phi &= (k_2x_2 - \omega_2t_2 + \phi_2) - (k_1x_1 - \omega_1t_1 + \phi_1) \\ &= (kx_2 - \omega t + \phi) - (kx_1 - \omega t + \phi) \\ &= k(\Delta x)\end{aligned}$$

If we pick a spot where $x_1 = x_2$ we would expect

$$\Delta\phi = 0$$

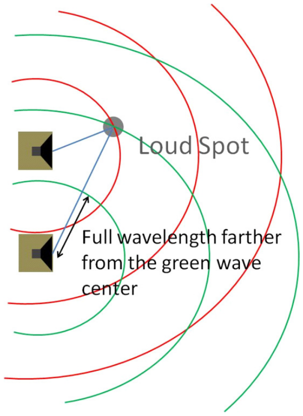
and since 0 is a multiple of 2π in our criteria, then we expect this will give us constructive interference.



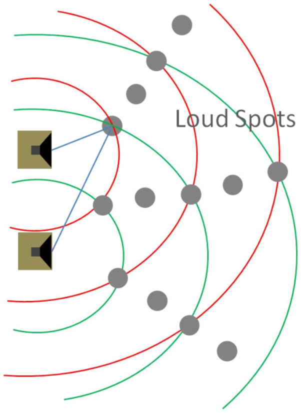
But there are more places where we expect constructive interference, because we know the sound wave is really spherical. Any time the path difference, $\Delta x = n\lambda$, then

$$\Delta\phi = \frac{2\pi}{\lambda}(n\lambda) = n2\pi$$

and we will have constructive interference The next figure shows an example where the path difference is one wavelength.



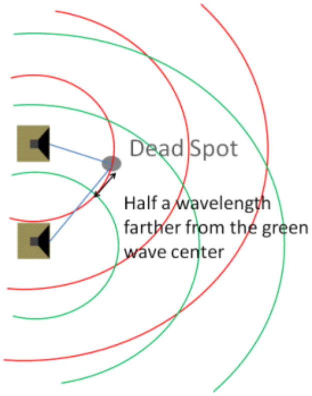
But any of the spots in the next figure will experience constructive interference. Note the loud spots are where there are two crests or two troughs together.



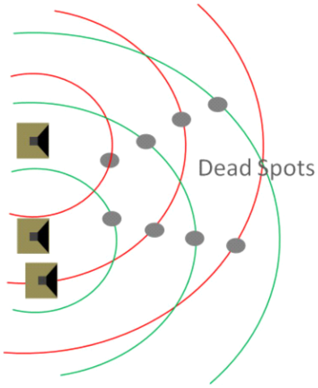
We also expect to see destructive interference. This should occur where path differences are multiples of $\lambda/2$ so that

$$\begin{aligned}\Delta\phi &= \frac{2\pi}{\lambda}\Delta x \\ &= \frac{2\pi}{\lambda}\left(n\frac{\lambda}{2}\right) \\ &= n\pi\end{aligned}$$

The next figure shows a case where $\Delta x = \lambda/2$



and the next figure shows many places where there will be destructive interference because the two waves are out of phase by half a wavelength.



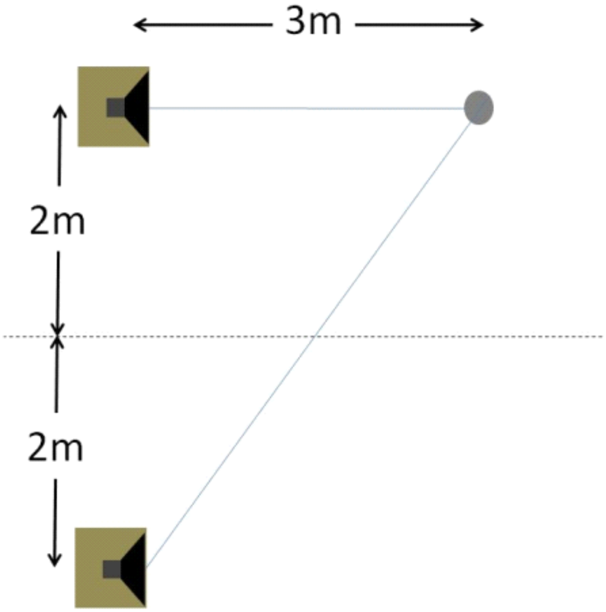
When you moved from one dimension to two dimensions in PH121, we changed from the variables x and y to the variable r where

$$r = \sqrt{x^2 + y^2}$$

Thus for this case our phase becomes

$$\Delta\phi = \left(2\pi\frac{\Delta r}{\lambda} + \Delta\phi_o\right)$$

In our dance example, suppose we have speakers that are $4\,\mathrm{m}$ apart and we are standing $3\,\mathrm{m}$ directly in front of one of the speakers. Further suppose that we play an A just above middle C which has a frequency of $440\,\mathrm{Hz}$. The speed of sound is $343\,\mathrm{m/\,s}$. Our speakers are connected to the same stereo with equal length wires. What is the phase difference at this spot?



From the geometry we can tell that the path from the second speaker must be $5\,\mathrm{m}$. So

$$\begin{aligned}\Delta x &= 5\,\mathrm{m} - 3\,\mathrm{m}\\ &= 2\,\mathrm{m}\end{aligned}$$

We can tell that the wavelength is

$$\begin{aligned}\lambda &= \frac{v}{f}\\ &= \frac{343\,\mathrm{m/\,s}}{440\,\mathrm{Hz}}\\ &= 0.779\,55\,\mathrm{m}\end{aligned}$$

Since the speakers are connected to the same stereo with equal length wires, $\Delta\phi_o = 0$. Then

$$\begin{aligned}\Delta\phi &= \frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\\ &= \frac{2\pi}{0.779\,55\,\mathrm{m}}\,(2\,\mathrm{m}) + 0\\ &= 5.\,131\,2\pi\\ &= 2\pi + 3.\,131\,2\pi\end{aligned}$$

We should ask, is this constructive or destructive interference? Well, it is neither purely constructive interference nor total destructive interference. Our amplitude would be

$$2A\cos\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}\Delta x + \Delta\phi_o\right)\right)$$

so in this case we get

$$2A\cos\left(\frac{1}{2}\left(2\pi + 3.\,131\,2\pi\right)\right) = -0.409\,27A$$

which is smaller (in magnitude) than A, so it is partial destructive interference. It would be quieter at this spot than if we just had one speaker playing.

You might guess that this sort of analysis plays a large part in design of concert halls. It also is important in mechanical designs. But you should have seen a deficit in what we have learned so far. Up to this point, we have only mixed waves that have the same frequency. Can we mix waves that have different frequencies?

Beats

Beat Demo

Up till now we only superposed waves that had the same frequency. But what happens if we take waves with different frequencies? Let's take the case where $\phi_2 = \phi_1 = \phi_o$, $k_2 = k_1 = k$ and let's let the waves mix at the same location $x_2 = x_1$ so

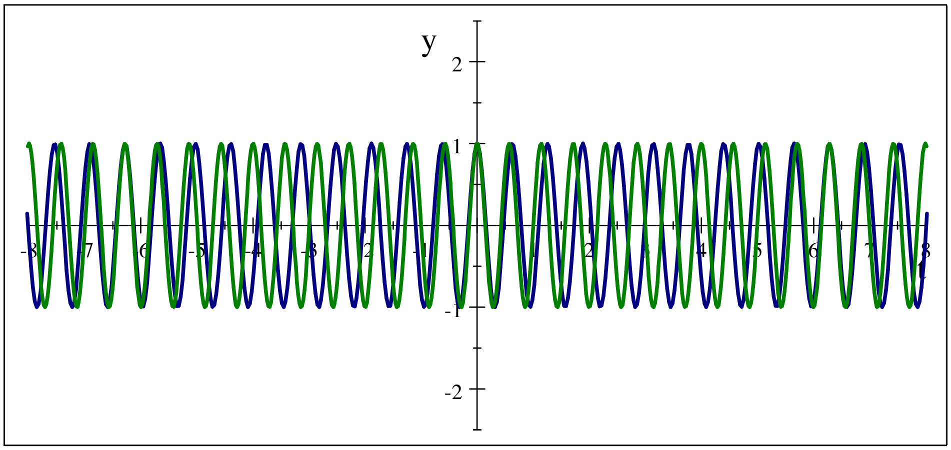
$$y_1 = y_{\max}\sin\left(kx - \omega_1 t + \phi_o\right)$$
$$y_2 = y_{\max}\sin\left(kx - \omega_2 t + \phi_o\right)$$
$$\begin{aligned}\Delta\phi &= \left(kx - \omega_1 t + \phi_o\right) - \left(kx - \omega_1 t + \phi_o\right)\\ &= \left(\omega_1 - \omega_2\right)t\end{aligned}$$

We can use our criteria for constructive and destructive interference, but before going on let's put this back into the total amplitude function for two wave mixing.

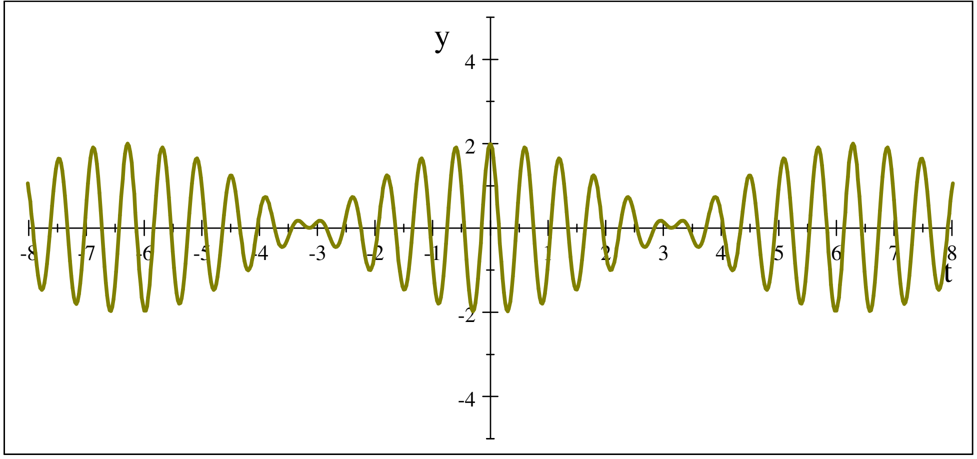
$$\begin{aligned}A &= 2y_{\max}\cos\left(\frac{1}{2}\Delta\phi\right)\\ &= 2y_{\max}\cos\left(\frac{1}{2}\left(\omega_1 - \omega_2\right)t\right)\end{aligned}$$

and we can see that our amplitude will change in time. Sometimes it will be constructive interference and sometimes destructive interference and often in between.

We can plot both waves on the same graph and see that this will happen.



Notice that there are places where the waves are in phase, and places where they are not. The superposition looks like this



where there is constructive interference, the resulting wave amplitude is large, where there is destructive interference, the resulting amplitude is zero. We get a traveling wave who's amplitude varies. We can find the amplitude function algebraically.

We can write out the entire resultant wave in our usually way. Our two waves are

$$y_1 = y_{\max} \sin \left(kx - \omega_1 t + \phi_o \right)$$
$$y_2 = y_{\max} \sin \left(kx - \omega_2 t + \phi_o \right)$$

and the resultant

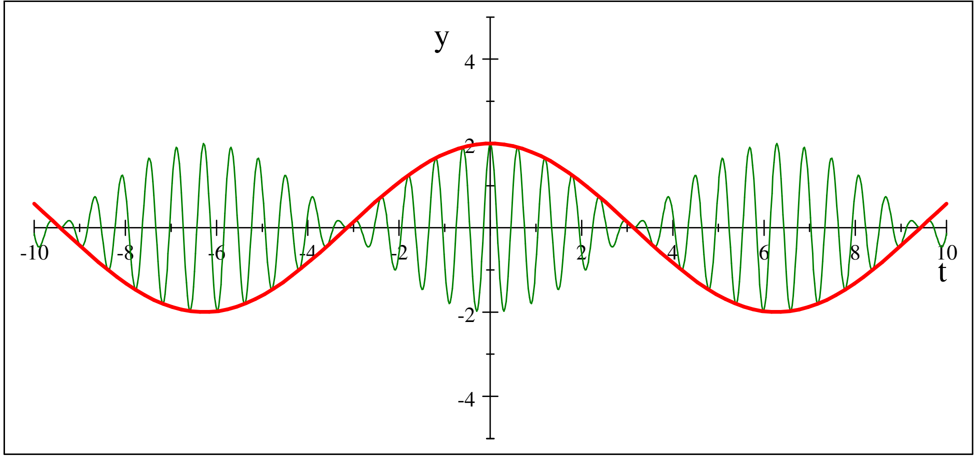
$$\begin{aligned} y_r 2y_{\max}\cos\left(\frac{(kx-\omega_2 t+\phi_o)-(kx-\omega_1 t+\phi_o)}{2}\right)\sin\left(\frac{kx-\omega_2 t+\phi_o+kx-\omega_1 t+\phi_o}{2}\right) \\ = \quad 2y_{\max}\cos\left(\frac{kx-2\pi f_2 t-(kx-2\pi f_1 t)}{2}\right)\sin\left(\frac{kx-2\pi f_2 t+kx-2\pi f_1 t+2\phi_o}{2}\right) \\ = \quad 2y_{\max}\cos\left(2\pi\frac{f_1-f_2}{2}t\right)\sin\left(kx-2\pi\frac{f_1+f_2}{2}t+\phi_o\right) \\ = \quad \left[2y_{\max}\cos\left(2\pi\frac{f_1-f_2}{2}t\right)\right]\sin\left(kx-2\pi\frac{f_1+f_2}{2}t+\phi_o\right) \end{aligned}$$

The sine part is a wave, it is a function of position and time. We see that it has a frequency that is the average of f_1 and f_2. This is the frequency we hear. But we have another complicated amplitude term, and this time it is a function of time just as we suspected. The amplitude has its own frequency that is half the difference of f_1 and f_2.

$$A_{\text{resultant}} = 2A\cos\left(2\pi\frac{f_1-f_2}{2}t\right)$$

So the sound amplitude will vary in time for a given position in the medium.

The situation is odder still. We have a cosine function, but it is really an envelope for the higher frequency motion of the air particles. The air molecules move back and forth for both the crest and the trough of the envelope function.



So we will hear two maxima for every period! This frequency with which we hear the sound get loud at a given location as the wave goes by is called the *beat frequency*. The red envelope (solid heavy line in the last figure) has a frequency of

$$f_A = \frac{f_1-f_2}{2}$$

but it is just the envelope. We can see that the green (thin line) wave will push and pull air molecules, and therefore our ear drums, with maximum loudness at twice this frequency. So our beat frequency is

$$f_{beat} = |f_1 - f_2|$$

Any time we mix waves of different frequencies we get beating. Often the beat frequency is very fast, and our hearing system can't track the beats, so we don't hear them. And if we mixed more than two waves the beats might not come at perfectly regular intervals. The mixing of waves can become quite complicated. Yet even a barbershop quartet is a mixing of at least four waves. So complicated superpositions are common. In the next lecture we will try to see how we could take on these complicated combined waves.

10 Non Sinusoidal Waves

Fundamental Concepts

You have probably wondered if all waves are sinusoidal. Can the universe really be described by such simple mathematics? The answer is both no, and yes. There are non-sinusoidal waves, in fact, most waves are not sinusoidal. But it turns out that we can use a very clever mathematical trick to make any shape wave out of a superposition of many sinusoidal waves. So our mathematics for sinusoidal waves turns out to be quite general.

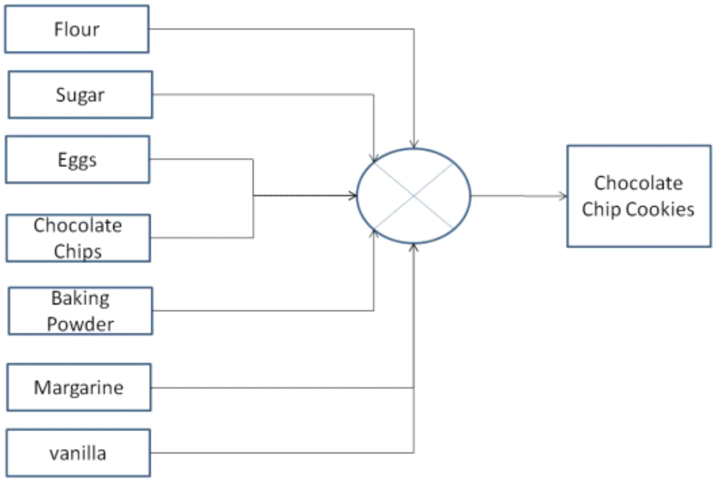
Music and Non-sinusoidal waves

Let's take the example of music.

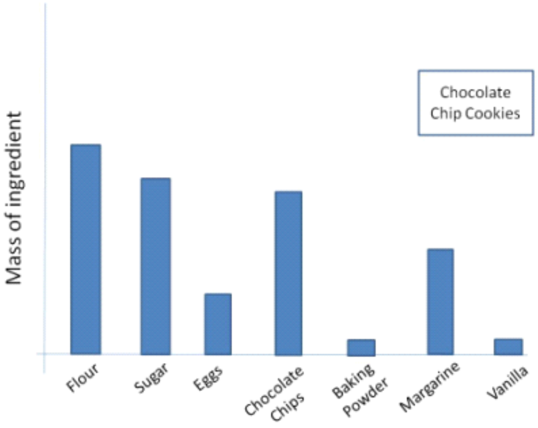
From our example of standing waves on strings, we know that a string can support a series of standing waves with discrete frequencies–the harmonic series. We have also discussed that usually we excite more than one standing wave at a time. The fundamental mode tends to give us the pitch we hear, but what are the other standing waves for?

To understand, lets take an analogy. Making cookies and cakes.

Here is the beginning of a recipe for cookies.

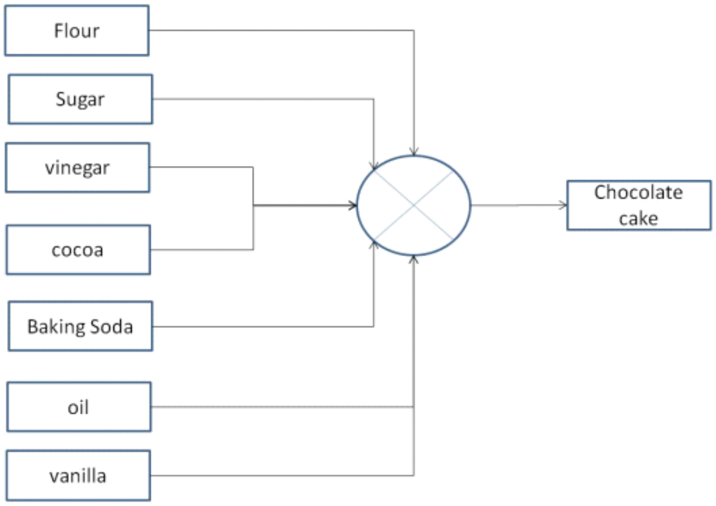


The recipe is a list of ingredients, and a symbolic instruction to mix and bake. The product is chocolate chip cookies. Of course we need more information. We need to know now much of each ingredient to use.



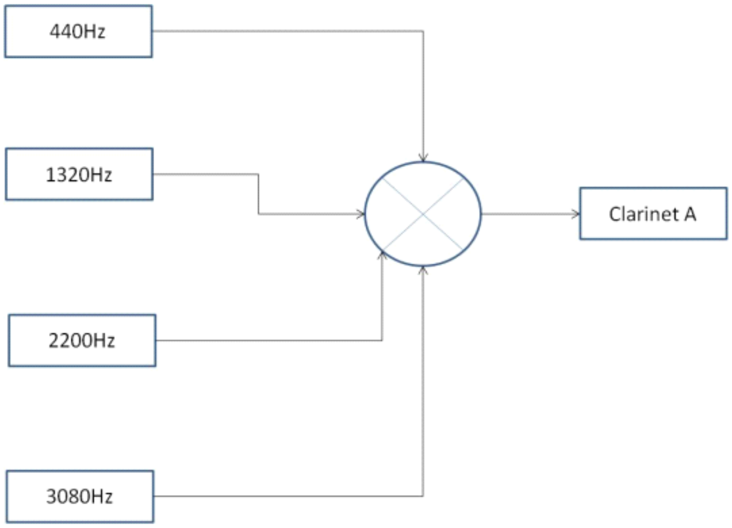
This graph gives us the amount of each ingredient by mass.

Now suppose we want chocolate cake.

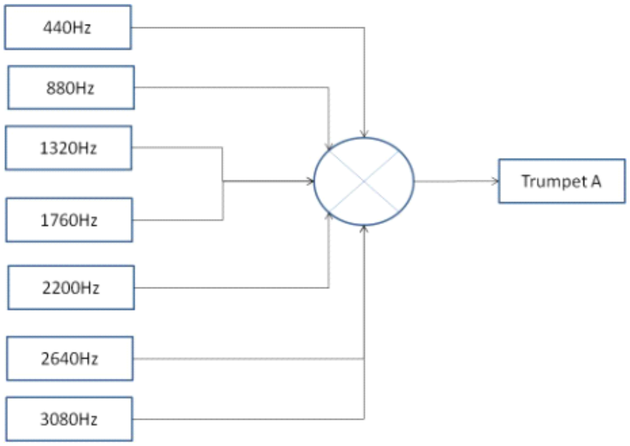


The predominant taste in each of these foods is chocolate. But chocolate cake and chocolate chip cookies don't taste exactly the same. We can easily see that the differences in the other ingredients make the difference between the "cookie" taste and the "cake" taste that goes along with the "chocolate" taste that predominates.

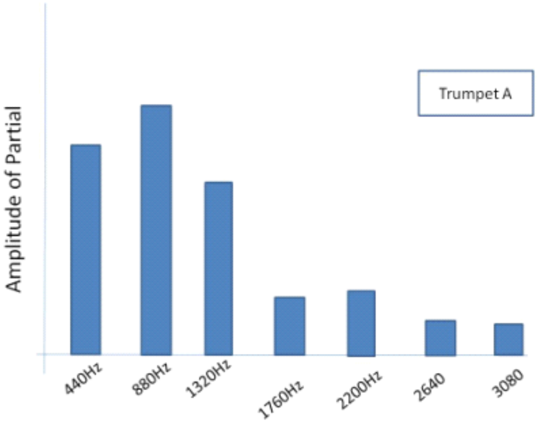
The sound waves produced by musical instruments work in a similar way. Here is a recipe for an "A" note from a clarinet.



and here is one for a trumpet playing the same "A" note.

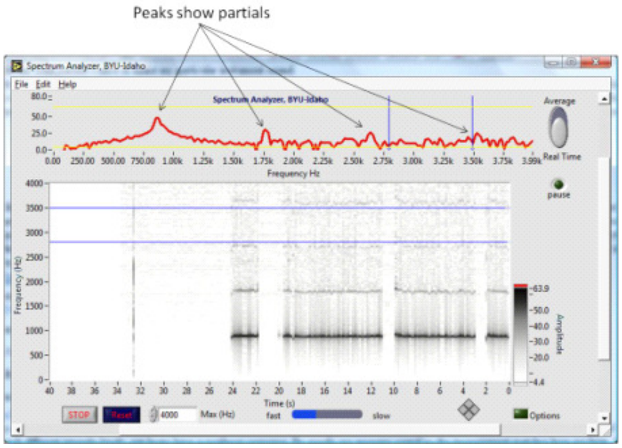


A trumpet sounds different than a clarinet, and now we see why. There are more harmonics involved with the trumpet sound than the clarinet sound. These extra standing waves make up the “brassiness” of the trumpet sound. As with our baking example, we need to know how much of each standing wave we have. Each will have a different amplitude. For our trumpet, we might get amplitudes as shown.

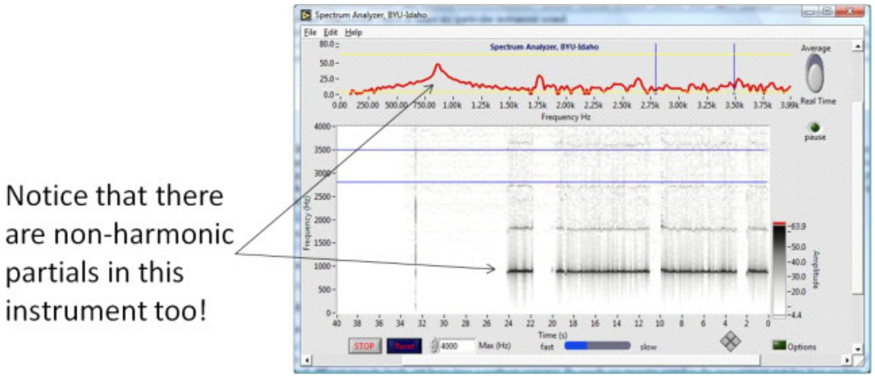


Note that the second harmonic has a larger amplitude, but we still hear the “A” as at 440 Hz. A fugal horn would still sound brassy, but would have a different mix of harmonics.

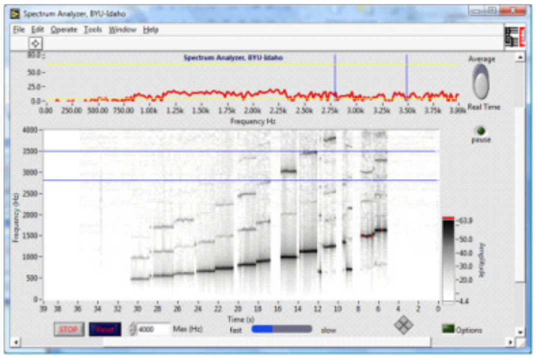
We have a tool that you can download to your PC to detect the mix of harmonics of musical instruments, or mechanical systems. In music, the different harmonics are called *partials* because they make up part of the sound. A graph that shows which harmonics are involved is called a *spectrum*. The next figure is the spectrum of a six holed bamboo flute. Note that there are several harmonics involved.


Peaks show partials

Note that our graph has two parts. One is the instantaneous spectrum, and one is the spectrum time history.

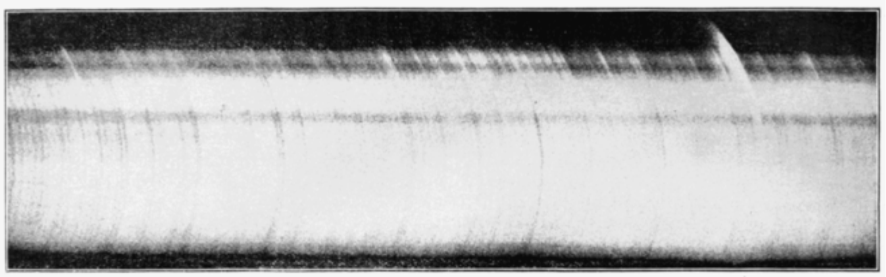

Notice that there are non-harmonic partials in this instrument too!

By observing the time history, we can see changes in the spectrum. We can also see that we don’t have pure harmonics. The graph shows some response off the specific harmonic frequencies. This six holed flute is very “breathy” giving a lot of wind noise along with the notes, and we see this in the spectrum. In the next picture, I played a scale on the flute.



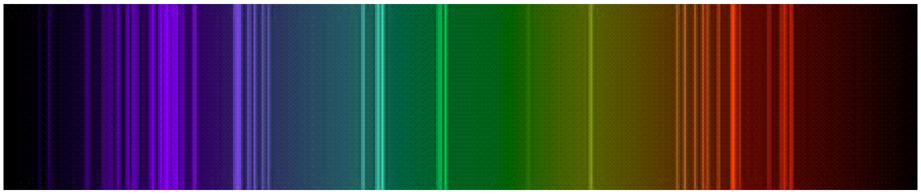
The instantaneous spectrum is not active in this figure (since it can't show more than one note at a time) but in the time history we see that as the fundamental frequency changes by shorting the length of the flute (uncovering holes), we see that every partial also goes up in frequency. The flute still has the characteristic spectrum of a flute, but shifted to new frequencies. We can use this fact to identify things by their vibration spectrum. In fact, that is how you recognize voices and musics within your auditory system!

The technique of taking apart a wave into its components is very powerful. With light waves, the spectrum is an indication of the chemical composition of the emitter. For example, the spectrum of the sun looks something like this



Solar coronal spectrum taken during a solar eclipse. The successive curved lines are each different wavelengths, and the dark lines are wavelengths that are absorbed. The pattern of absorbed wavelengths allows a chemical analysis of the corona. (Image in the Public Domain, orignally published in Bailey, Solon, L, Popular Science Monthly, Vol 60, Nov. 1919, pp 244)

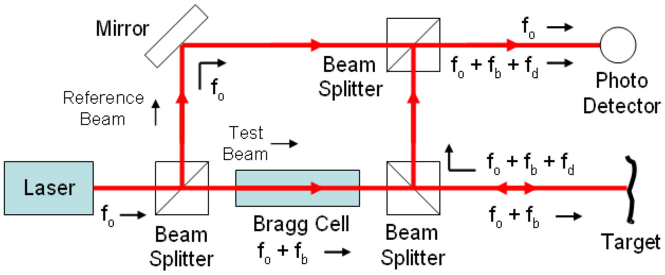
The lines in this graph show the amplitude of each harmonic component of the light. Darker lines have larger amplitudes. The harmonics come from the excitation of electrons in their orbitals. Each orbital is a different energy state, and when the electrons jump from orbital to orbital, they produce specific wave frequencies. By observing the mix of dark lines in pervious figure, and comparing to laboratory measurements from each element (see next figure) we can find the composition of the source. This figure shows the emission spectrum for Calcium. because it is an emission spectrum the lines are bright instead of dark. We can even see the color of each line!



Emission spectrum of Calcium (Image in the Public Domain, courtesy NASA)

Vibrometry

Just like each atom has a specific spectrum, and each instrument, each engine, machine, or anything that vibrates has a spectrum. We can use this to monitor the health of machinery, or even to identify a piece of equipment. Laser or acoustic vibrometers are available commercially.



Laser Vibrometer Schematic (Public Domain Image from Laderaranch: http://commons.wikimedia.org/wiki/File:LDV_Schematic.png)

They provide a way to monitor equipment in places where it would be dangerous or even impossible to send a person. The equipment also does not need to be shut down, a great benefit for factories that are never shut down, or for a satellite system that cannot be reached by anyone.

Fourier Series: Mathematics of Non-Sinusoidal Waves

We should take a quick look at the mathematics of non-sinusoidal waves.

Let' start with a superposition of many sinusoidal waves. The math looks like this

$$y\left(t\right)=\sum_{n}\left(A_n\sin\left(2\pi f_n t\right)+B_n\cos\left(2\pi f_n t\right)\right)$$

where A_n and B_n are a series of coefficients and f_n are the harmonic series of frequencies.

Example: Fourier representation of a square wave.

For example, we could represent a function $f(x)$ with the following series

$$\begin{aligned} f(x) &= C_o + C_1 \cos\left(\frac{2\pi}{\lambda}x + \varepsilon_1\right) && (10.1)\\ &\quad +C_2 \cos\left(\frac{2\pi}{\frac{\lambda}{2}}x + \varepsilon_2\right) && (10.2)\\ &\quad +C_3 \cos\left(\frac{2\pi}{\frac{\lambda}{3}}x + \varepsilon_3\right) && (10.3)\\ &\quad +\ldots && (10.4)\\ &\quad +C_n \cos\left(\frac{2\pi}{\frac{\lambda}{n}}x + \varepsilon_n\right) && (10.5)\\ &\quad +\ldots && (10.6) \end{aligned}$$

where we will let $\varepsilon_i = \omega_i t + \phi_i$

The $C's$ are just coefficients that tell us the amplitude of the individual cosine waves. The more terms in the series we take, the better the approximation we will have, with the series exactly matching $f(x)$ when the number of terms, $N \rightarrow \infty$.

Usually we rewrite the terms of the series as

$$C_m \cos(mkx + \varepsilon_m) = A_m \cos(mkx) + B_m \sin(mkx) \qquad (10.7)$$

where k is the wavenumber

$$k = \frac{2\pi}{\lambda} \qquad (10.8)$$

and λ is the wavelength of the complicated but still periodic function $f(x)$. Then we identify

$$\begin{aligned} A_m &= C_m \cos(\varepsilon_m) && (10.9)\\ B_m &= -C_m \sin(\varepsilon_m) && (10.10) \end{aligned}$$

then

$$f(x) = \frac{A_o}{2} + \sum_{m-1}^{\infty} A_m \cos(mkx) + \sum_{m-1}^{\infty} B_m \sin(mkx) \qquad (10.11)$$

where we separated out the $A_o/2$ term because it mikes things nicer later.

Fourier Analysis

The process of finding the coefficients of the series is called *Fourier analysis*. We start by integrating equation (10.11)

$$\int_0^\lambda f(x)\,dx = \int_0^\lambda \frac{A_o}{2}dx + \int_0^\lambda \sum_{m-1}^{\infty} A_m \cos(mkx)\,dx + \int_0^\lambda \sum_{m-1}^{\infty} B_m \sin(mkx)\,dx \tag{10.12}$$

We can see immediately that all the sine and cosine terms integrate to zero (we integrated over a wavelength) so

$$\int_0^\lambda f(x)\,dx = \int_0^\lambda \frac{A_o}{2}dx = \frac{A_o}{2}\lambda \tag{10.13}$$

We solve this for A_o

$$A_o = \frac{2}{\lambda}\int_0^\lambda f(x)\,dx \tag{10.14}$$

To find the rest of the coefficients we need to remind ourselves of the orthogonality of sinusoidal functions

$$\int_0^\lambda \sin(akx)\cos(bkx)\,dx = 0 \tag{10.15}$$

$$\int_0^\lambda \cos(akx)\cos(bkx)\,dx = \frac{\lambda}{2}\delta_{ab} \tag{10.16}$$

$$\int_0^\lambda \sin(akx)\sin(bkx)\,dx = \frac{\lambda}{2}\delta_{ab} \tag{10.17}$$

where δ_{ab} is the Kronecker delta.

To find the coefficients, then, we multiply both sides of equation (10.11) by $\cos(lkx)$ where l is a positive integer. Then we integrate over one wavelength.

$$\int_0^\lambda f(x)\cos(lkx)\,dx = \int_0^\lambda \frac{A_o}{2}\cos(lkx)\,dx \tag{10.18}$$

$$+\int_0^\lambda \sum_{m-1}^{\infty} A_m \cos(mkx)\cos(lkx)\,dx \tag{10.19}$$

$$+\int_0^\lambda \sum_{m-1}^{\infty} B_m \sin(mkx)\cos(lkx)\,dx \tag{10.20}$$

which gives

$$\int_0^\lambda f(x)\cos(mkx)\,dx = \int_0^\lambda A_m \cos(mkx)\cos(mkx)\,dx \tag{10.21}$$

that is, only the term with two cosine functions where $l = m$ will be non zero. So

$$\int_0^\lambda f(x)\cos(mkx)\,dx = \frac{\lambda}{2}A_m \tag{10.22}$$

solving for A_m we have

$$A_m = \frac{2}{\lambda}\int_0^\lambda f(x)\cos(mkx)\,dx \tag{10.23}$$

We can perform the same steps to find B_m only we use $\sin(lkx)$ as the multiplier. Then we find

$$B_m = \frac{2}{\lambda}\int_0^{\lambda} f\left(x\right)\sin\left(mkx\right)dx \qquad (10.24)$$

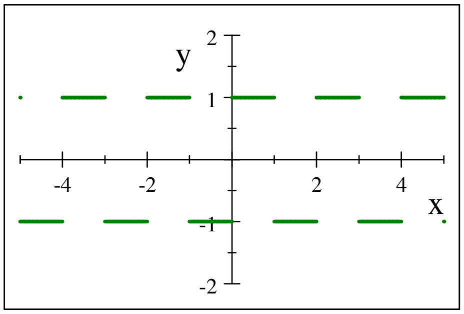
Square wave

Let's find the series for a square wave using our Fourier analysis technique.

Let's take

$$\lambda = 2 \qquad (10.25)$$

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \frac{\lambda}{2} \\ -1 & \text{if} \quad \frac{\lambda}{2} < x < \lambda \end{cases} \qquad (10.26)$$



since $f\left(x\right) is$ odd, $A_m = 0$ for all m. We have

$$B_m = \frac{2}{\lambda}\int_0^{\frac{\lambda}{2}}\left(1\right)\sin\left(mkx\right)dx + \frac{2}{\lambda}\int_{\frac{\lambda}{2}}^{\lambda}\left(-1\right)\sin\left(mkx\right)dx \qquad (10.27)$$

so

$$B_m = \frac{1}{m\pi}\left(-\cos\left(mkx\right)|_0^{\frac{\lambda}{2}} + \frac{1}{m\pi}\left(\cos\left(mkx\right)|_{\frac{\lambda}{2}}^{\lambda}\right.\right. \qquad (10.28)$$

Which is

$$B_m = \frac{1}{m\pi}\left(1\cos\left(m\frac{2\pi}{\lambda}x\right)\right|_0^{\frac{\lambda}{2}} + \frac{1}{m\pi}\left(\cos\left(m\frac{2\pi}{\lambda}x\right)\right|_{\frac{\lambda}{2}}^{\lambda} \qquad (10.29)$$

so

$$B_m = \frac{1}{m\pi}\left(\left(-\cos\left(m\frac{2\pi}{\lambda}\frac{\lambda}{2}\right)\right) + \cos\left(m\frac{2\pi}{\lambda}\left(0\right)\right)\right) \qquad (10.30)$$

$$+\frac{1}{m\pi}\left(\left(\cos\left(m\frac{2\pi}{\lambda}\lambda\right) - \cos\left(m\frac{2\pi}{\lambda}\frac{\lambda}{2}\right)\right)\right) \qquad (10.31)$$

which is

$$B_m = \frac{2}{m\pi}\left(1 - \cos\left(m\pi\right)\right) \qquad (10.32)$$

Our series is then just

$$f(x) = \sum_{m-1}^{\infty} \frac{2}{m\pi}\left(1 - \cos(m\pi)\right)\sin(mkx) \tag{10.33}$$

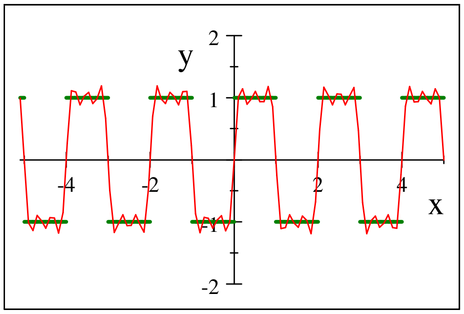
and we can write a few terms

Term	
1	$\frac{4}{\pi}\sin(kx)$
2	0
3	$\frac{4}{3\pi}\sin(3kx)$
4	0
5	$\frac{4}{5\pi}\sin(5kx)$

(10.34)

then the partial sum up to $m = 5$ looks like

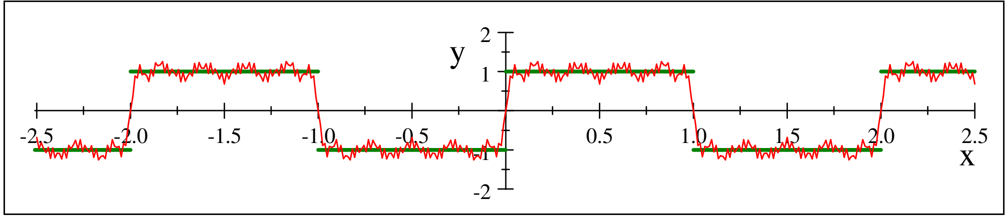
$$f(x) = \frac{4}{\pi}\sin(kx) + \frac{4}{3\pi}\sin(3kx) + \frac{4}{5\pi}\sin(5kx) \tag{10.35}$$



If we take many terms,

$$\begin{aligned} f(x) &= \frac{4}{\pi}\sin(kx) + \frac{4}{3\pi}\sin(3kx) + \frac{4}{5\pi}\sin(5kx) + \frac{4}{7\pi}\sin(7kx) + \frac{4}{9\pi}\sin(95kx) \\ &\quad + \frac{4}{11\pi}\sin(11kx) + \frac{4}{13\pi}\sin(13kx) + \frac{4}{15\pi}\sin(15kx) + \frac{4}{17\pi}\sin(17kx) + \frac{4}{19\pi}\sin(19kx) \end{aligned} \tag{10.36}$$

We see the function get closer and closer to a square wave.



In the limit of infinitely many waves, the match would be perfect. But we don't usually need an infinite number of terms. we can pick the part of the spectrum that best represents the phenomena we desire to observe. For example, oil based compounds all have specific spectral signatures in the wavelength range between $3 - 5$ micrometers. If you wish to tell the difference between gasoline and crude oil, you can restrict your study to these wavelengths alone.

11 Interference of Light Waves

Fundamental Concepts

What is Light?

Before the 19th century (1800's) light was assumed to be a stream of particles. Newton was one of the chief proponents of this theory. The theory was able to explain reflection of light from mirrors and other objects and therefore explain vision. In 1678 Huygens showed that wave theory could also explain reflection and vision.

In 1801 Thomas Young demonstrated that light had attributes that were best explained by wave theory. We will study Young's experiment later today. The crux of his experiment was to show that light displayed constructive and destructive interference–clearly a wave phenomena! The theory of the nature of light took a dramatic shift

In 1805 Joseph Smith was born in Sharon, Vermont.

In September of 1832 Joseph Smith received a revelation that said in part :

> For the word of the Lord is truth, and whatsoever is truth is light, and whatsoever is light is Spirit, even the Spirit of Jesus Christ. And the Spirit giveth light to every man that cometh into the world; and the Spirit enlighteneth every man through the world, that hearkeneth to the voice of the Spirit. (D&C 84:45-46)

In December of 1832 Joseph Smith received another revelation that says in part:

> This Comforter is the promise which I give unto you of eternal life, even the glory of the celestial kingdom; which glory is that of the church of the Firstborn, even of God, the holiest of all, through Jesus Christ his Son—He that ascended up on high, as also he descended below all things, in that he comprehended all things, that he might be in all and through all things,

> the light of truth; which truth shineth. This is the light of Christ. As also he is in the sun, and the light of the sun, and the power thereof by which it was made. As also he is in the moon, and is the light of the moon, and the power thereof by which it was made; as also the light of the stars, and the power thereof by which they were made; and the earth also, and the power thereof, even the earth upon which you stand. And the light which shineth, which giveth you light, is through him who enlighteneth your eyes, which is the same light that quickeneth your understandings; which light proceedeth forth from the presence of God to fill the immensity of space—the light which is in all things, which giveth life to all things, which is the law by which all things are governed, even the power of God who sitteth upon his throne, who is in the bosom of eternity, who is in the midst of all things. (D&C 88:5-12)

Light, even real, physical light, seems to be of interest to members of the Restored Church.

In 1847 the saints entered the Salt Lake Valley.

In 1873 Maxwell published his findings that light is an electromagnetic wave (something we will try to show before this course is over!).

Planck's work in quantization theory (1900) was used by Einstein In 1905 to give an explantation of the photoelectric effect that again made light look like a particle.

Current theory allows light to exhibit the characteristics of a wave in some situations and like a particle in others. We will study both before the end of the semester.

The results of Einstein's work give us the concept of a *photon* or a quantized unit of radiant energy. Each "piece of light" or photon has energy

$$E = hf \tag{11.1}$$

where f is the frequency of the light and h is a constant

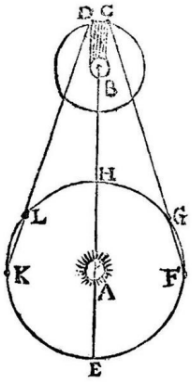
$$h = 6.63 \times 10^{-34}\,\mathrm{J\,s} \tag{11.2}$$

The nature of light is fascinating and useful both in physical and religious areas of thought.

Measurements of the Speed of Light

One of the great foundations of modern physical theory is that the speed of light is constant in a vacuum. Galileo first tried to measure the speed of light. He used two towers in town and placed a lantern and an assistant on each tower. The lanterns had shades. The plan was for one assistant to remove his shade, and then for the assistant on the other tower to remove his shade as soon as he saw the light from the first lantern. Back at the first tower, the first assistant would use a clock to determine the time difference between when the first lantern was un-shaded, and when they saw the light from the second tower. The light would have traveled twice the inter-tower distance. Dividing that distance by the time would give the speed of light. You can probably guess that this did not work. Light travels very quickly. The clocks of Galileo's day could not measure such a small time difference. Ole Rømer was the first to succeed in measuring the speed of light.

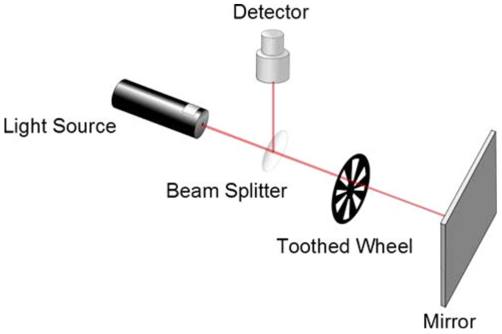
Rømer's Measurement of the speed of light



A diagram illustrating Rømer's determination of the speed of light. Point A is the Sun, piont B is Jupiter. Point C is the immersion of Io into Jupiter's shadow at the start of an eclipse

Rømer performed his measurement in 1675, 269 years before digital devices existed!. He used the period of revolution of Io, a moon of Jupiter, as Jupiter revolved around the sun. He first measured the period of Io's rotation about Jupiter, then he predicted an eclipse of Io three months later. But he found his calculation was off by $600\,\mathrm{s}$. After careful thought, he realized that the Earth had moved in its orbit, and that the light had to travel the extra distance due to the Earth's new position. Given Rømer's best estimate for the orbital radius of the earth and his time difference, Rømer arrived at a estimate of $c = 2.3 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$. Amazing for 1675!

Fizeau's Measurement of the speed of light



Hippolyte Fizeau measured the speed of light in 1849 using the apparatus indicated in the figure above. He used a toothed wheel and a mirror and a beam of light. The light passed through the open space in the wheel's teeth as the wheel rotated. Then was reflected by the mirror. The speed would be

$$v = \frac{\Delta x}{\Delta t}$$

We just need Δx and Δt.

It is easy to see that

$$\Delta x = 2d$$

because the light travels twice the distance to the mirror (d) and back. If the wheel turned just at the right angular speed, then the reflected light would hit the next tooth and be blocked. Think of angular speed

$$\omega = \frac{\Delta \theta}{\Delta t}$$

so the time difference would be

$$\Delta t = \frac{\Delta \theta}{\omega}$$

we find $\Delta\theta$ by taking 2π and dividing by the number of teeth on the wheel.

$$\Delta \theta = \frac{2\pi}{N_{teeth}}$$

Then the speed of light must be

$$\begin{aligned} c &= v = \frac{2d}{\frac{\Delta\theta}{\omega}} \\ &= \frac{2d\omega}{\Delta\theta} \\ &= \frac{2d\omega N_{teeth}}{2\pi} \\ &= \frac{d\omega N_{teeth}}{\pi} \end{aligned}$$

then if we have 720 teeth and ω is measured to be $d = 7500\,\mathrm{m}$

$$\begin{aligned} c &= \frac{(7500\,\mathrm{m})\,(172.79\,\mathrm{Hz})\,(720)}{\pi} \\ &= 2.97 \times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

which is Fizeau's number and it is pretty good!

Modern measurements are performed in very much the same way that Fizeau did his calculation. The current value is

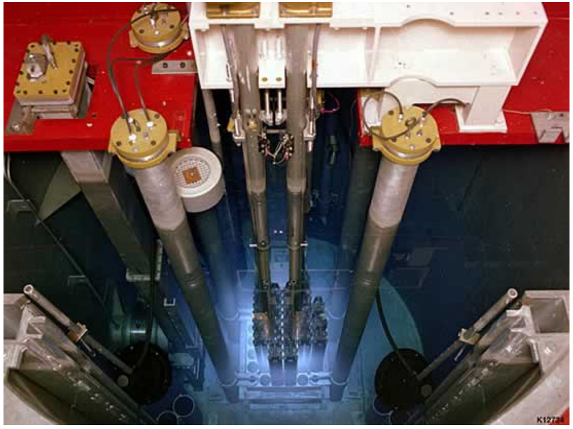
$$c = 2.9979 \times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \tag{11.3}$$

Faster than light

The speed of light in a vacuum is constant, but in matter the speed of light changes.

Pass the photon (ball) demo

We will study this in detail when we look at refraction. But for now, a dramatic example is Cherenkov radiation. It is an eerie blue glow around the core of nuclear reactors. It occurs when electrons are accelerated past the speed of light in the water surrounding the core. The electrons emit light and the light waves form a Doppler cone or a light-sonic boom! The result is the blue glow.

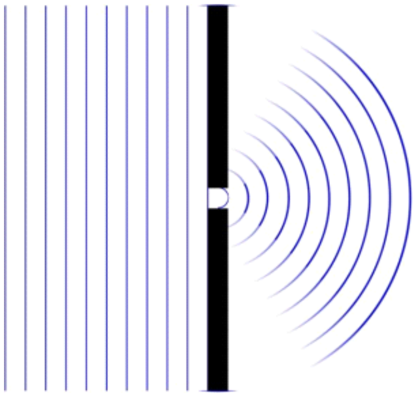


Cherenkov radiation (United States Department of Energy, image in the public domain)

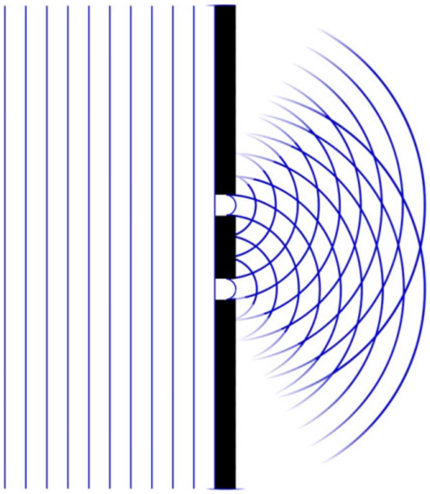
This does bring up a problem in terminology. What does the word "medium" mean? We have used it to mean the substance through which a wave travels. This substance must have the property of transferring energy between it's parts, like the coils of a spring can transfer energy to each other, or like air molecules can transfer energy by collision. For light the wave medium is the electromagnetic field. This field can store and transfer energy (PH220). But many books on physics call materials like glass a "medium" through which light travels. The water in our last example is a “medium” in this sense. Are glass and water wave mediums for light? The answer is an emphatic NO! Light does not need any matter to form it’s wave. The wave medium is the electromagnetic field. So we will have to keep this in mind as we allow light to travel through matter. I will try to say that the light enters a new “material” to describe something like light entering a piece of glass. But some books may call the matter a “medium.” You must remember that by this they don’t mean the wave medium.

Interference and Young’s Experiment

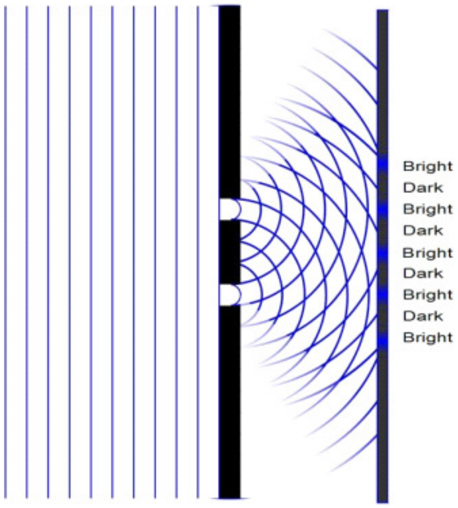
Waves do some funny things when they encounter barriers. Think of a water wave. If we pass the wave through a small opening in a barrier, the wave can’t all get through the small hole, but it can cause a disturbance right at the opening. We know that a small disturbance will cause a wave. But this wave will be due to a very small–almost point–source. Point sources make spherical waves. So the waves on the other side of the small opening will be nearly spherical. The smaller the opening the more pronounced the curving of the wave, because the source (the hole) is more like a point source.



Now suppose we have two of these openings. We expect the two sources to make curved waves and those waves can interfere.

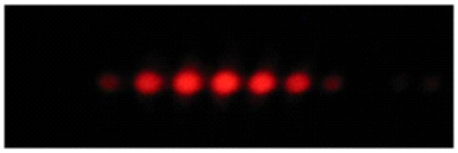


In the figure, we can already see that there will be constructive and destructive interference were the waves from the two holes meet. Thomas young predicted that we should see constructive and destructive interference in light (he drew figures very like the ones we have used to explain his idea).



Young set up a coherent source of light and placed it in front of this source a barrier with two very thin slits cut in it to test his idea.. He set up a screen beyond the barrier and observed the pattern on the screen formed by the light. This (in part) is what he saw

Young's Experiment demo



We see bright spots (constructive interference) and dark spots (destructive interference). Only wave phenomena can interfere, so this is fairly good evidence that light is a wave.

Question 123.26.5

Constructive Interference

We can find the condition for getting a bright or a dark band if we think about it a bit. The two waves coming from the two slits are just two different waves. We can use our two wave mixing analysis with our constructive and destructive interference criteria. For constructive interference, the difference in phase must be a multiple of 2π. Let's review this, but instead of $y_{\max}$ being the displacement of the waves, let's write them as $E_{\max}$ because it is the electric field that is carrying the wave and it doesn't really go up or down, it just gets a higher value or lower value. Our waves will be

$$\begin{aligned} E_1 &= E_{\max}\sin\left(k_2r_2-\omega_2t_2+\phi_2\right) \\ E_2 &= E_{\max}\sin\left(k_1r_1-\omega_1t_1+\phi_1\right) \end{aligned}$$

and the resulting wave will be

$$\begin{aligned} E_r &= E_{\max}\sin\left(k_2r_2-\omega_2t_2+\phi_2\right)+A\sin\left(k_1r_1-\omega_1t_1+\phi_1\right) \\ &= 2E_{\max}\cos\left(\frac{(k_2r_2-\omega_2t_2+\phi_2)-(k_1r_1-\omega_1t_1+\phi_1)}{2}\right) \\ &\quad\times\sin\left(\frac{(k_2r_2-\omega_2t_2+\phi_2)+(k_1r_1-\omega_1t_1+\phi_1)}{2}\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}\left[(k_2r_2-\omega_2t_2+\phi_2)-(k_1r_1-\omega_1t_1+\phi_1)\right]\right) \\ &\quad\times\sin\left(\frac{(k_2r_2-\omega_2t_2+\phi_2)+(k_1r_1-\omega_1t_1+\phi_1)}{2}\right) \end{aligned}$$

As we are now well aware, the sine part is a combined wave, and the cosine part is part of the amplitude. The amplitude can be written as

$$A=2E_{\max}\cos\left(\frac{1}{2}\left[(k_2r_2-\omega_2t_2+\phi_2)-(k_1r_1-\omega_1t_1+\phi_1)\right]\right)$$

We should pause and think about which of our values will change. The frequency won't change, $\omega_2=\omega_1=\omega$. There is no slower material, so the wavelength won't change, $k_2=k_1=k$. We want the waves to mix at the same time so $t_2=t_1=t$. And the new waves are created from the old wave hitting the slits. As long as the original wave hits both slits at once, then $\phi_2 = \phi_1 = \phi_o$. We are left with

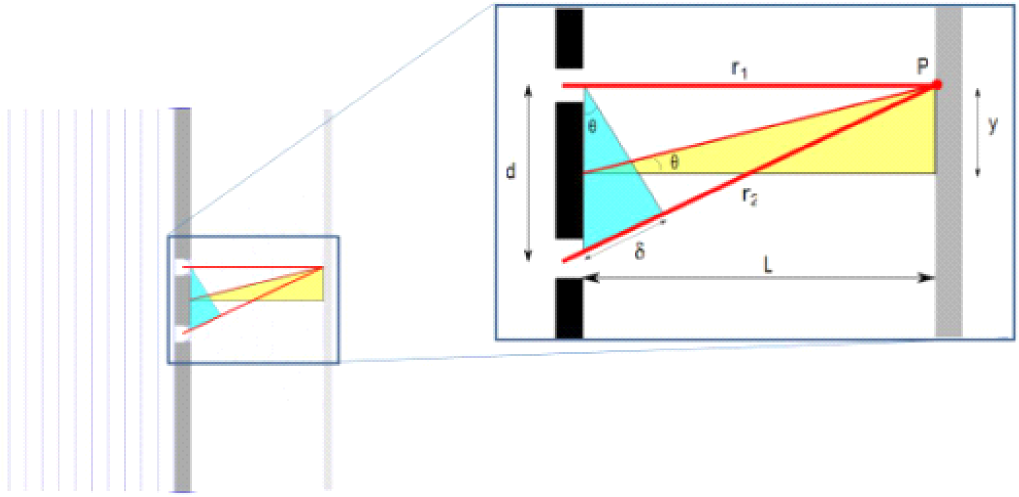
$$\begin{aligned} A &= 2E_{\max}\cos\left(\frac{1}{2}\left[(kr_2 - \omega t + \phi_o) - (kr_1 - \omega t + \phi_0)\right]\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}\left[(kr_2) - (kr_1)\right]\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}k\left(r_2 - r_1\right)\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}k\Delta r\right) \end{aligned}$$

That means the path difference between the two slit-sources must be an even number of wavelengths. We have been calling the path difference in the total phase Δx, or for spherical waves Δr, but in optics it is customary to call this path difference δ. So

$$\delta = \Delta r$$

Then we can write the amplitude function as

$$A = 2E_{\max}\cos\left(\frac{1}{2}k\delta\right)$$



so our total phase equation becomes

$$\Delta\phi = k\delta$$

We want to use this in our criteria for constructive or destructive interference. But first, let's express $\delta = \Delta r$ in terms of geometry that is easy to measure in our experiment. In this set up, the screen is much farther away than d, the slit distance, we can say that the blue triangle is almost a right triangle, and then δ is

$$\delta = r_2 - r_1 \approx d\sin\theta$$

Our wave repeats every 2π radians or every wavelength, λ, then we have constructive interference (a bright spot) when

$$\Delta\phi = k\delta \approx kd\sin\theta = 2\pi m \quad (m = 0, \pm 1, \pm 2 \ldots)$$

We can solve this for δ

$$\delta = d\sin\theta = m\lambda \quad (m = 0, \pm 1, \pm 2 \ldots)$$

and we can give m a name. It is called the *order number*. That is, if we are off by any number of whole wavelengths then our total phase due to path difference will be a multiple of 2π. Question 123.26.6

If we assume that $\lambda \ll d$ we can find the distance from the axis for each fringe more easily. This guarantees that θ will be small. Using the yellow triangle we see

$$\tan\theta = \frac{y}{L}$$

but if θ is small this is just about the same as

$$\sin\theta = \frac{y}{L}$$

because for small angles $\tan\theta \approx \sin\theta \approx \theta$. So if θ is small then

$$\begin{aligned} \delta &= d\sin\theta \\ &= d\frac{y}{L} \end{aligned}$$

and for a bright spot or *"fringe"* we find

$$d\frac{y}{L} = m\lambda$$

Solving for the position of the bright spots gives

$$y_{bright} \approx \frac{\lambda L}{d} m \quad (m = 0, \pm 1, \pm 2 \ldots) \tag{11.4}$$

We can measure up from the central spot and predict where each successive bright spot will be.

Destructive Interference

We can also find a condition for destructive interference. We know that a path difference of an odd multiple of a half wavelength will give distractive interference. so

$$\Delta\phi = k\delta \approx kd\sin\theta = \left(m + \frac{1}{2}\right) 2\pi \qquad m = 0, \pm 1, \pm 2, \pm 3, \cdots$$

or using $k = 2\pi/\lambda$

$$\delta = d\sin\theta = \left(m + \frac{1}{2}\right)\lambda \quad (m = 0, \pm 1, \pm 2 \ldots)$$

will give a dark fringe. The location of the dark fringes will be

$$y_{dark} \approx \frac{\lambda L}{d}\left(m + \frac{1}{2}\right) \quad (m = 0, \pm 1, \pm 2 \ldots) \tag{11.5}$$

Double Slit Intensity Pattern

The fringes we have seen are not just points, but are patterns that fade from a maximum intensity. We can calculate the intensity pattern. We need to know a little bit about electric fields to do this.

We can represent an electromagnetic wave using just the electric field (the magnetic field pattern is very similar and can be derived from the electric field pattern) as we did at the beginning of this lecture

$$\begin{aligned} E_1 &= E_{\max}\sin\left(k_2 r_2 - \omega_2 t_2 + \phi_2\right) \\ E_2 &= E_{\max}\sin\left(k_1 r_1 - \omega_1 t_1 + \phi_1\right) \end{aligned}$$

and the resulting wave will be

$$\begin{aligned} E_r &= E_{\max}\sin\left(k_2 r_2 - \omega_2 t_2 + \phi_2\right) + A\sin\left(k_1 r_1 - \omega_1 t_1 + \phi_1\right) \\ &= 2E_{\max}\cos\left(\frac{(k_2 r_2 - \omega_2 t_2 + \phi_2) - (k_1 r_1 - \omega_1 t_1 + \phi_1)}{2}\right) \\ &\quad \times\sin\left(\frac{(k_2 r_2 - \omega_2 t_2 + \phi_2) + (k_1 r_1 - \omega_1 t_1 + \phi_1)}{2}\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}\left[(k_2 r_2 - \omega_2 t_2 + \phi_2) - (k_1 r_1 - \omega_1 t_1 + \phi_1)\right]\right) \\ &\quad \times\sin\left(\frac{(k_2 r_2 - \omega_2 t_2 + \phi_2) + (k_1 r_1 - \omega_1 t_1 + \phi_1)}{2}\right) \end{aligned}$$

but now we know that we can simplify this because $\omega_2 = \omega_1 = \omega$, $k_2 = k_1 = k$, $t_2 = t_1 = t$, and $\phi_2 = \phi_1 = \phi_o$.

$$\begin{aligned} E_r &= 2E_{\max}\cos\left(\frac{1}{2}k\delta\right)\sin\left(\frac{kr_2 + kr_1}{2} - \omega t + \phi_o\right) \\ &= 2E_{\max}\cos\left(\frac{1}{2}\frac{2\pi}{\lambda}d\sin\theta\right)\sin\left(k\frac{r_2 + r_1}{2} - \omega t + \phi_o\right) \end{aligned}$$

We have a combined wave at point P that is a traveling wave $\left(\sin\left(k\frac{(r_2+r_1)}{2} - \omega t + \phi_o\right)\right)$ but with amplitude $\left(2E_o\cos\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right)\right)$ that depends on our total phase $\Delta\phi = \frac{2\pi}{\lambda}d\sin\theta$.

But the situation is more complicated because of how we detect light. Our eyes, film, and most detectors measure the intensity of the light. We know that

$$I = \frac{\mathcal{P}}{A}$$

In PH 220 you will learn that the power is proportional to the square of the electric field wave amplitude.

$$\mathcal{P} \propto E^2 \tag{11.6}$$

Then the intensity must also be proportional to the amplitude of the electric field squared.

$$\begin{aligned} I &= \frac{P}{A} \propto E^2 \\ &\propto 4E_o^2 \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right)\sin^2\left(\frac{k\left(r_2+r_1\right)}{2}-\omega t+\phi_o\right) \end{aligned}$$

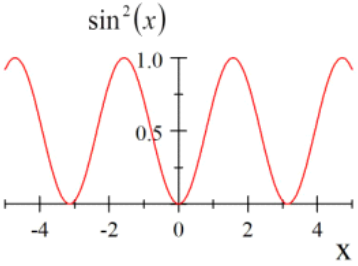
Light detectors collect power for a set amount of time. So most light detection will be a value averaged over a set *integration time*. This means that the detector sums up (or integrates) the amount of power received over the detector time. Usually the integration time is much longer than a period, so we need to time-average our intensity.

$$\begin{aligned} \int_{\text{many periods}} I dt &\propto = \int_{\text{many periods}} 4E_o^2 \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right)\sin^2\left(\frac{k\left(r_2+r_1\right)}{2}-\omega t+\phi_o\right)dt \\ &= 4E_o^2 \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right)\int_{\text{many periods}} \sin^2\left(\frac{k\left(rx_2+r_1\right)}{2}-\omega t+\phi_o\right)dt \end{aligned}$$

but the term

$$\int_{\text{many periods}} \sin^2\left(\frac{k\left(r_2+r_1\right)}{2}-\omega t+\phi o\right)dt = \frac{1}{2} \tag{11.7}$$

To convince yourself of this, think that $\sin^2(x)$ has a maximum value of 1 and a minimum of 0. Looking at the graph



should be believable that the average value over a period is $1/2$. The average over many periods will still be $1/2$.

So we have

$$\bar{I} = \int_{\text{many periods}} I dt \propto 2E_o^2 \cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right) \tag{11.8}$$

where $\bar{I}$ is the time average intensity. The important part is that the time varying part has averaged out.

So, usually in optics, we ignore the fast fluctuating parts of such calculations because we can't see them and so we write

$$I = I_{\max}\cos^2\left(\frac{1}{2}\left(\frac{2\pi}{\lambda}d\sin\theta\right)\right)$$

where we have dropped the bar from the I, but it is understood that the intensity we report is a time average over many periods.

We should remind our selves, our intensity pattern

$$I = I_{\max}\cos^2\left(\frac{1}{2}\frac{2\pi}{\lambda}d\sin\theta\right)$$

is really

$$I = I_{\max}\cos^2\left(\frac{\Delta\phi}{2}\right)$$

Which is just our amplitude squared for the mixing of two waves. All we have done to find the intensity pattern is to find and expression for the phase difference $\Delta\phi$.

Our intensity pattern should give the same location for the center of the bright spots as we got before. Let's check that it works. We used the small angle approximation before. It is still valid, so let's use it again now. For for small angles

$$\begin{aligned} I &= I_{\max}\cos^2\left(\frac{\pi d}{\lambda}\theta\right) \\ &= I_{\max}\cos^2\left(\frac{\pi d}{\lambda}\frac{y}{L}\right) \end{aligned}$$

Then we have constructive interference when

$$\frac{\pi d}{\lambda}\frac{y}{L} = m\pi$$

or

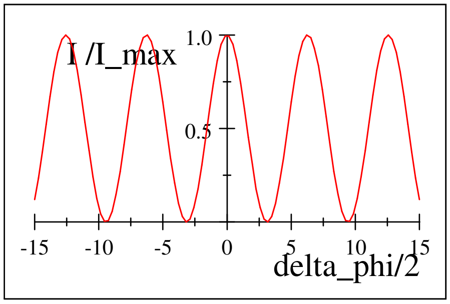
$$y = m\frac{L\lambda}{d}$$

which is what we found before.

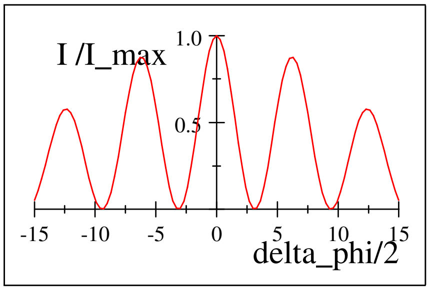
The plot of normalized intensity

$$\frac{I}{I_{\max}} = \cos^2\left(\frac{\Delta\phi}{2}\right)$$

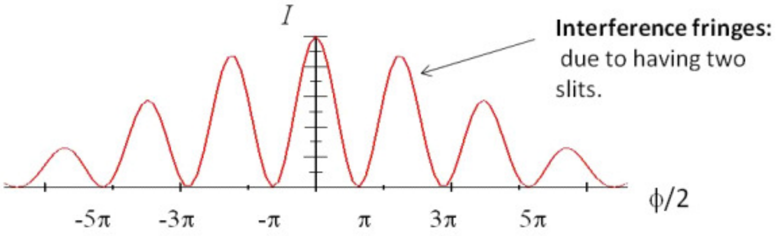
verses $\Delta\phi/2$ is given next,

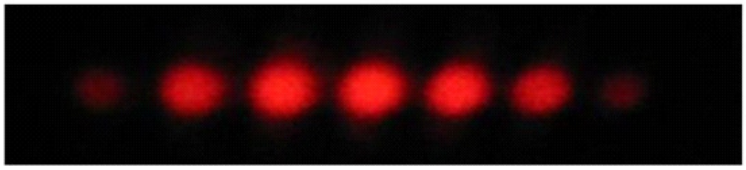


but we will find that we are not quite through with this analysis. Next time we will find that there is another compounding factor that reduces the intensity as we move away from the midpoint.



Let's pause to remember what this pattern means. This is the intensity of light due to interference. It is instructive to match our intensity pattern to the pattern Young saw with our graph.





The high intensity peaks are the bright fringes and the low intensity troughs are the dark fringes. The pattern moves smoothly and continuously from bright to dark.

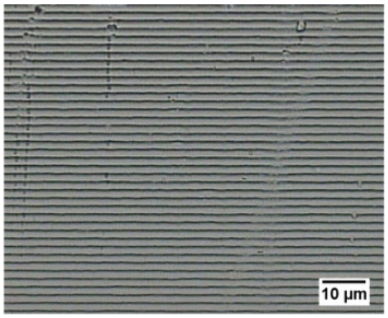
12 Many Slits, and Single Slits

Fundamental Concepts

Diffraction Gratings

Rainbow Glasses

A diffraction grating is an optical element with many many parallel slits spaced very close together. Here is a typical diffraction grating created by etching lines in a piece of glass. The etchings scatter the light, but the un-etched part allows the light to pass through. The un-etched parts are essentially a series of slits.



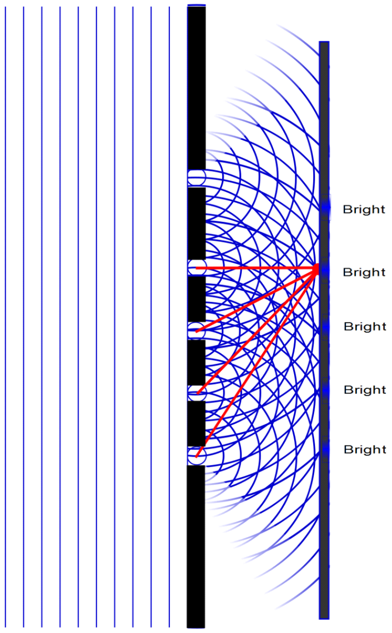
Surface of a diffraction grating (600 lines/mm). Image taken with optical transmission microscope. (Image in the public domain courtesy Scapha)

A typical grating might have 5000 slits per unit centimeter. You have probably used a diffraction grating to see rainbow colors in a beginning science class. Gratings are usually made by cutting parallel grooves in a flat surface.

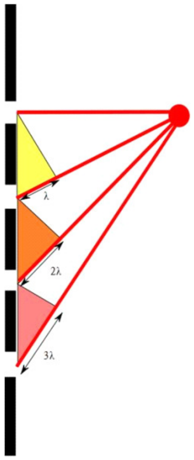
If we use $5000\frac{grooves}{\mathrm{cm}}$ for an example, we see that the slit spacing is

$$\begin{aligned} d &= \frac{1}{5000}\,\mathrm{cm} \qquad (12.1) \\ &= 2.0\times 10^{-6}\,\mathrm{m} \qquad (12.2) \end{aligned}$$

Take a section of diffraction grating as shown below



At some point, two of the slits will have a path difference that is a whole wavelength, and we would expect a bright spot. But what about the other slits? If we have a slit spacing such that each of the succeeding slits has a path difference that is just an additional wavelength, then each of the slits will contribute to the constructive interference at our point, and the point will become a bright spot.

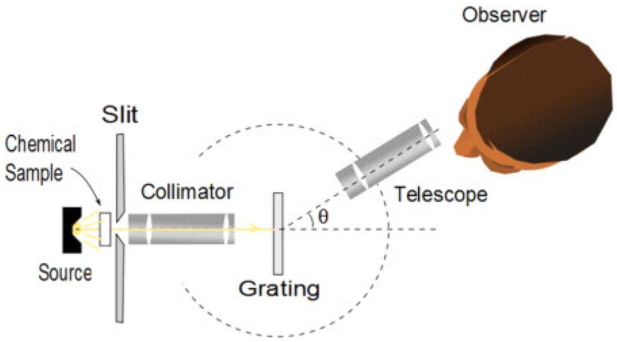


Let's look at just two slits. The light leaves each slit in phase with the light from the rest of the slits, but at some distance L away and at some angle θ we will have a path difference

$$\delta = d\sin\left(\theta_{bright}\right) = m\lambda \quad m = 0, \pm 1, \pm 2, \ldots \tag{12.3}$$

because the path lengths are not all the same.

This equation tells us that each wavelength, λ, will experience constructive interference at a slightly different angle θ_{bright}. Different frequencies will create bright spots at different angles. We have found a way to create a spectrum with light waves. We often call a spectrum made with visible light frequencies a rainbow. Knowing d and θ allows an accurate calculation of λ. This may seem a silly thing to do, but suppose we add into our system a sample of a chemical to identify



We could then record the intensity of the transmitted light as a function of angle, which is equivalent to λ. We can again generate a spectrum. This is a traditional way to build a spectrometer and many such devices are available today.

Demo a student spectrometer with a gas tube

Resolving power of diffraction gratings

We noticed that with two slits, we got a bright spot for a particular wavelength, but we didn't just get one bright spot. We got several. The same is true for diffraction gratings. So we expect to get a rainbow, but we really expect to get a series of rainbows. The integer m tells us which rainbow we have in the series. The integer m is called the order number.

For two nearly equal wavelengths λ_1 and λ_2, we say that the diffraction grating can resolve the wavelengths if we can distinguish the two using the grating. The *resolving power* of the grating is defined as

$$R = \frac{\left(\lambda_1 + \lambda_2\right)}{2\left(\lambda_1 - \lambda_2\right)} = \frac{\bar{\lambda}}{\Delta\lambda} \tag{12.4}$$

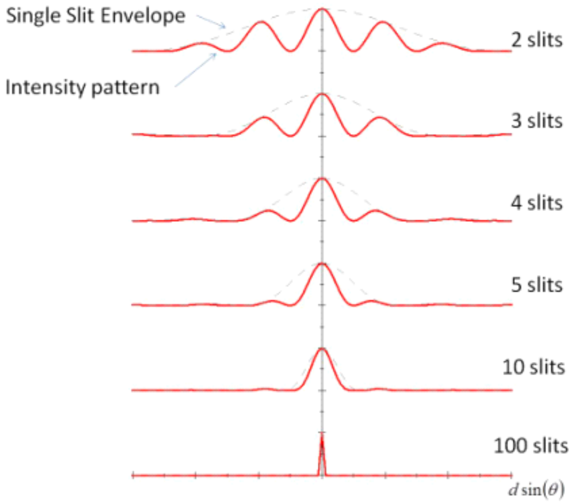
We can show that for the m-th order diffraction, the resolving power is

$$R = Nm \tag{12.5}$$

where N is the number of slits. So our ability to distinguish wavelengths increases with the number of slits and with the order (which is related to how far off-axis we look).

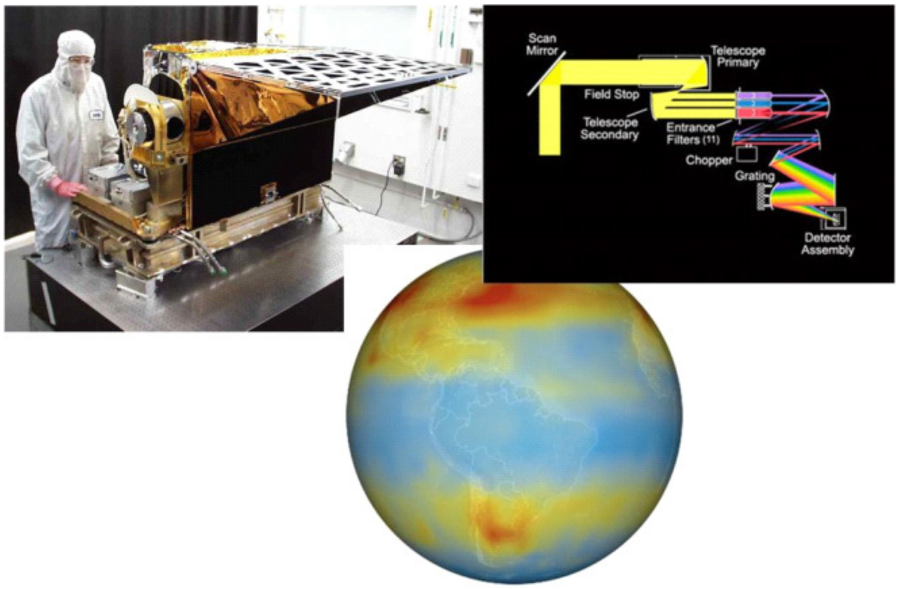
Note that for $m = 0$ we have no ability to resolve wavelengths. The central peak is a mix of all wavelengths and usually looks white for normal illumination.

That the resolution depends on the number of slits, N, means that we can improve our spectrometer by using more lines. Here is a representation of what happens as we increase N



we can see that the peaks get narrower as N increases. These graphs are for a particular λ. If the peaks for a particular λ get narrower, then there will be less overlap with adjacent $\lambda's$ which means that each wavelength can more easily be resolved.

Spectrometers are used in many places. On that has some public interest today is monitoring the atmosphere. Instruments like the one shown below detect the amount of special gasses in the atmosphere using IR spectrometers.



AIRS sensor, spectrometer design, and global CO_2 data. (Images in the Public Domain courtesy NASA)

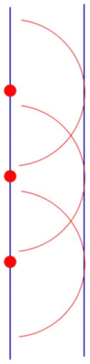
The instrument shown is the AIRS spectrometer. You can see in the diagram that it uses a grating spectrometer. The picture of the Earth is a composite of AIRS data showing the northern and southern bands of CO_2.

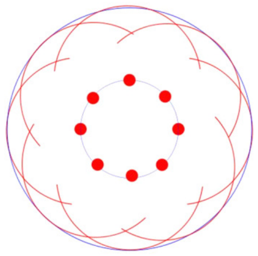
Single Slits

We have looked at interference from two slits, and for many slits. The two slits acted like two coherent sources. We might expect that a single slit will give only a single bright spot. But let's consider a single slit very closely. To do this, let's return to the work of Huygens.[9] His idea for the nature of light was simple. He suggested that every point on the wave front of a light wave was the source (the disturbance) for a new set of small spherical waves. In optics, the crests of the waves are often called wavefronts. The next wavefront would be formed by the superposition of the little "wavelets." Here is an example for a plane wave and a spherical wave.

---

[9] Huygens method is technically not a correct representation of what happens. The actual wave leaving the single opening is a superposition of the original wave, and the wave scattered from the sides of the opening. You can see this scattering by tearing a small hole in a piece of paper and looking through the hole at a light source. You will see the bright ring around the hole where the edges of the paper are scattering the light. But the mathematical result we will get using Huygens method gives a mathematically identical result for the resulting wave leaving the slit with much less high power math. So we will stick with Huygens in this class.





In each case we have drawn spots on the wave front and drawn spherical waves around those spots. where the wavefronts of the little wavelets combine, we have new wave front of our wave. This is sort of what happens in bulk matter. Remember that light is absorbed and re-emitted by the atoms of the material. This is why light slows down in a material. Because of the time it is absorbed, it effectively goes slower. But the light is not necessarily re-emitted in the same direction. Sometimes it is, but sometimes it is not. This creates a small, spherical wave (called a wavelet) that is emitted by that atom. So Huygens idea is not too bad.

We can use this idea for a single slit and look at what happens as the light goes through. Here is such a slit.

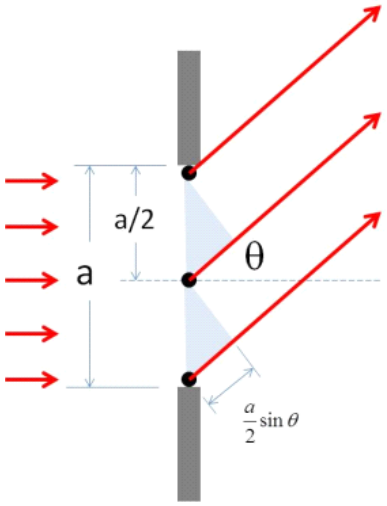
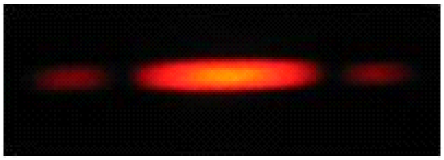


Figure 12.2.

In the figure above, we have divided a single slit of width a into two parts, each of size

$a/2$. According to Huygens' principle, each position of the slit acts as a source of light rays. So we can treat half a slit as two coherent sources. These two sources should interfere. So what do we see when we perform such an experiment?

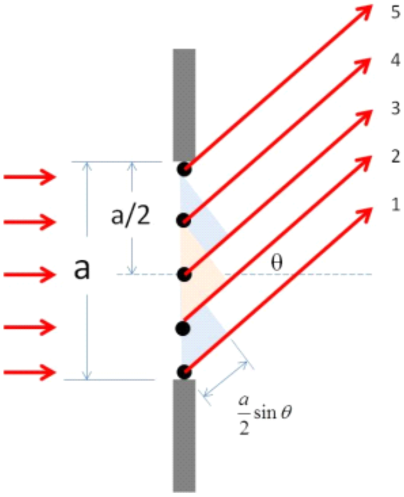


The figure shows a diffraction pattern for a thin slit. There are several terms that are in common use to describe the pattern

1. Central Maximum: The broad intense central band.
2. Secondary Maxima:The fainter bright bands to both sides of the central maxima
3. Minima: The dark bands between the maxima

Narrow Slit Intensity Pattern

Let's use figure 12.2 to find the dark minima of the single slit pattern. First we should notice that figure 12.2 could have another set of rays that contribute to the bright spot because they will also have a path difference of $(a/2)\sin\theta$. Let's fill these in. They are rays 2 and 4 of the next figure.



Before we started with what we are now calling rays 1 and 3. Ray 1 travels a distance

$$\delta = \frac{a}{2}\sin\left(\theta\right) \tag{12.6}$$

farther than ray 3. As we just argued, rays 2 and 4 also have the same path difference, and so do rays 3 and 5. If this path difference is $\lambda/2$ then we will have destructive interference. The condition for a minima is then

$$\frac{a}{2}\sin\left(\theta\right) = \pm\frac{\lambda}{2} \tag{12.7}$$

or

$$\sin\left(\theta\right) = \pm\frac{\lambda}{a} \tag{12.8}$$

Now we could also divide the slit into four equal parts. Then we have a path difference of

$$\delta = \frac{a}{4}\sin\left(\theta\right) \tag{12.9}$$

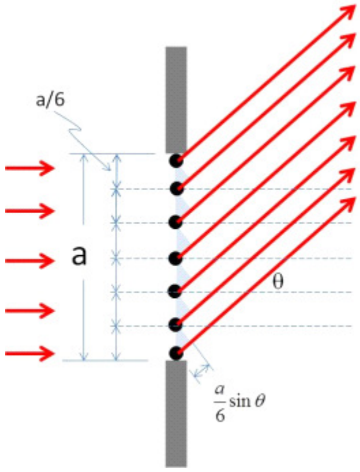
and to have destructive interference we need this path difference to be $\lambda/2$

$$\frac{a}{4}\sin\left(\theta\right) = \pm\frac{\lambda}{2} \tag{12.10}$$

or

$$\sin\left(\theta\right) = \pm\frac{2\lambda}{a} \tag{12.11}$$

We can keep going to find a minima at



$$\sin\left(\theta\right) = \pm\frac{3\lambda}{a} \tag{12.12}$$

and in general at

$$\sin\left(\theta\right) = m\frac{\lambda}{a} \quad m = \pm1, \pm2, \pm3\ldots \tag{12.13}$$

You might object that we did not find the bright spots, only the dark spots. That's fine because the bright spots have to be in between the dark spots. But we can do better. Let's look at the full intensity pattern for a single slit.

Intensity of the single-slit pattern

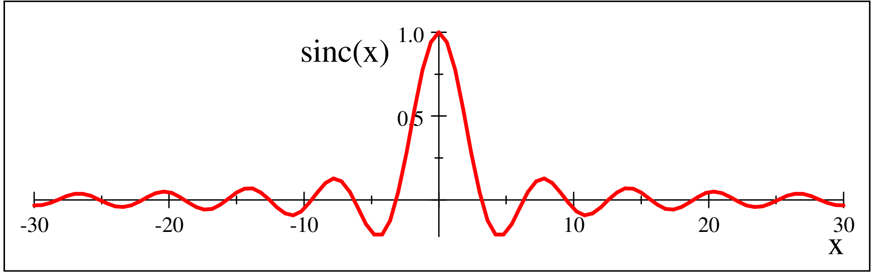
I will not derive the intensity pattern for the single slit (though it is not really too hard to do) but I will give it here

$$I = I_{\max}\left(\frac{\sin\left(\frac{\pi}{\lambda}a\sin\theta\right)}{\frac{\pi}{\lambda}a\sin\theta}\right)^2 \tag{12.14}$$

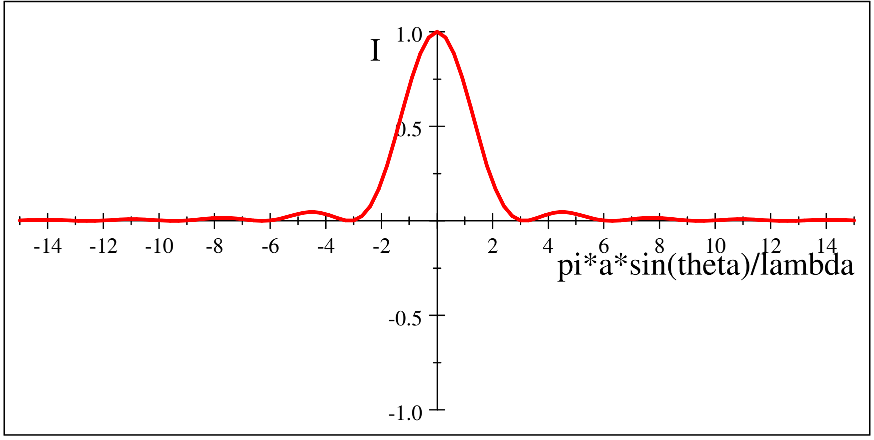
Notice this has the form

$$\frac{\sin x}{x}$$

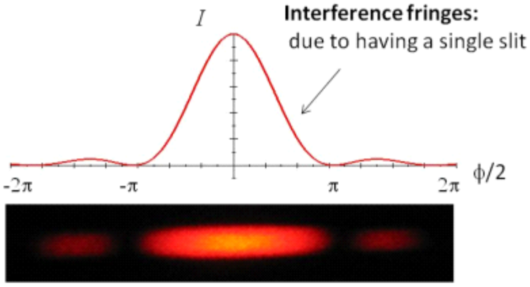
which has a distinctive shape.



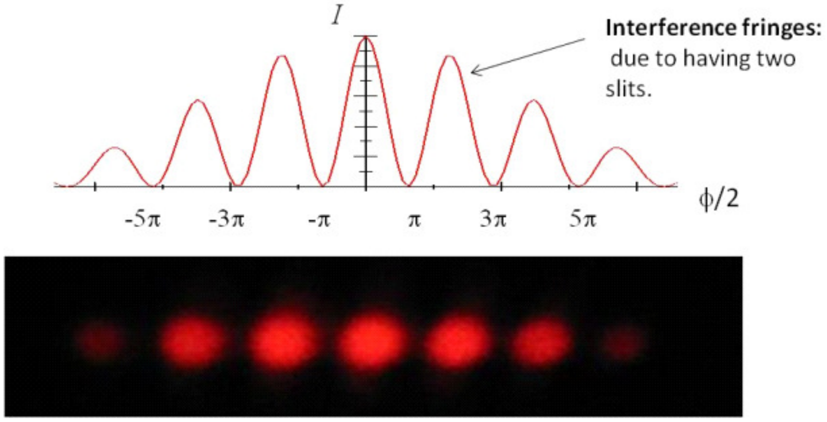
this is known as a sinc function (pronounced like "sink"). It has a central maximum as we would expect. Of course our pattern has a sinc squared



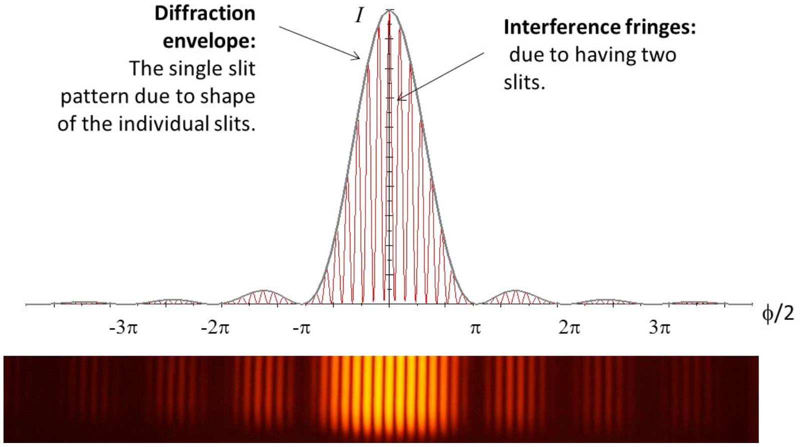
You can see the central maximum and the much weaker minima produced by this function. Indeed, it seems to match what we saw very well. Putting it all together, our pattern looks like this.



This is really an interesting result. You might wonder why, when we found the two slit interference pattern, there was no evidence of the single slit fringing that we discovered in this chapter. After all, a double slit system is made from single slits. Shouldn't there be some effect due to the fact that the slits are individually single slits? The answer is that we did see some hint of the single slit pattern. Remember the figure below.



The intensity of the peaks seems to fall off with distance from the center. We dealt with only the center-most part of the pattern. If we draw the pattern for larger angles, we see the following.



It takes a bright laser or dark room to see the secondary groups of fringes easily, but we can do it. We can also graph the intensity pattern. It is the combination of the two slit and single slit pattern with the single slit pattern acting and an envelope.

$$I = I_{\max}\cos^2\left(\frac{\pi d\sin(\theta)}{\lambda}\right)\left(\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{\frac{\pi a\sin(\theta)}{\lambda}}\right)^2 \tag{12.15}$$

Note that one of the double slit maxima is clobbered by a minimum in the single slit pattern. We can find out the order of the missing maximum. Recall that

$$d\sin(\theta) = m\lambda$$

describes the maxima from the double slit. But

$$a\sin(\theta) = \lambda$$

describes the minimum from the single slit. Dividing these yields

$$\begin{aligned}\frac{d\sin(\theta)}{a\sin(\theta)} &= \frac{m\lambda}{\lambda}\\ \frac{d}{a} &= m\end{aligned}$$

so the

$$m = \frac{d}{a} \tag{12.16}$$

double slit maximum will be missing.

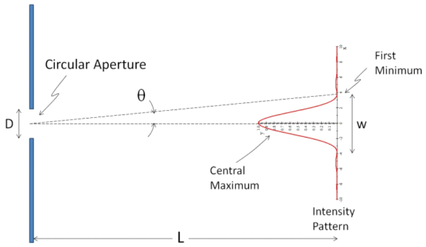
Circular Apertures

Question 123.27.4

Question 123.27.5

Question 123.27.6

Our analysis of light going through holes has been somewhat limited by squarish holes or slits. But most optical systems, including our eyes don't have square holes. So what happens when the hole is round? The situation is as shown in the next figure.



Before we discuss this situation, let's think about the width of a single slit pattern. We remember that

$$\sin(\theta) = (1)\frac{\lambda}{a}$$

for the first minima,or that

$$\theta \approx \frac{\lambda}{a}$$

and from the figure we can see that

$$\theta \approx \frac{y}{L}$$

so long as θ is small, then we find the position of the first minimum to be

$$y \approx \frac{\lambda}{a}L$$

This is the distance from the center bright spot to the first dark spot. The width of the bright spot is twice this distance

$$w \approx 2\frac{\lambda}{a}L$$

We expect something like this for our circular aperture. The derivation is not to hard, but it involves Bessel functions, which are beyond the math requirement for this course. So I will give you the answer

$$\theta \approx 1.22\frac{\lambda}{D}$$

That's right, the circular aperture (hole) only adds a factor of 1.22. And as with the slit

$$\theta \approx \frac{y}{L}$$

so

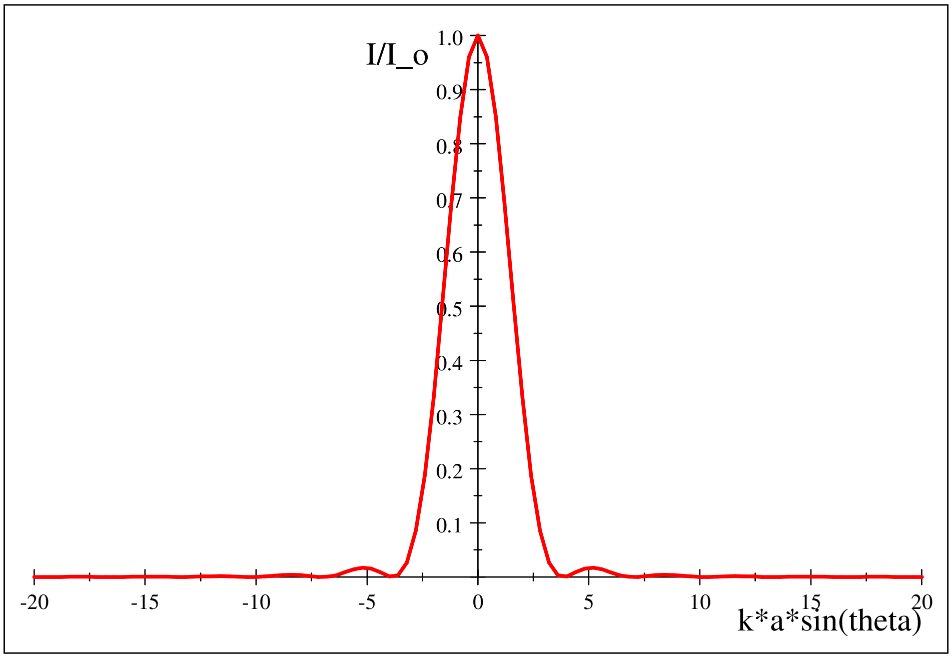
$$y \approx 1.22\frac{\lambda}{D}L \tag{12.17}$$

and

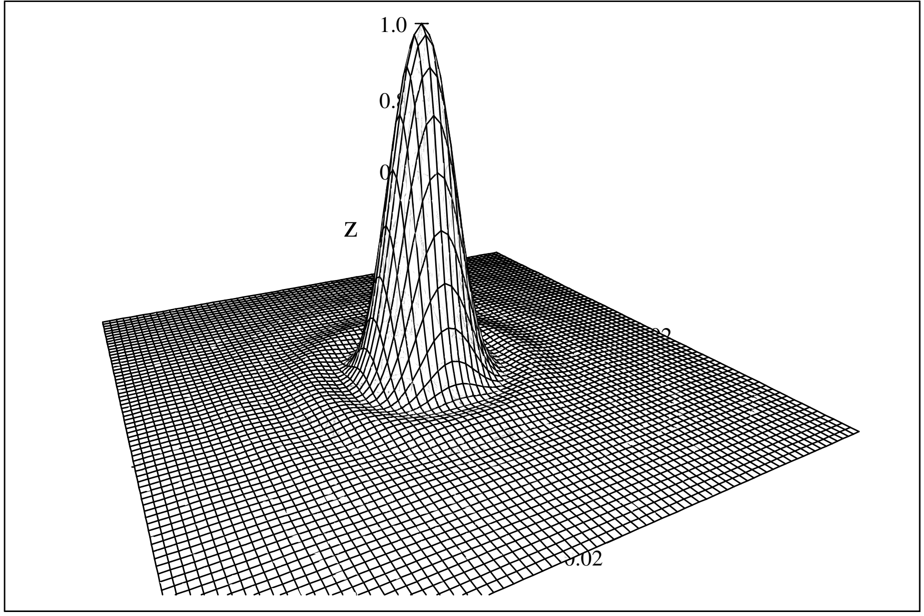
$$w \approx 2.44\frac{\lambda}{D}L \tag{12.18}$$

The picture in most books is a little bit deceptive. The pattern looks a little like the slit pattern. But the secondary maxima are actually very small for the circular aperture case. Much smaller than the secondary maxima in the slit case. Here is a larger version.

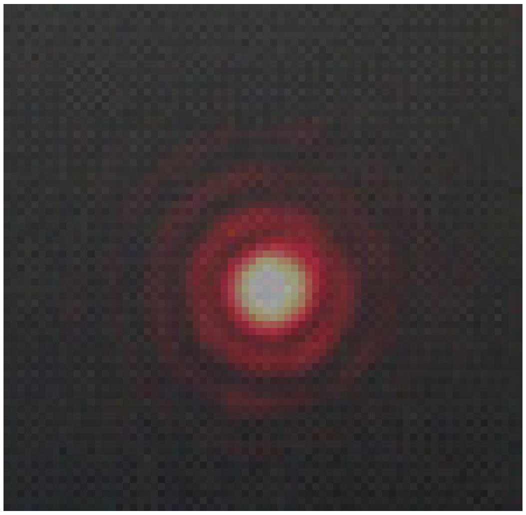
Airy Pattern Demo



A three dimensional version of the intensity pattern from the circular aperture.



With a bright enough laser, they pattern becomes visible.



13 Interferometers and Rays

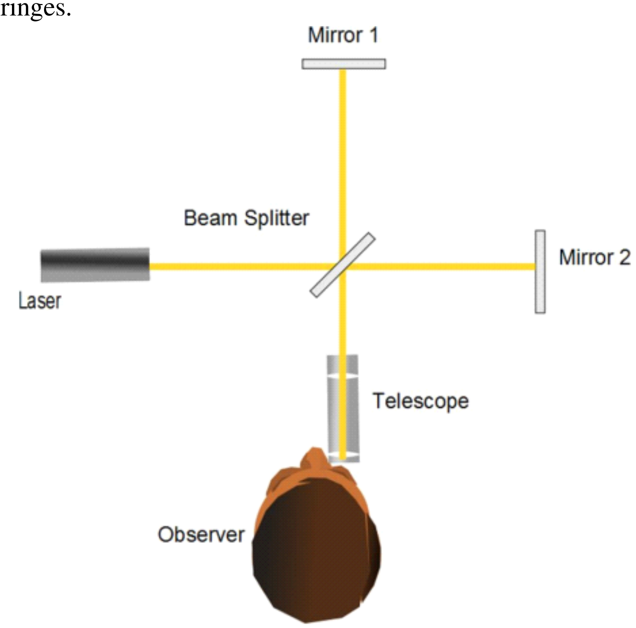
Fundamental Concepts

Interferomenter Demo

Before we leave wave properties of physics and go to the ray approximation, we should study some devices that use interference.

The Michelson Interferometer

The Michelson interferometer is another device that uses path differences to create interference fringes.



The device is shown in the figure. A coherent light source is used. The light beam is split into two beams that are usually at 90° apart. The beams are reflected off of two mirrors back along the same path and are mixed at the telescope. The result (with perfect alignment) is a target fringe pattern like the first two shown below.

Bright Center Dark Center Small Off-Center Fringes

If the alignment is off, you get smaller fringes, but the system can still work. This is shown in the last image in the previous figure.

In the figure, we have constructive interference in the center, but if we move one of the mirrors half a wavelength, we would have destructive interference and would see a dark spot in the center. This device gives us the ability to measure distances on the order of the wavelength of the light. When the distance is continuously changed, the pattern seems to grow from the center (or collapse into the center).

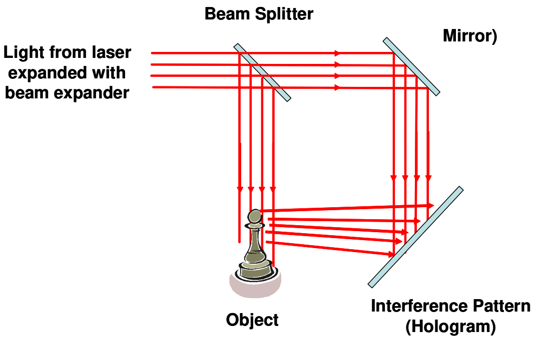
Notice that if the mirror is moved $\frac{\lambda}{2}$, the path distance changes by λ because the light travels the distance to the mirror and then back from the mirror (it travels the path twice!).

Holography

Hologram demo-picture of woman

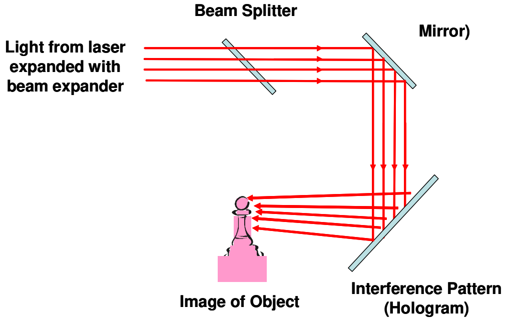
Hologram demo-chess pieces

You may have seen holograms in the past. We have enough understanding of light to understand how they are generated now.



A device for generating a hologram is shown in the figure above. Light from a laser or other coherent source is expanded and split into two beams. One travels to a photographic plate, the other is directed to an object. At the object, light is scattered and the scattered light also reaches the photographic plate. The combination of the direct and scattered beams generates a complicated interference pattern.

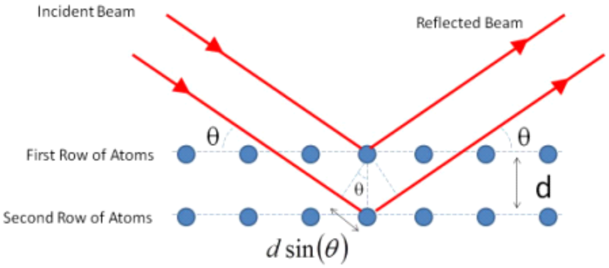
The pattern can be developed (like you develop photographic film). Once developed, it can be re- illuminated with a direct beam. The emulsion on the plate creates complicated patterns of light transmission, which combine to create interference. It is like a very complicated slit pattern or grating pattern. The result is a three-dimensional image generated by the interference. The interference pattern generates an image that looks like the original object.



Diffraction of X-rays by Crystals

If we make the wavelength of light very small, then we can deal with very small diffraction gratings. This concept is used to investigate the structure of crystals with x-rays. The crystal latus of molecules or atoms creates the regular pattern we need for a grating. The pattern is three dimensional, so the patterns are complex.

Let's start with a simple crystal with a square regular latus. $NaCl$ has such a structure.



If we illuminate the crystal with x-rays, the x-rays can reflect off the top layer of atoms, or off the second layer of atoms (or off any other layer, but for now let's just consider two layers). If the spacing between the layers is d, then the path difference will be

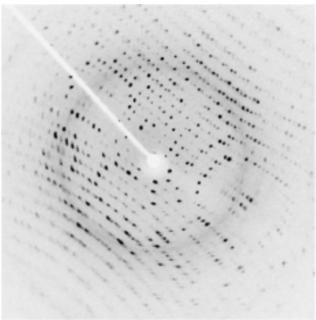
$$\delta = 2\left(d\sin\left(\theta\right)\right) \tag{13.1}$$

then for constructive interference

$$2d\sin\left(\theta\right) = m\lambda \qquad m = 1, 2, 3, \ldots \tag{13.2}$$

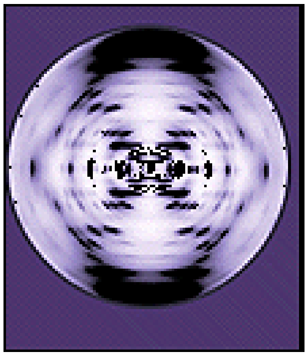
This is known as *Bragg's law.* This relationship can be used to measure the distance between the crystal planes.

A resulting pattern is given in the following figure.



Diffraction image of protein crystal. Hen egg lysozyme, X-ray souce Bruker IμS, $\lambda = 0.154188\,\mathrm{nm}$, $45\,\mathrm{kV}$, Exposure $10\,\mathrm{s}$. (image in the public domain)

DNA makes in interesting diffraction pattern.

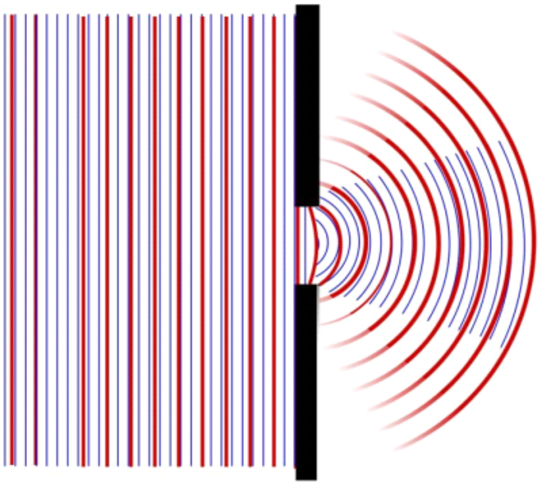


X-ray diffraction pattern of DNA (image courtesy of the National Institute of Health, image in the public domain)

Transition to the ray model

Question 223.28.1

In the next figure, two waves of different wavelengths go through a single opening. The wave representing the central maxima is shown in each case, but not the secondary maxima.



Notice that the smaller wavelength has a narrower central maxima as we would expect from

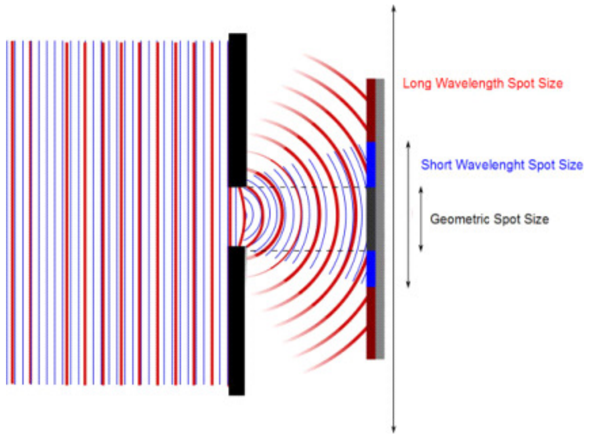
$$\sin\left(\theta_{dark}\right) = 1.22\frac{\lambda}{D}$$

or

$$\theta_{dark} \approx 1.22\frac{\lambda}{D}$$

we see that the ratio of the wavelength to the hole size determines the angular extent of the central maxima. The smaller the ratio, the smaller the central region. We can use this to explain why the wave nature of light was so hard to find.

The patch of light on a screen that is created by light passing through the aperture is created by the central maximum.



For the long wavelength (red) the central maximum is larger than the screen. The short wavelength spot will be wholly on the screen as shown. The geometric spot is what we would see if the light traveled straight through the opening. Notice that the short wavelength spot is closer to the size of the geometric spot. In the limit that

$$\lambda \ll a$$

or for circular openings

$$\lambda \ll D$$

then

$$\theta \approx \frac{\lambda}{a} \approx 0$$

or

$$\theta \approx \frac{\lambda}{D} \approx 0$$

and the spot size would be very nearly equal to the geometric spot size.

This is the limit we will call the *ray approximation*.

For most of mankind's time on the Earth, it was very hard to build holes that were comparable to the size of a wavelength of visible light. So it is no wonder that the waviness of light was missed for so many years.

But this ray limit is very useful for apertures the size of camera lenses. So we will begin to use this small λ, large aperture approximation.

The Ray Approximation in Geometric Optics

In the last section we said that when the geometric spot size was larger than the spot due to diffraction, we could ignore diffraction and use the simpler ray model. This is usually true in our personal experiences. But this may not be true in experiments or devices we design. We should see where the crossover point is.

Intuitively, if the aperture and the spot are the same size, that ought to be some sort of critical point. That is when the aperture size is equal to the spot size. We found that for a circular aperture the spot width is

$$w \approx 2.44 \frac{\lambda}{D_{aperture}} L$$

We want the case where $w = D_{aperture}$

$$D_{aperture} = 2.44 \frac{\lambda}{D_{aperture}} L$$

This gives

$$D_{aperture} = \sqrt{2.44 \lambda L}$$

Of course this is for round apertures, but for square apertures we know we remove the 2.44. This gives about a millimeter for visible wavelengths.

$$\begin{aligned} D_{aperture} &= \sqrt{2.44\,(500\,\mathrm{nm})\,(1\,\mathrm{m})} \\ &= 1.104\,5 \times 10^{-3}\,\mathrm{m} \end{aligned}$$

for apertures much larger than a millimeter, we expect interference effects due to diffraction through the aperture to be much harder to see. We expect them to be easy to see if the aperture is smaller than a millimeter. But what about when the aperture is about a millimeter in size? That is a subject for PH375, and so we will avoid this case in this class. But this is not too restrictive. Most good optical systems have apertures larger than $1\,\mathrm{mm}$. Cell phone cameras may be an exception (but I don't consider cell phone cameras to be good optical systems). Even our eyes have an aperture that varies from about $2\,\mathrm{mm}$ to about $7\,\mathrm{mm}$, so most common experiences in visible wavelengths will work fine with what we learn. Note that for microwave or radio wave systems this may really not be true!

How about the other extreme? Suppose $\lambda \gg D$. This is really beyond our class (requires partial differential equations), but in the extreme case, we can use reason to find out what happens. If the opening is much smaller than the wavelength, then the wave does not see the opening, and no wave is produced on the other side. This is the case of a microwave oven door. If the wavelength is much larger than the spacing of the little dots or lines that span the door, then the waves will not leave the interior of the microwave oven. Of course as the wavelength becomes closer to D this is less true, and this case is more challenging to calculate, and we will save it for a 300 level electrodynamics course.

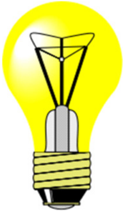
To summarize

$\lambda \ll D$	Wave nature of light is not visible
$\lambda \approx D$	Wave nature of light is apparent
$\lambda \gg D$	Little to no penetration of aperture by the wave

The ray model and phase

Question 123.28.2

There is a further complication that helps to explain why the wave nature of light was not immediately apparent. Let's consider a light source.



For a typical light source, the filament is larger than about a millimeter, so we should expect that diffraction should be hard to see. But the filament is made of hot metal. The atoms of the hot metal emit light because of the extra energy they have. The method of producing this light is that the atom's excited electrons are in upper shells because of the extra thermal energy provided by the electricity flowing through the filament. But the electrons eventually fall to their proper shell, and in doing so they give off the extra energy as light. It is not too hard to believe that this process of exciting electrons and having them fall back down is a random process. Each electron that moves starts a wave. The atoms have different positions, so there will be a path difference Δr between each atom's waves. There will also be a time difference Δt between when the waves start. We can model this with a $\Delta \phi_o$.

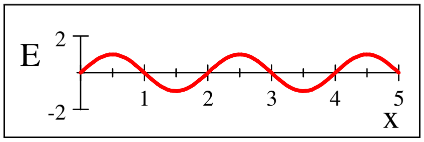
Question 123.28.3

It is also true that not all of the electrons fall from the same shell. This gives us different frequencies, so we expect beating between different waves from different atoms. It is also true that we have millions of atoms, so we have millions of waves.

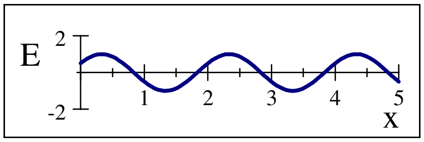
Let's look at just two of these waves

$$\begin{aligned}
&\lambda = 2 \\
&k = \tfrac{2\pi}{\lambda} \\
&\omega = 1 \\
&\phi_o = \tfrac{\pi}{6} \\
&t = 0 \\
&E_o = 1 \tfrac{\mathrm{N}}{\mathrm{C}}
\end{aligned}$$

$$E_1 = E_o \sin(kx - \omega t)$$



$$E_2 = E_o \sin(kx - \omega t + \phi_o)$$

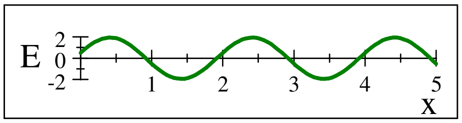


then

$$E_r = E_o \sin(kx - \omega t) + E_o \sin(kx - \omega t + \phi_o)$$

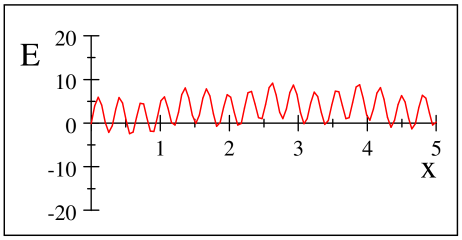
We found for these two waves

$$E_r = 2E_o \cos\left(\frac{\phi_o}{2}\right) \sin\left(kx - \omega t + \frac{\phi_o}{2}\right)$$

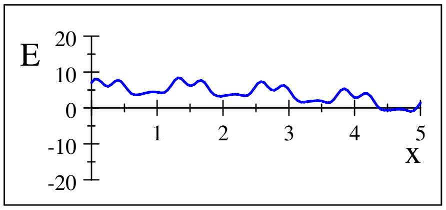


But suppose we complicate the situation by sending lots of waves at random times, each with different amplitudes and wavelengths, down the rope. If we look at a single point for a specific time, we might be experiencing interference, but it would be hard to tell. Lets try this mathematically. I will combine many waves with random phases, some coming from the right and some coming from the left.

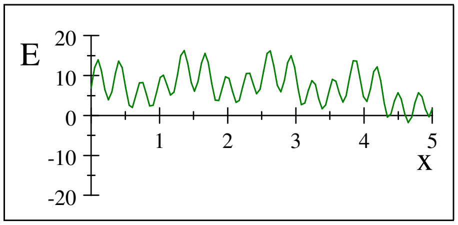
$$\begin{aligned} E_1 \quad = \quad & E_o \sin\left(5x - \omega t + \frac{\pi}{4}\right) + 0.5E_o \sin\left(0.2x - \omega t - \frac{\pi}{6}\right) \\ & +3.6E_o \sin\left(.4x - \omega t + \frac{\pi}{10}\right) + 4E_o \sin\left(20x - \omega t - \frac{\pi}{7}\right) \\ & +.2E_o \sin(15x - \omega t + 1) + 0.7E_o \sin(.7x - \omega t - .25) \end{aligned}$$



$$\begin{aligned} E_2 &= E_o \sin\left(0.2x + \omega t + \pi\right) + 2E_o \sin\left(5x + \omega t + \frac{\pi}{6}\right) \\ &\quad + 6E_o \sin\left(0.4x + \omega t + \frac{\pi}{3.5}\right) + 0.4E_o \sin\left(20x + \omega t - 0\right) \\ &\quad + E_o \sin\left(15x + \omega t + 1\right) + 0.7E_o \sin\left(.7x + \omega t - 4\right) \end{aligned}$$



Then $E_1 + E_2$ looks like



In this example, you could think about the superposition of E_1 and E_2 and predict the outcome, but if there were millions of waves, each with it's own wavelength, phase, and amplitude, the situation would be hopeless. Note that the fluctuations in these waves are much more frequent than our original waves. With all the added waves, we get a rapid change in amplitude.

Now if these waves are light waves, our eyes and most detectors are not able to react fast enough to detect the rapid fluctuations. So if there is constructive or destructive interference that might be simple enough to distinguish, we will miss it due to our detection systems' integration times. To describe this rapidly fluctuating interference pattern that we can't track with our detectors, we just say that light bulbs emit *incoherent light*. The ray approximation assumes incoherent light.

But then light bulbs and hot ovens and most things must emit incoherent light. Does any thing emit coherent light? Sure, today the easiest source of coherent light is a laser. That is why I have used lasers in the class demonstrations so far. Really though, even a laser is not perfectly coherent. One property of the laser is that it produces light with a long *coherence length*, or it produces light that can be treated under most circumstances as begin monochromatic and having a single phase across the wave for much of the beam length. Radar and microwave transmitters emit coherent light (but at frequencies we can't see) and so do radio stations.

In the past, one could carefully create a monochromatic beam with filters. Then split the beam into two beams and remix the two beams. This would generate two mostly coherent sources if the distances traveled were not too large. This is what Young did.

Coherency

Question 123.28.4

To be coherent,

1. A given part of the wave must maintain a constant phase with respect to the rest of the wave.
2. The wave must be monochromatic

These are very hard criteria to achieve. Most light, like that from our light bulb, is not coherent.

In the next lecture, we will leave behind the wave nature of light and consider in-coherent beams of light. Our goal will be to understand how light moves around us to allow us to see things.

14 Reflection and Refraction

Question 223.13.4

In the movie *Star Wars* inter-galactic star ships blast each other with laser cannons. The laser beams streak across the screen. This is dramatic, but not realistic. For us to see the light, some of the light must get to our eyes. The light must either travel directly to our eyes from the source, or it must bounce off of something. We can make a laser beam visible by providing dust for the light to bounce off of so it travels to our eyes. But unless that galaxy far far away is really dusty, Hollywood doesnt understand light waves very well. Let's start our study of geometric optics with reflection, the bouncing of light off of an object (like dust, or a mirror) and then consider what happens when light enters a material like glass or water.

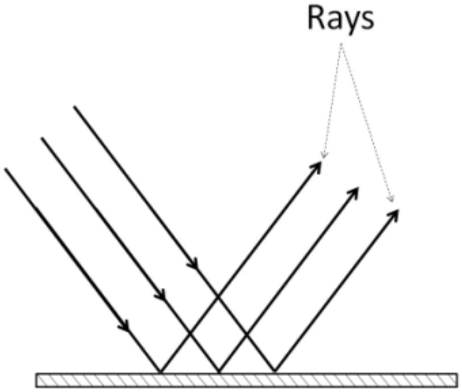
Fundamental Concepts

- Reflection
- Specular and Diffuse reflection
- Refraction
- Total internal reflection

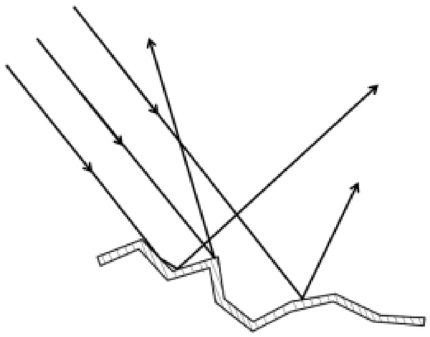
Question 223.13.5

Using the ray approximation we wish to find what happens when a bundle of rays reaches a boundary between materials. If the material boundary is very smooth, then the rays are reflected (bounce off) in a uniform way. This is called *specular* reflection

Specular and Diffuse Reflector Demo



If it is not smooth, then something different happens. The rays are reflected, but they are reflected randomly



This is called *diffuse* reflection

Question 223.13.6

This difference can be seen in real life. In the next figure, the surface on the left is a specular reflector and you can see the reflected light beam. But the surface on the right photo is diffuse, no reflected beam is seen.

We said the surface must be smooth for there to be specular reflection. What does smooth mean? Generally the size of the rough spots must be much smaller than a wavelength to be considered smooth. So suppose we have a red laser. How small do the surface variations have to be for the surface to be considered smooth? The wavelength of a $HeNe$ laser is

$$\lambda_{HeNe} = 633\,\mathrm{nm}$$

This is very small. Modern optics for remote sensing are often manufactured to $1/10$ of a wavelength, which would be $63\,\mathrm{nm}$. Mirrors are smooth for visible light.

How about a microwave beam of light like your cell phone uses?

$$\begin{aligned} c &= \lambda f \\ \lambda &= \frac{c}{f} = \frac{3 \times 10^8 \, \frac{\mathrm{m}}{\mathrm{s}}}{1\,\mathrm{GHz}} \\ &= 0.3\,\mathrm{m} \end{aligned}$$

Rough brick walls are smooth for cell phone light!

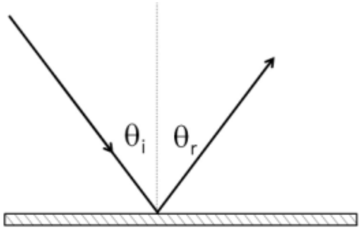
We can see that we must be careful in our definition of “smooth.”

Law of reflection

Ball Bounce Demo

Experience shows that if we do have a smooth surface, that light bounces much like a ball. This is why Newton though light was a particle. Suppose we take a flat surface and we shine a light on it. We have a ray that approaches at an angle θ_i measured from the normal. Then the reflected ray will leave the surface with an angle θ_r measured from the normal such that

$$\theta_r = \theta_i$$



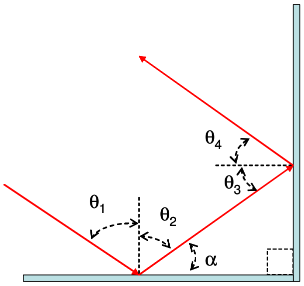
This is called the *law of reflection.*

Question 223.13.7

Question 223.13.8

Retroreflection

Let’s take and example



Let's take our system to be two mirrors set at a right angle. We have a beam of light incident at angle θ_1. By the law of reflection, it must leave the mirror at $\theta_2 = \theta_1$. We can see that α must be $90\,^{\circ} - \theta_2$ and it is clear that $\theta_3 = \alpha$. By the law of reflection, $\theta_3 = \theta_4$. Then, since

$$\begin{aligned} 90\,^{\circ} &= \theta_2 + \alpha \\ &= \theta_2 + \theta_3 \end{aligned}$$

and

$$90\,^{\circ} = \theta_1 + \theta_4$$

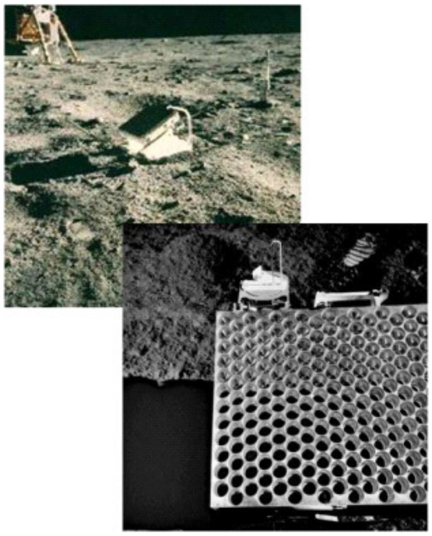
then the total angular change is

$$90\,^{\circ} + 90\,^{\circ} = 180\,^{\circ}$$

or the outgoing ray is sent back toward the source! If we do this in three dimensions we have a corner cube. Here is a picture of a radar corner cube array.

Radar retroreflector tower located in the center of Yucca Flat dry lake bed. Used as a radar target by maneuvering aircraft during "inert" contact fusing bomb drops at Yucca Flat. Sandia National Laboratories conducted the tests on the lake bed from 1954 to 1956. (Image in the Public Domain in the United States)

And we left visible corner cube arrays on the Moon so we can bounce laser beems off of them and monigor the Earth-Moon distance.



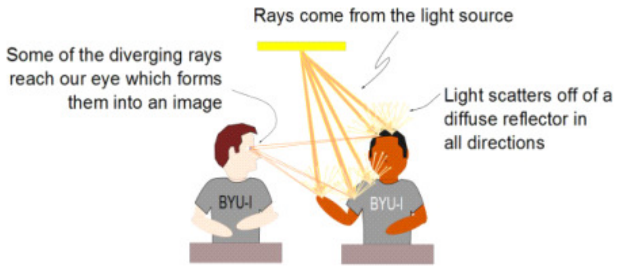
Apollo Retroreflector (Images in the Public Domain courtesy NASA)

Question 223.13.9

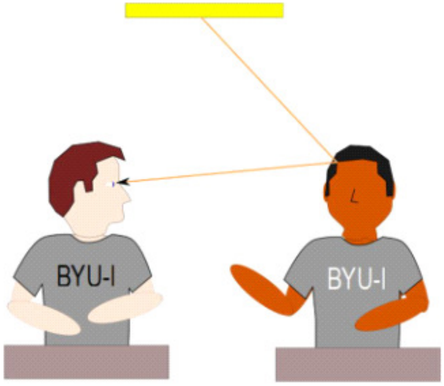
Reflections, Objects, and seeing

Armed with the law of reflection, we can start to understand how we see things. Using the ray concept, we can say that a ray of light must leave the light source. That ray then reflects from something. Suppose you look at the person sitting next to you in class. We should wonder, how is it that we can see them? We can only detect (see) light that gets to our eyes. Let's trace the light from it's source (the light fixture in our room) to our eyes to see how we see our neighbors.

Light comes from the ceiling lights and goes in all directions. Some of that light hits our neighbor. And some of that light that hits our neighbor will reflect. But is the person a specular or diffuse reflector? Once again, we can only give an answer relative to the wavelength of light. For visible light, your neighbors do not look like mirrors. The are diffuse reflectors. Light bounces off of them in every direction. Some of that light that reflects from your neighbor reflects in the direction of your eyes. That light can be detected. Your eye is designed to take this diverging set of rays and condense it into a picture of the person that your brain can interpret.

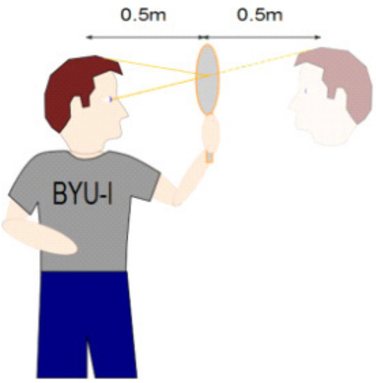


We tend to not draw the rays that bounce off the diffuse reflector but that don't get to our eyes, because we don't see them. So a ray diagram is usually much simpler.

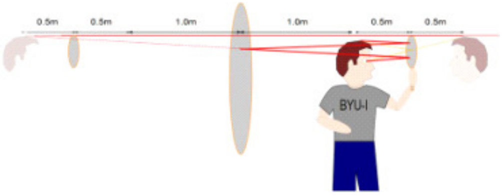


This is easy to understand, but we must keep in mind the wildly fluctuating waviness that is masked by our macroscopic view. Question 223.13.10

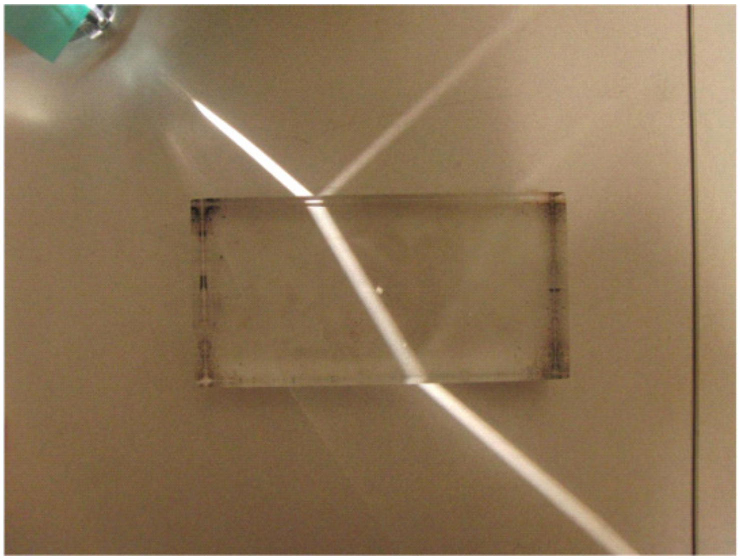
We can use the idea of a ray diagram to solve problems. Suppose you hold a mirror half a meter in front of you and look at your reflection. Where would the reflection appear to be?



Knowing that rays travel in straight lines and that our mind interprets rays as going in straight lines, then we can use rays to see where the light appears to be from. The image is half a meter behind the mirror. Now suppose we look at an image of that image in a mirror behind us.

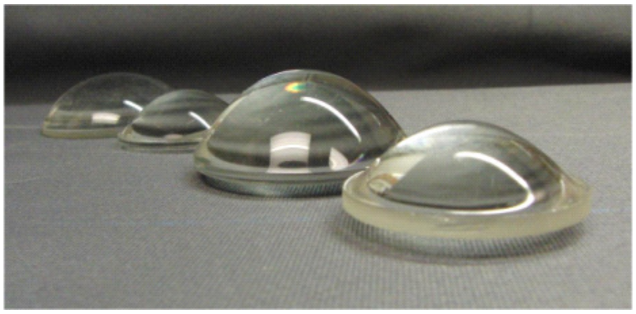


The ray diagram makes it easy to see that the image will appear to be $2\,\mathrm{m}$ behind the big mirror. You might not feel that this was so easy, but you might find it is not so bad in a problem in your near future.



Refraction

Not all surfaces reflect all the light. Some, like the lenses shown below, reflect some light at visible wavelengths, but are transparent so most of the light travels through them.

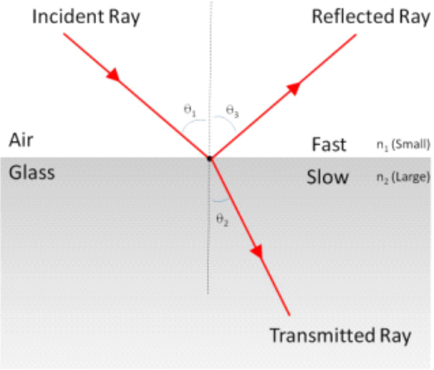


We need a way to deal with transparent materials. This is tricky, because different wavelengths of light penetrate different materials in different ways. As an example, this is also a lens

Infrared lens, but visible mirror (Image courtesy US Navy, image in the public domain)

but it clearly is not transparent at visible wavelengths. It is transparent in the infrared. So what might be transparent at one wavelength might not be at another.

When light travels into a material, we say it is transmitted. The situation is shown schematically below.



In the figure we see a ray incident on an air-glass boundary. Some of the light is reflected just as we saw before. But some passes into the glass. Notice that the angle between the normal and the new transmitted ray is *not* equal to the incident ray. We say the ray has been bent or *refracted* by the change in material. Many experiments were performed to find a relationship between the incident and the refracted angles. It was found that

$$\frac{\sin(\theta_2)}{\sin(\theta_1)} = \frac{v_2}{v_1} = \text{constant} \tag{14.1}$$

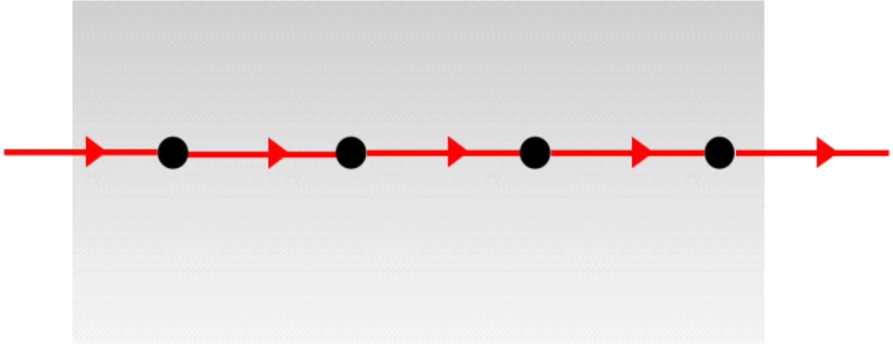
Many optics books write this as

$$\frac{\sin(\theta_t)}{\sin(\theta_i)} = \frac{v_2}{v_1} = \text{constant} \tag{14.2}$$

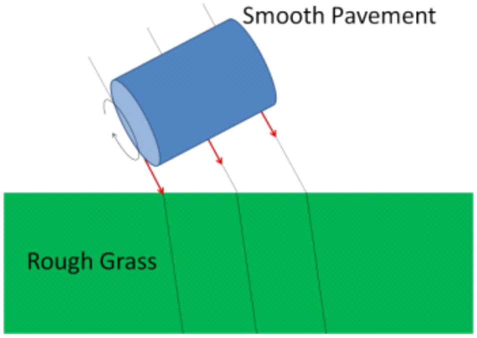
where the subscript i stands for "incident" and the subscript t stands for "transmitted." Note that we are using the fact that the speed of light changes in a material. We should probably recall why this should occur

Speed of light in a material

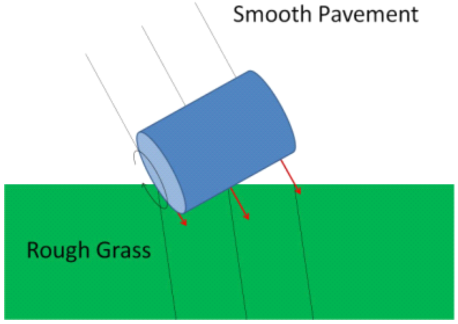
In a vacuum, light travels as a disturbance in the electromagnetic field with nothing to encounter. In a material (like glass) the light waves continually hit atoms. We have not studied antennas, but I think many of you know that an antenna works because the electrons in the metal act like driven harmonic oscillators. The incoming radio waves drive the electron motion. Here each atom has electrons, and the atoms act like little antennas, their electrons moving and absorbing the light. But the atom cannot keep the extra energy (PH433), so it is readmitted. It travels to the next atom and the process repeats. Quantum mechanics tells us that there is a time delay in the re-emission of the light. This causes the propagation of the light to slow down. Thus the speed of light is slower in a material.



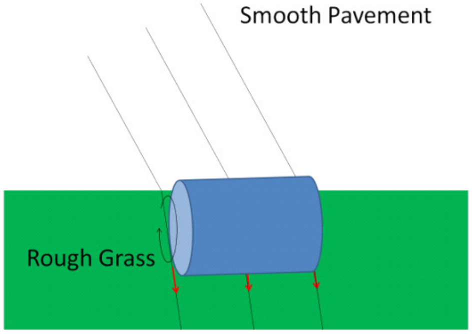
As a mechanical analog, consider a rolling barrel.



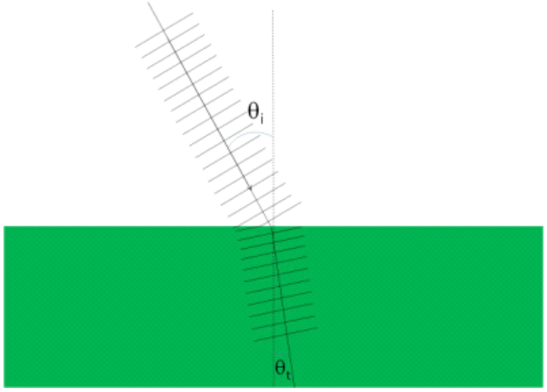
As the barrel rolls from a flat low-friction concrete to a higher-friction grass lawn, the friction slows the barrel. If the barrel hits the lawn parallel to the boundary (so it's velocity vector is perpendicular to the boundary), then the barrel continues in the same direction at the slower speed. But if it hits at an angle, the leading edge is slowed first.



This makes the trailing edge travel faster than the leading edge, and the barrel turns slightly.



We expect the same behavior from light.



We can see that the left hand side of the wave hits the slower (green) material first and slows down. The rest of the wave front moves quicker. The result is the turning of the wave.[10]

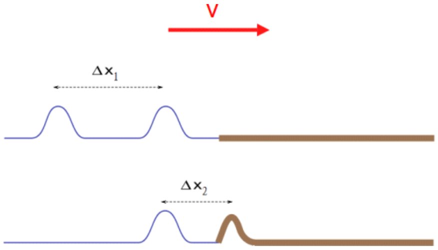
Question 223.14.1

Change of wavelength

We have found that when a wave enters a material, its speed may change. But we remember from wave theory

$$v = \lambda f \tag{14.3}$$

But it is time to review: does λ change, or does f change? If you will recall, we found that the change in speed at the boundary changes the wavelength. Recall that if we go from a fast material to a slow material, the forward part of the wave slows and the rest of the wave catches up to it.



This will comperes pulses, and lower the wavelength. Now that we know more about light we can also argue that f cannot change because

$$E = hf$$

If f changed, then we would either require an input of energy or we would store energy

---

[10] Once again this is a bit of a simplification, but it will due for now. If you are lucky enough to take a junior level optics class, you will revisit this.

at the boundary because

$$\Delta f = \frac{\Delta E}{h}$$

This can't be true. If the wavelength changes, there is no such change in energy.

Since

$$v_1 = \lambda_1 f$$

and

$$v_2 = \lambda_2 f$$

then the ratio

$$\frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2}$$

and we again have our solution for the wavelength in the material

$$\lambda_2 = \lambda_1 \frac{v_2}{v_1}$$

which agrees with our previous analysis.

Index of refraction and Snell's Law

Question 223.14.2

Question 223.14.3

Because the equation

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_2}{v_1} = \text{constant} \tag{14.4}$$

has a constant ratio of velocities, it is convenient to define a term that represents that ratio. We already have a concept that can help. The *index of refraction* is just such a term. It assumes that one speed is the speed of light in vacuum, *c*.

$$n \equiv \frac{c}{v} \tag{14.5}$$

Then for our example

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{1}{n} \tag{14.6}$$

Suppose we don't have a vacuum (or air that is close to a vacuum). We can write our formula as

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{14.7}$$

where we have determined

$$n_1 \equiv \frac{c}{v_1} \tag{14.8}$$

and

$$n_2 \equiv \frac{c}{v_2} \tag{14.9}$$

This is called *Snell's law of refraction*.

Using the index of refraction we can write our equation relating the ratio of velocities and wavelengths as

$$\frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2} = \frac{\frac{2}{n_1}}{\frac{c}{n_2}} = \frac{n_2}{n_1} \tag{14.10}$$

which gives

$$\lambda_1 n_1 = \lambda_2 n_2 \tag{14.11}$$

and if we have vacuum and a single material we can find the index of refraction from

$$n = \frac{\lambda}{\lambda_{in}} \tag{14.12}$$

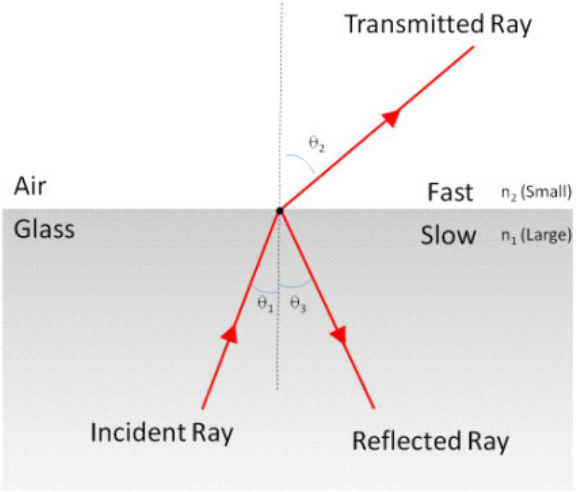
where λ_{in} is the wavelength in the material. Question 223.14.4

Question 223.14.5

Total Internal Reflection

Question 223.14.6

Up to now we have assumed that light was coming from a region of low index of refraction into a region of high index of refraction. We should pause to look at what can happen if we go the other way.



We start with Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{14.13}$$

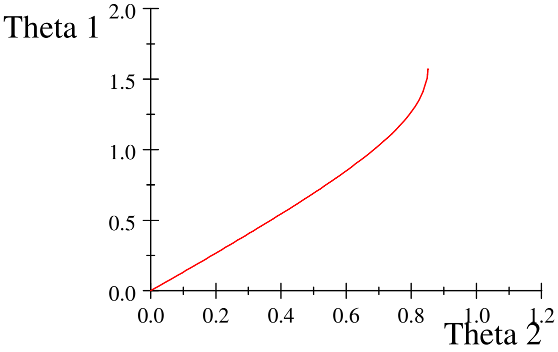
but this time $n = n_1$ and $n_2 = 1$ so

$$n \sin\theta_1 = \sin\theta_2 \tag{14.14}$$

which gives

$$\theta_2 = \sin^{-1}\left(n \sin\theta_1\right) \tag{14.15}$$

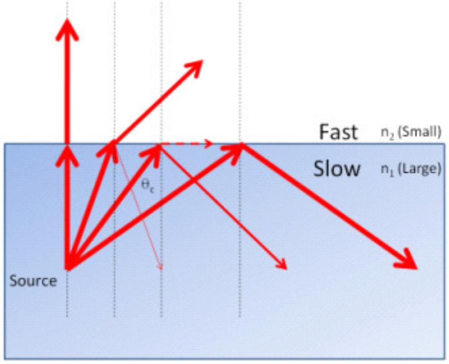
If we take $n = 1.33$ (water) we can plot this expression as a function of θ_1



we see that at $\theta_1 = 0.850\,91\,\mathrm{rad}\ (48.\,754\,^{\circ})$ the curve becomes infinitely steep. If we use this value in our equation this gives

$$\begin{aligned}
\theta_2 &= \sin^{-1}\left(n\sin\left(0.850\,91\right)\right) && (14.16)\\
&= 1.\,570\,8\,\mathrm{rad} && (14.17)\\
&= 90\,^{\circ} && (14.18)
\end{aligned}$$

The light skims along the edge of the water!



We can find the value of θ_1 that makes this happen without graphing. Set $\theta_2 = 90\,^{\circ}$ then

$$\theta_1 = \theta_c \equiv \sin^{-1}\left(\frac{1}{n}\right) \tag{14.19}$$

We give this value of θ_1 a special name. It is the *critical angle* for internal reflection. But what happens if we go farther than this ($\theta_1 > \theta_c$). We will no longer have a transmitted ray. The ray will be reflected. This is why when you dive into a pool and look up, you see a region of the roof of the pool area (or sky) but off to the side of the pool the surface looks mirrored. It is also why you sometimes see the sides of a fish tank appear to be mirrored when you look through the front.

More importantly, it is why cut gems (like diamonds) sparkle. They capture the light with facets that are cut at angles that create total internal reflection. The light that enters the gem comes back out the front (We will study how to make the pretty colored sparkles next time).

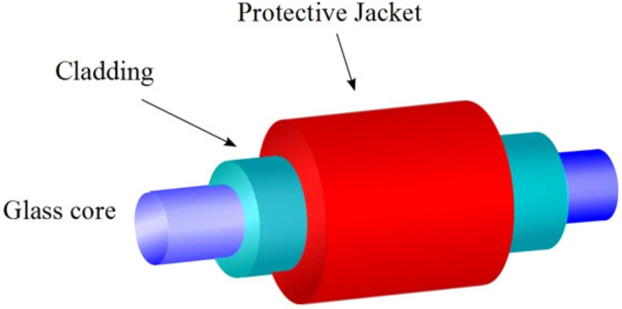
Question 223.14.7

Question 223.14.8

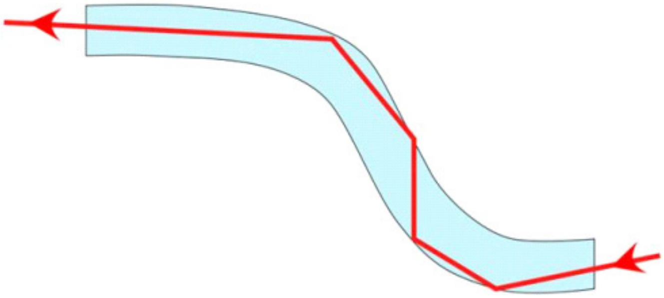
Question 223.14.9

Fiber Optics

Beyond pretty pebbles, this effect is very useful! It is the heart and soul of fiber optics.

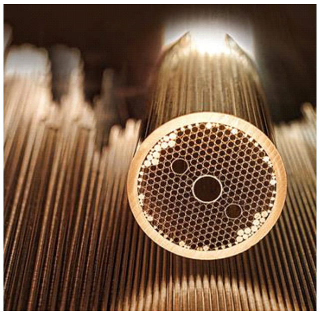


An interior material with a lower index of refraction is inclosed in a cladding with a higher index. This creates a light pipe that traps the light in the fiber.



Modern fibers don't always have a hard boundary. The fibers have a gradual change in index of refraction that changes the direction of the light gradually. This keeps the light in the fiber but tends to direct along the fiber so the beam is not crisscrossing as it goes.

The cutting edge of fiber design today uses hollow fibers or fibers filled with different index material.



Hollow-Core Fiber (Courtesy Defense Advanced Research Projects Agency (DARPA), image in the public domain)

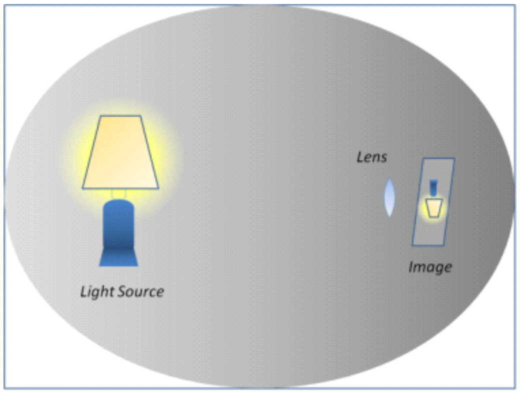
15 Images and Color

Fundamental Concepts

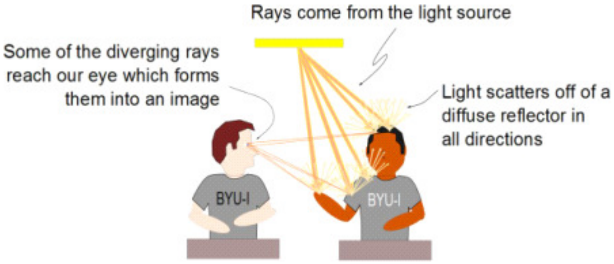
- An image is a diverging set of rays in a recognizable pattern
- Images can be formed by refraction
- The index of refraction is wavelength dependent
- That different wavelengths bend different amounts when refracted is called *dispersion*
- White light is a superposition of many other frequencies

Let's think about what an image is. Take a piece of paper and a lens, and hold up the lens in a darkened room that has some bright object in it. Move the lens or the paper back and forth, and at just the right distance, a miniature picture of the bright object will appear. We should think about what the word "picture" means in this sense.

Make Images with Lens Demo



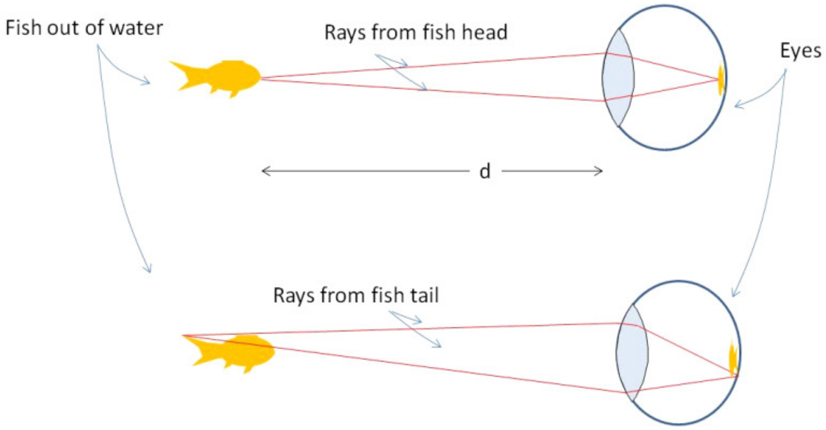
We have talked about how we see objects. Remember the BYU-I guys from last time.



Our eyes gather rays that are diverging from the object because light has bounced off of the object. Our eyes intersect a diverging set of rays that form a definite pattern. That diverging set of rays forming a pattern is the picture of the object.

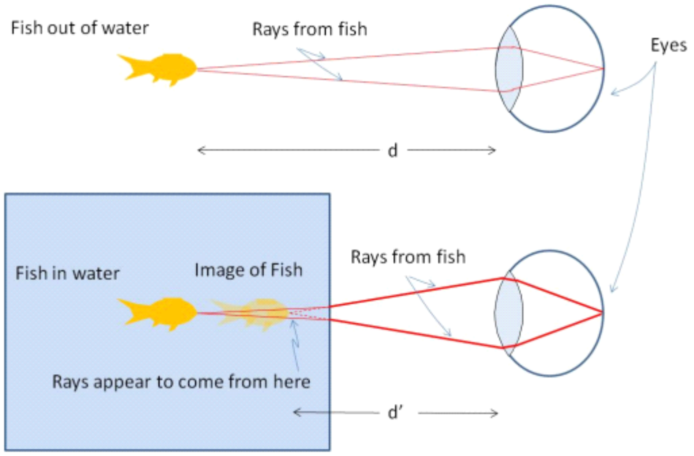
So when we say that the lens has formed a miniature picture of our object, we mean that the lens has somehow formed a diverging set of rays that form a pattern that looks like the pattern formed by the diverging set of rays coming from the object, itself. In other words, the object forms a diverging set of rays. And our lens forms a duplicate set of rays in the same pattern, so we see the same thing. The lens' version is smaller, upside down, and backwards, but it is still essentially the same pattern.

As a first step to see how this works, consider our fish tank again. It would be bad on the fish, but think about looking at a fish in air. The room light would bounce off of the fish, and we would have a diverging set of rays from every point on the fish (see next figure).



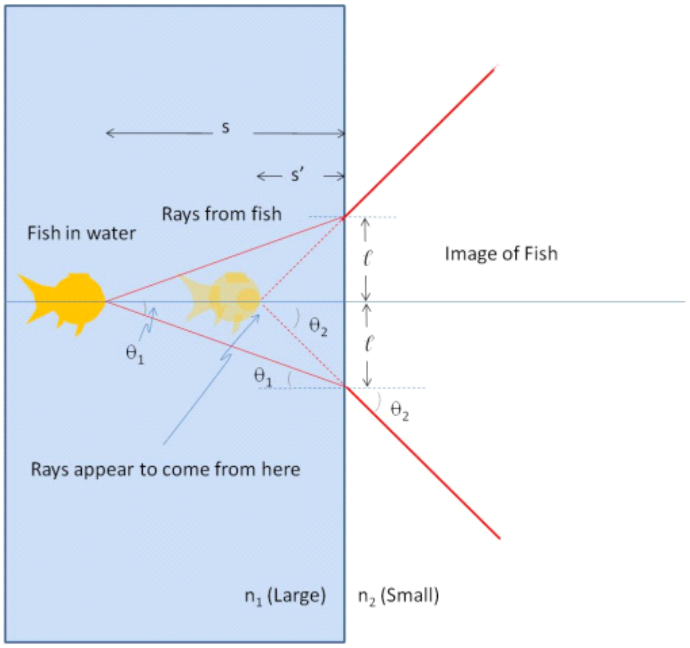
We can see that the picture is made from every point on the fish being "imaged" to a point on our retina. We collect the rays leaving every point on the fish, and bring them to corresponding points on the retina to make the picture.

It will take us a few lectures to see exactly how this is done, but as a first step, let's consider the fish tank, itself. Put the fish back in the tank and look at it (next figure).



Rays still come from the fish. But we now know that the change from a slow material to a fast material will bend the light. These bent rays are collected by our eyes, and the picture of the fish is formed on the retina just as before. But our eyes interpret the light as though it went in straight lines with no bends (dotted lines in the last figure). our mind is designed to believe light travels in straight lines, so our mind tells us there is a fish, but that the fish head (and every other part of the fish) is closer than it really is. We call this apparent fish at the closer location an image of the fish, because this is where we think the diverging set of rays come from that form the fish pattern.

The next figure shows the details of the rays leaving a dot on the fish head (with the angles exagerated so it’s easier to see them).



A spot on the fish head is our object for this set of rays. The distance from the fish-head dot and the edge of the water/air boundary is called the *object distance* and is given the symbol s.

The distance from the image of the fish-head dot to the edge of the water/air boundary is called the *image distance* and is given the symbol s'. Note that this is not a derivative, it is just a distance like s, because it appears to be where the rays come from, but it is a different distance because of the refraction of the rays, so to make it look difference we put a prime mark on it.

We can find where the image is (s') knowing s. We can see from the figure that

$$\begin{aligned} \ell &= s\tan\theta_1 \\ &= s'\tan\theta_2 \end{aligned}$$

so

$$s\tan\theta_1 = s'\tan\theta_2$$

or

$$s\frac{\tan\theta_1}{\tan\theta_2} = s'$$

from Snell's law, we know that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$$

Usually we can take the small angle approximation. This would limit our analysis to rays that are near the central axes. Let's call this central axis the *optics axis* and the rays that are not too far away from this axis *paraxial rays.* Then for our small angles we can write

$$\tan\theta_i \approx \sin\theta_i$$

so

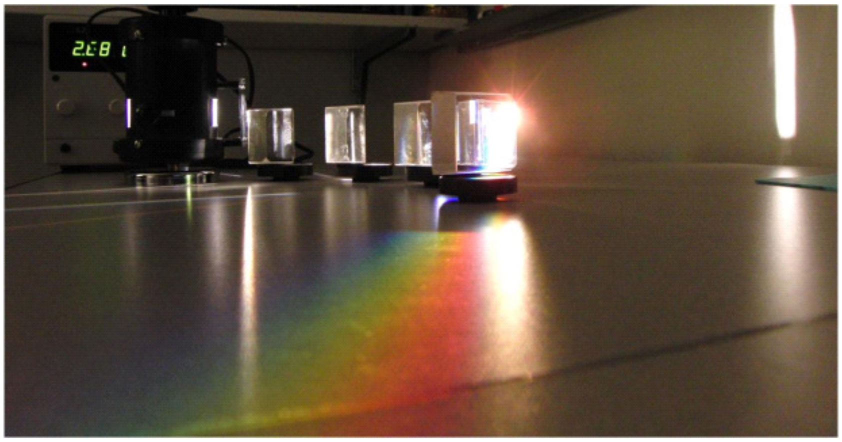
$$s\frac{\tan\theta_1}{\tan\theta_2} \approx s\frac{\sin\theta_1}{\sin\theta_2} = s\frac{n_2}{n_1} = s'$$

and we have the image distance

$$s' = s\frac{n_2}{n_1}$$

This is not so useful unless you have some burning need to know where your fish are in a tank. But we now have the vocabulary to discuss the larger problem of how a lens works, which we will take up next time.

Dispersion



Question 123.15.1

Who hasn't played with a prism? We immediately recognize a rainbow. But why does the prism make a rainbow? The secret lies in the nature of the refractive index.

Notice that in the figure, the index of refraction depends on wavelength. This means that as light enters a material, different wavelengths will be refracted at different angles. White light is a superposition of many wavelengths of light. Thus white light is pulled apart by refraction into a rainbow. This process is called *dispersion*.

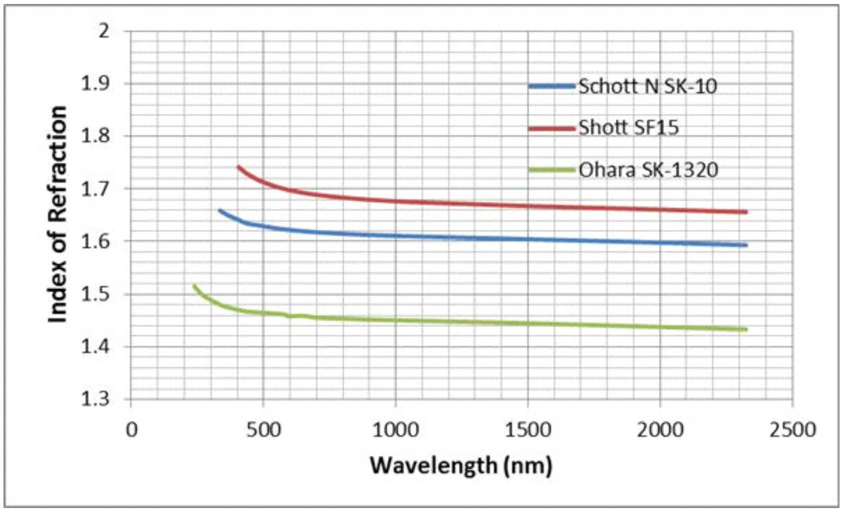
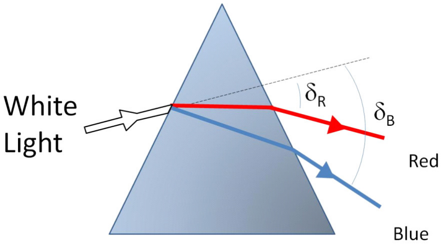


Figure 15.3.Index of refraction as a function of wavelength (Ohara optical glass http://www.oharacorp.com/fused-silica-quartz.html data and Schott optical glass data http://www.uqgoptics.com/materials_glasses_schott.aspx)

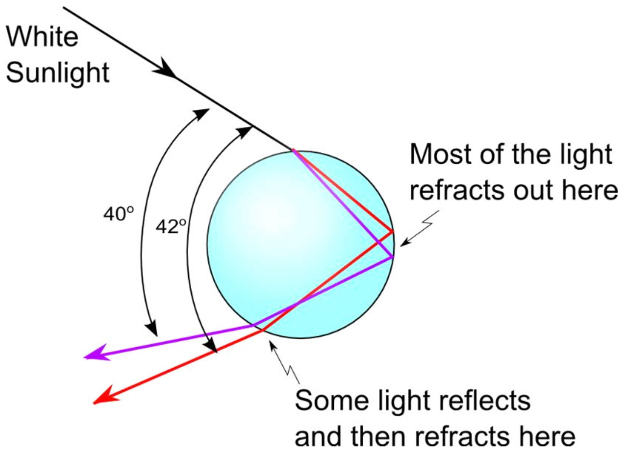
The graph tells us that blue light bends more than red light.



We call the change in direction measured from the original direction of travel, δ, the *angle of deviation*. The colors we can see are called the visible spectrum. Note that this is a second way to make a visible light spectrum. The first way used a diffraction grating and the wave nature of light. Although this dispersive element (prism) method of making a spectrum uses the ides of geometric optics, it is still the wave nature of light that makes the waves of different wavelengths refract differently. Question 123.15.1

Question 123.15.2

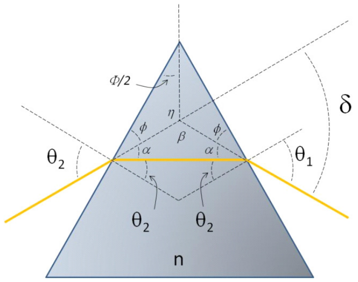
Let's look at a natural rainbow. The dispersion is caused buy small droplets of water. The white sunlight enters the drop and is dispersed. It bounces off the back of the drop and then leaves the drop, again being dispersed. Red light leaves the drop at about $42\,^\circ$ from its input direction, and blue light leaves at about $40\,^\circ$.



This effectively separates all the different wavelengths and we see a rainbow at angles between $40\,^\circ$ and $42\,^\circ$ from the incoming light direction. Question 123.15.3

Calculation of n using a prism

Let's do a problem using the idea of refraction in a prism. Let's find the index of refraction of a the prism material. Suppose we make a prism as shown. We know the angle Φ and can measure the exit angle δ. In terms of these two variables, what is n?



Using the notation indicated in the figure, we choose θ_1 such that the interior ray is horizontal.[11] This is a refraction problem, so we will want to use Snell's law.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Thus, we need to find θ_1 and θ_2. Knowing Φ and δ, and realizing $n_1 = 1$, we can find θ_2 and θ_1. Then use

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

---

[11] WARNING! in the upcoming homework problem you can't make the same assumptions!

to find n_2. Let's see one way to do this. In geometry it is fair game to extend lines and even make some new lines of our own. By doing this we can realize that

$$\theta_1 = \theta_2 + \alpha \tag{15.1}$$

and that

$$\delta = 180 - \beta \tag{15.2}$$

and it is also true that

$$180 = \beta + 2\alpha \tag{15.3}$$

Then

$$\delta = 2\alpha \tag{15.4}$$

and

$$\alpha = \frac{\delta}{2} \tag{15.5}$$

Now also realize that

$$90 = \alpha + \theta_2 + \phi \tag{15.6}$$

and

$$180 = \Phi + 2\alpha + 2\phi \tag{15.7}$$

or

$$90 = \frac{\Phi}{2} + \alpha + \phi \tag{15.8}$$

Then

$$\begin{aligned} \alpha + \theta_2 + \phi &= \frac{\Phi}{2} + \alpha + \phi && (15.9) \\ \theta_2 &= \frac{\Phi}{2} && (15.10) \end{aligned}$$

We can put these in our equation for θ_1

$$\begin{aligned} \theta_1 &= \theta_2 + \alpha && (15.11) \\ &= \frac{\Phi}{2} + \frac{\delta}{2} && (15.12) \\ &= \frac{\Phi + \delta}{2} && (15.13) \end{aligned}$$

Now we can use Snell's Law

$$\begin{aligned} \sin(\theta_1) &= n \sin(\theta_2) \quad (15.14) \\ \sin\left(\frac{\Phi+\delta}{2}\right) &= n \sin\left(\frac{\Phi}{2}\right) \quad (15.15) \end{aligned}$$

then

$$n = \frac{\sin\left(\frac{\Phi}{2}\right)}{\sin\left(\frac{\Phi+\delta}{2}\right)} \quad (15.16)$$

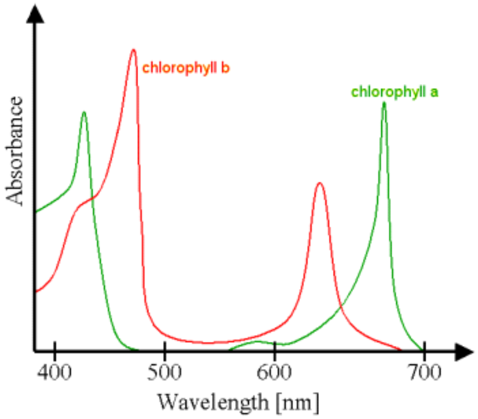
and since we know Φ and δ, we can find n.

Filters and other color phenomena

Of course, we have assumed without statement, that white light is made up of all the colors of the rainbow. We should ask why a red shirt is red, or why passing light through a green film makes the light look green as it leaves.

Both of these phenomena are examples of removing wavelengths from white light.

In the case of the red shirt, the red dye in the cloth absorbs all of the visible colors except red. The red is reflected, so the shirt looks red. The filter is much the same. The green pigment in the film causes nearly all visible colors to be absorbed except green. So only green light is transmitted. This is why leaves are green. Chlorophyll absorbs red and blue wavelengths, so the green is reflected or transmitted.



Chlorophyll Spectrum (Public Domain image courtesy Kurzon)

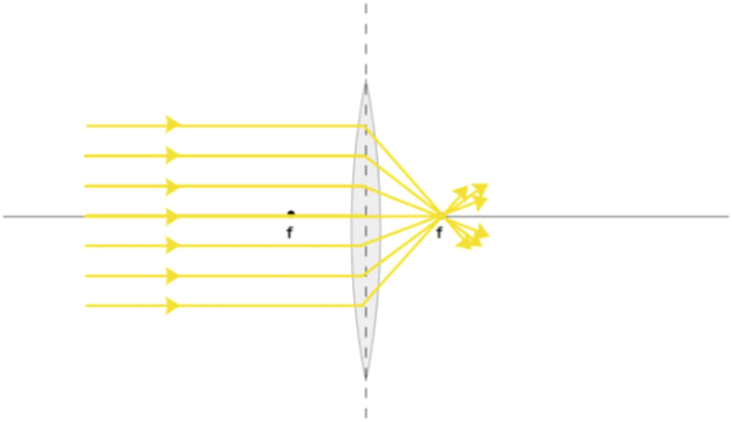
16 Ray Diagrams and Thin Lenses

Back in Newton's day, there were no electronic computers, or calculators, or mechanical adding machines. Early optics researchers did math with a pen and paper. This is one reason they liked small angle approximations. The approximation allowed them to do harder problems using easy math. And so long as the things they built worked, the approximations were good enough. We are going to use another approximation in this lecture. It is called the thin lens approximation. It will make the math that describes lenses for, say, telescopes or microscopes, much easier. We will also introduce a way to draw lenses and light that will tell you how the lens works. We will use this drawing approach to do homework and examination problems. Professional optical designers use computer codes that do this drawing very accurately to help the optical engineer understand the system they are designing and to look for mistakes. So this drawing scheme is very useful. Because it makes optical systems so much easier to understand, let's start with the drawings and then work toward a mathematical description of thin lenses.

Fundamental Concepts

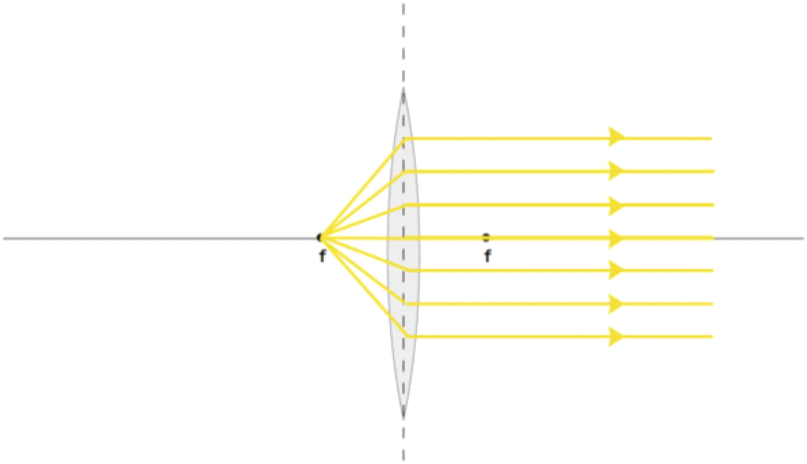
1. We can describe how a lens operates with just three easy-to-draw rays.
2. The magnification is given by $m \equiv -\frac{h'}{h} = -\frac{s'}{s}$
3. A semi-infinite bump can be described by the equation $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{R}$
4. For a thin lens, we can describe where the light goes using $\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$where $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$
5. There is a sign convention for all of these equations.

Before we do a lot of math to describe how lenses work, lets think about our early childhood experiences. You may have burned things with a magnifying glass[12]. Using the idea of a ray diagram, here is what happens.



The rays from the Sun come from so far away that they are essentially parallel. We know that these rays come together to a fine point that can start a fire. The point where these rays converge is important to us. We will call this the *focal point*.

Knowing that the light will follow the same paths either direction, we would expect that if we put a light source at the focal distance, the rays should come out parallel.

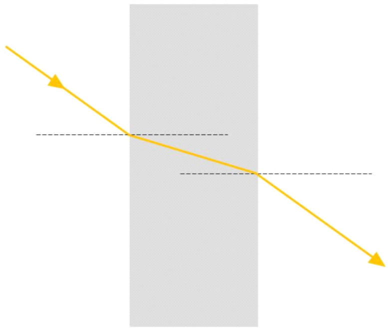


This is one way manufacturers make LED flashlights.

We need one other bit of information that we already have seen from basic refraction.

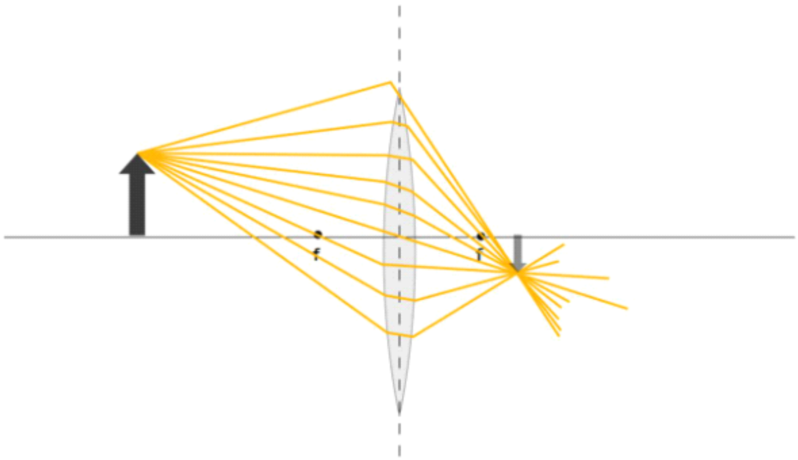
---

[12] If you didn’t do this as a child, consider trying it now. Be responsible and safe, but it is valuable to see how this works.

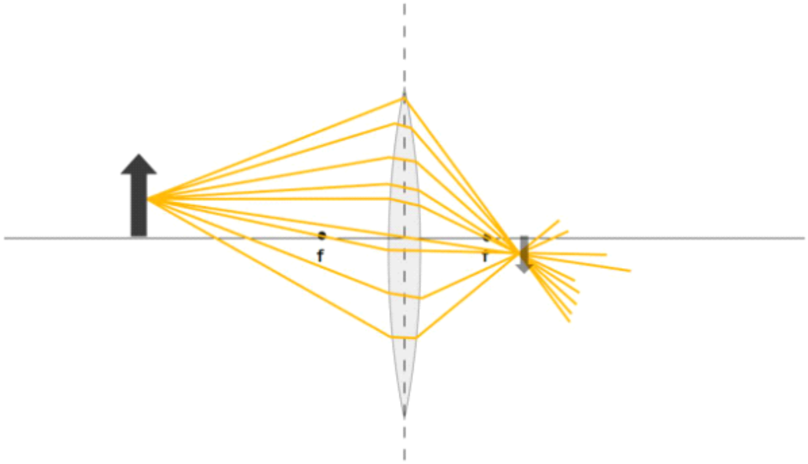


A flat block does refract the light, but when the light leaves the block it is only displaced, it retains the original direction. With these three ray scenarios, we can describe how a lens works.

We know that for every point on the object, we get millions of reflected rays that diverge. In the next figure, rays of light are reflecting off of the large, black arrow. The lens must collect these rays together to form the corresponding point on the image.

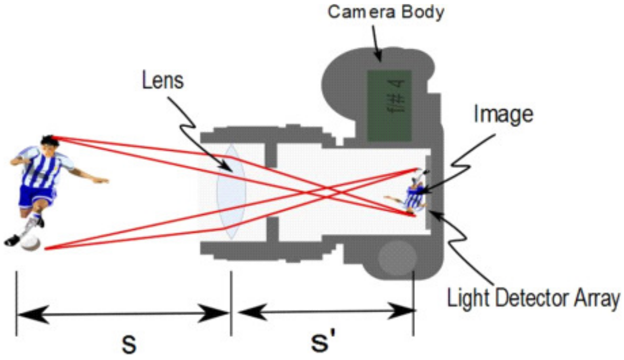


Of course, this happens for every point on the image.



But we usually pick the top of the object. If we place the bottom of the object on the *optic axis*[13], the bottom of the image will also be on the optic axis. So knowing where the bottom of the image is, and finding the top of the image gives a pretty good idea of where the rest of the image must be. So we will draw diagrams for the top of the object to find the top of the image.

But suppose this is not true? For example, when we use a camera, we do not align the optical system on an axis before we shoot.



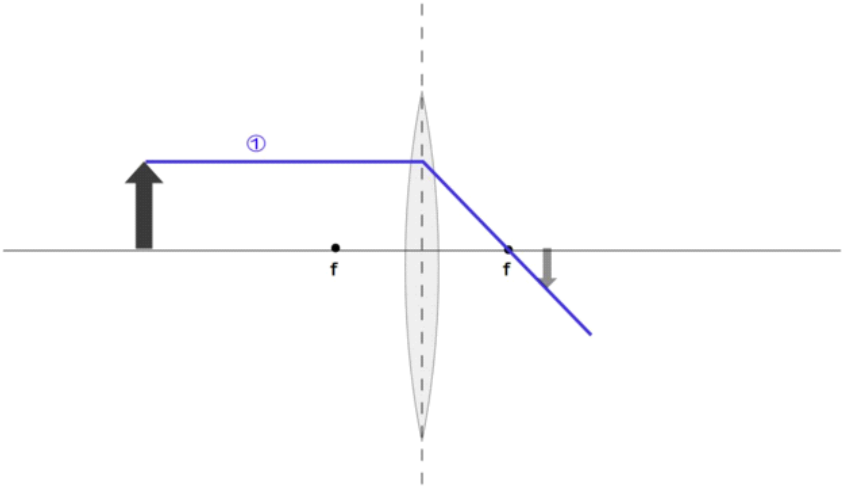
We can, of course, trace two rays for the bottom of the image in this case and find the location of the bottom of the image.

Drawing millions of rays is impractical, and fortunately, not needed. We can choose three simple rays that leave the top of the object , and see where these rays converge to form the top of the image. Let's start with a ray that travels from the top of the object and travels parallel to the optic axis. We recognize this ray as being like one of the rays from the Sun. It comes in parallel, so it will leave the lens and travel through the focal

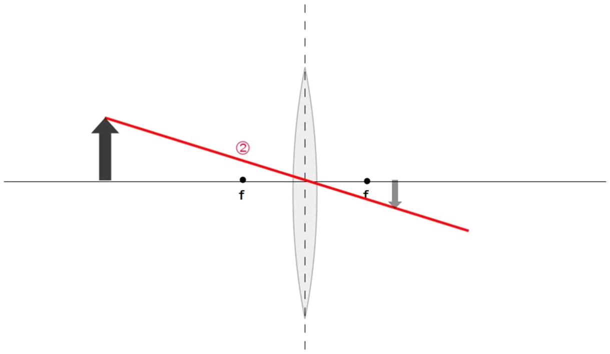
---

[13] The line drawn on the figure thorugh the middle of the lens.

point.

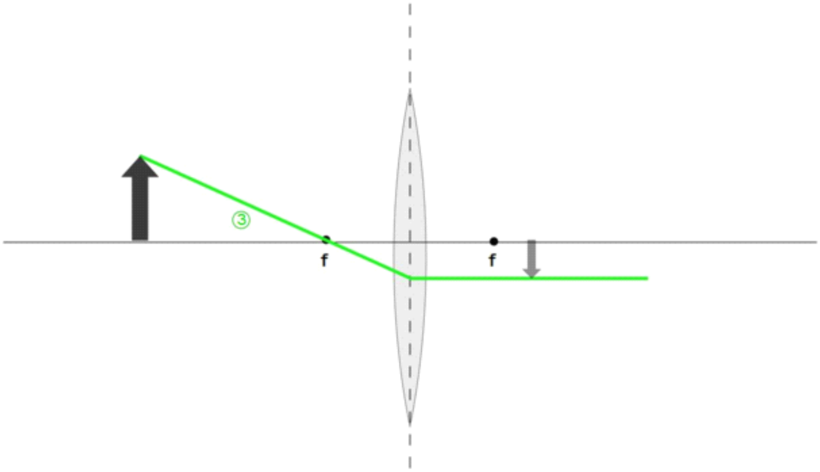


For a second easy ray, lets take the case that is like our flat block. Near the center of the lens, the sides are nearly flat. So we expect that the ray will leave in about the same direction as it was going before it struck the lens.

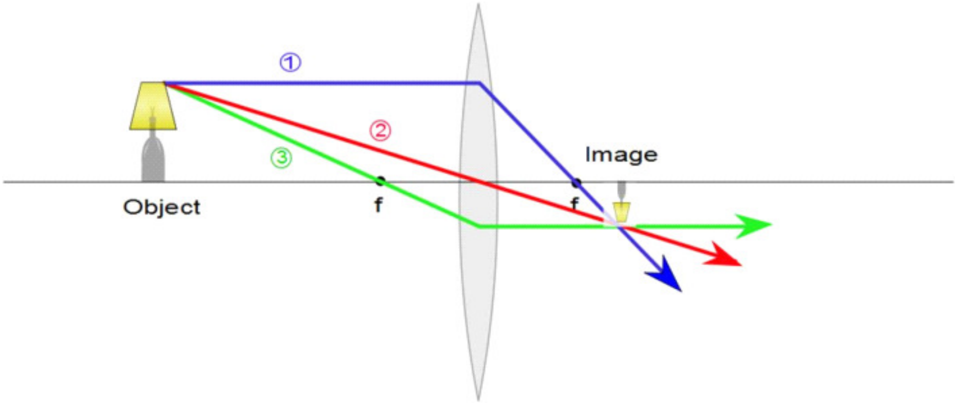


Technically this ray should be shifted. This is where our thin lens approximation comes in. If the lens is thin, then the ray going through the center of the lens won't be shifted much, and we can ignore the shift. We just dray ray 2 as a straight line.

Two rays are really enough to determine where the top of the image will be, but there is a third ray that is easy to draw, so let's draw it to give us more confidence in our answer. That ray is one that leaves the top of the object and passes through the focal point on the object side of the lens. This situation we also recognize. Since the ray goes through the focal point, it is as though the light came from that point. This is like our LED flashlight case. This ray will leave the lens parallel to the optic axis.



Where all three rays intersect, we will have the top of the image.



Notice that in this case, the image is upside down. That is normal. Also notice that it is smaller than the object. We say that the image is magnified, which may seem a little bit strange. But in optics, a magnification of greater than one means that the image is bigger than the object. This is like a movie projector that makes a large image of a small film segment. The magnification can be equal to one, meaning the object and image are the same size. And finally the magnification can be less than one. This means that the image is smaller than the object. This is a convenient definition, because then we can use the same equation to describe all three situations.

$$m \equiv \frac{\text{Image height}}{\text{Object height}} = -\frac{h'}{h}$$

where h is the object height, and h' is the image height. Notice the negative sign. By convention (meaning physicists got together and voted on this) we say that an upside down image has a negative magnification. You just have to memorize this, there is no obvious reason for this except it is mathematically convenient.

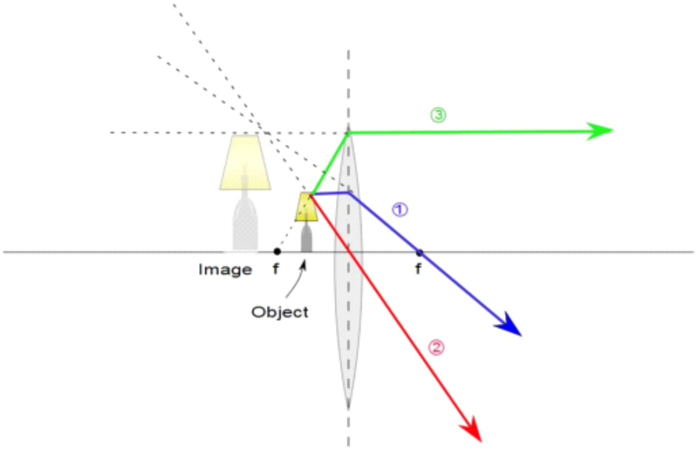
We will find that

$$m = -\frac{s'}{s}$$

Question 123.15.4

Virtual Images with Lenses

Lets take another case and draw a ray diagram. This time let's place the object closer than the focal distance. This is the case when we use a lens as a magnifying glass. The rays will look like this.



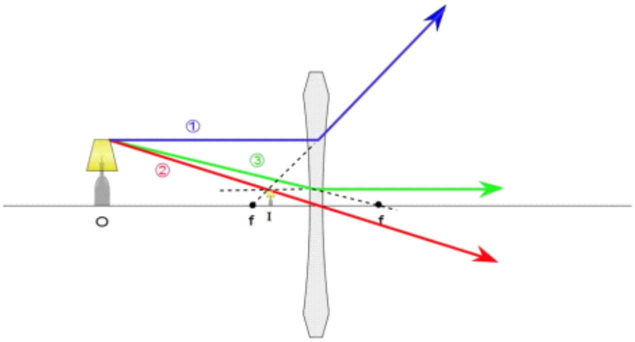
Notice that these rays never converge! We won't get an image that could project on a paper. But we know that there is an image, we can look through the lens and see it! And that is the key. The image does not really exist. There is no place where light gathers and then diverges into a pattern that we recognize. But we look through the lens, and our mind interprets the diverging rays coming from the lens as though they had only traveled in straight lines. If we extend these rays backwards along straight lines, they appear to come from a common point. This is the point they would have had to have come from if there were no lens. We believe we see an image at this location. But no light really goes there! Because this image is not really made from light diverging from this position, we call it a *virtual image*. The image we formed before that could be projected on a screen is called a *real image*.

By convention, we say the distance from the lens to the virtual image is a negative value.

Diverging Lenses

So far our lenses have only been the sort that work as magnifying glasses. We call these *converging lenses.* These lenses are fatter in the middle and thinner on the edges. Because of this they are sometimes called *convex lenses*. By convention, we say the focal distance for this type of lens is positive. For this reason, they are often called *positive lenses.*

But what if we make a lens that is thinner in the middle and thicker on the edges. We can call this sort of lens a *concave lens*, and we will give it a negative focal length by convention, so we can also call it a *negative lens*. But what would this lens do? If we think about our three rays, ray 1 won't be bent toward the optic axis for this type of lens. In fact, if we observe an object through this lens, ray number 1 will appear to come from the focal point. Ray number 2 will still go through the middle of the lens, and if the lens is thin enough, ray 2 will pass through undeviated.



finally ray three leaves the object in the direction of the far focal point. It will hit the lens and leave parallel to the optic axis. From the figure we see that these three rays will never converge. We expect they will form a virtual image. If we extend the rays backward as shown, we see that the extensions all meet at a point. The rays leaving the lens appear to come from this point. This is the location of the top of the virtual image of the object.

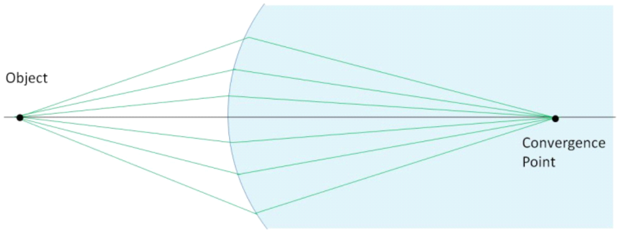
You might wonder what good such a lens could do, but we will find that this type of lens is used to correct vision for nearsighted people. It's likely that many people in our class have this type of lens with them either on their eye (contacts) or on their nose (eye glasses).

Semi-Infinite Lenses and the Semi-Infinite Lens Image Equation

Question 123.165.1

We learned how to find an image location graphically, now let's do it algebraically. Let's start by thinking of a special case for refraction. A curved surface on a very large piece of glass. We will assume that the piece of glass is semi-infinate, but all it has to be is very large.

We can call this a semi-infinate bump of glass.



Take a point object that either glows, or has rays of light reflecting from it. The rays leave the object and reach the surface of the glass. The rays will refract at the surface. Each bends toward the normal, but because of the curvature of the glass, the rays all converge toward the center. We can identify this convergence point as the image of the point object. At the surface we can find the refracted angles using Snell's law

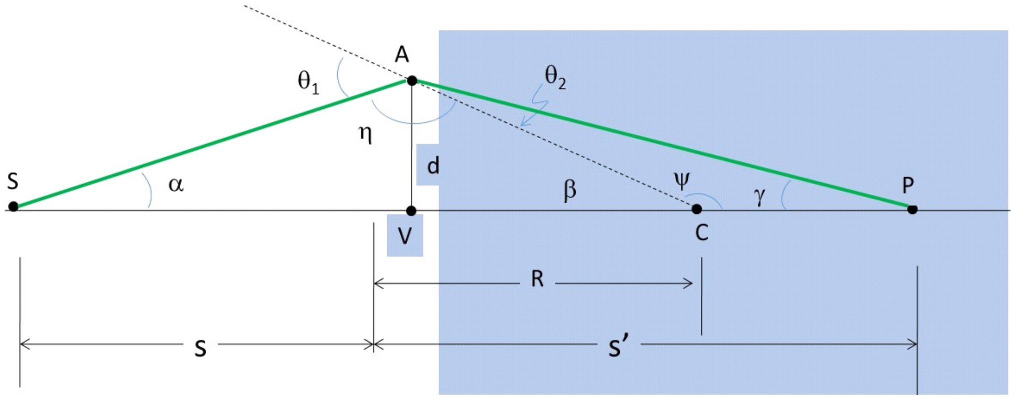
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

We will again use the small angle approximation. then θ_1 and θ_2 are small and none of the rays are very far away from the axis. This is called the *paraxial*[14] approximation. Snell's law becomes

$$n_1\theta_1 = n_2\theta_2$$

---

[14] If you speak Spanish (or Latin) you will recognize this as meaning "by the axis."



Using the more detailed figure above, we observe triangles SAC and PAC. We can see that for triangle SAC the top angle labeled η, plus θ_1 must be 180.

$$\eta+\theta_1=180\,^{\circ}$$

We also know that the sum of interior angles of a triangle must equal 180. So

$$180\,^{\circ}=\alpha+\beta+\eta$$

then

$$\begin{aligned}\eta+\theta_1 &= \alpha+\beta+\eta\\ \theta_1 &= \alpha+\beta\end{aligned}$$

Likewise, from triangle PAC,

$$\beta+\psi=180\,^{\circ}$$

$$180\,^{\circ}=\psi+\gamma+\theta_2$$

so then

$$\beta=\theta_2+\gamma$$

then,

$$\theta_2=\beta-\gamma$$

and we can write our paraxial Snell's law as

$$\begin{aligned}n_1\theta_1 &= n_2\theta_2\\ n_1\left(\alpha+\beta\right) &= n_2\left(\beta-\gamma\right)\\ n_1\alpha+n_1\beta &= n_2\beta-n_2\gamma\\ n_1\alpha+n_2\gamma &= n_2\beta-n_1\beta\\ n_1\alpha+n_2\gamma &= \beta\left(n_2-n_1\right)\end{aligned}$$

Looking at the figure. We see that d is a leg of three different right triangles (SAV,

ACV, and PAV). The ray in the figure is clearly not a paraxial ray. If we use a paraxial ray, then the point V will approach the air-glass boundary. When this happens, then $SV = s$, $VC = R$, and $VP = s'$. So we can write

$$\begin{aligned} \tan\alpha &\approx \alpha \approx \frac{d}{s} \\ \tan\beta &\approx \beta \approx \frac{d}{R} \\ \tan\gamma &\approx \gamma \approx \frac{d}{s'} \end{aligned}$$

so our Snell's law becomes

$$\begin{aligned} n_1\alpha + n_2\gamma &= \beta\left(n_2 - n_1\right) \\ n_1\frac{d}{s} + n_2\frac{d}{s'} &= \frac{d}{R}\left(n_2 - n_1\right) \end{aligned}$$

We can divide out the $d's$

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{\left(n_2 - n_1\right)}{R} \tag{16.1}$$

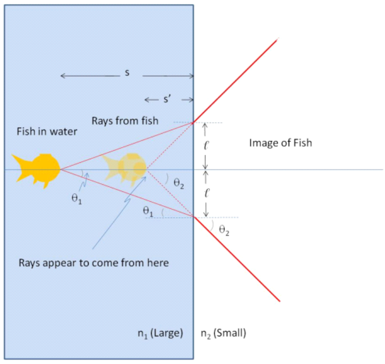
We can use this formula to convince ourselves that no matter what the angle is (providing it is small), the rays will form an image at P.

Real images will be in the glass for this special case (we will fix this with non-infinite lenses soon) so we need to change sign conventions.

Quantity	**Positive if**	**Negative if**
Object location (s)	Object is in front of interface surface	Object is in back of interface surface (virtual object)
Image location (s')	Image is in back of interface surface (real image)	Image is in front of interface surface (virtual image)
Image height (h')	Image is upright	Image is inverted
Radius (R)	Center of curvature is in back of interface surface	Center of curvature is in front of interface surface

We could go through the entire derivation and switch the indices of refraction. It turns out we get the same equation. The results will be different, but the equation is the same.

Flat Refracting surfaces



Let's return to our fish tank. The fish tank has an interface, but it is flat. Can we use our equation (16.1) to describe this?

The answer is yes, if we let $R = \infty$. This makes sense for a flat surface. If we have an infinitely large sphere, then our small part of that spherical surface that makes up the fish tank wall will be very flat.

Then

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{\infty}$$

or

$$\frac{n_1}{s} + \frac{n_2}{s'} = 0$$

we see that

$$s' = -s\frac{n_2}{n_1}$$

This is what we got before for this case, except before we just got the distance, and now we have included the effects of our sign convention. The negative sign means that the image is in front of the surface. By "in front" we always mean to follow the light from the source (fish) to the optical boundary. This boundary is the water/air boundary of the tank, so the fact that our image is in the water means that our image is in front of the optical boundary. As we know, this means the image is virtual.

Thin Lenses

Lets' find an equation for a spherical surface once more. But this time, let's let it be more practical and not make the "lens" semi-infinate. We will need to deal with two sides of the lens because (usually) both will be curved.

We found that for refraction

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{R}$$

but we did this for a spherical bump on a semi-infinite piece of glass. For this new problem let's make a few assumptions:

1. We have two spherical surfaces, with R_1 and R_2 as the radii of curvature
2. We have only paraxial rays
3. The image formed by one refractive surface serves as the object for the second surface
4. The lens is not very thick (the thickness is much smaller than both R_1 and R_2)

The answer we will get is quite simple

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \tag{16.2}$$

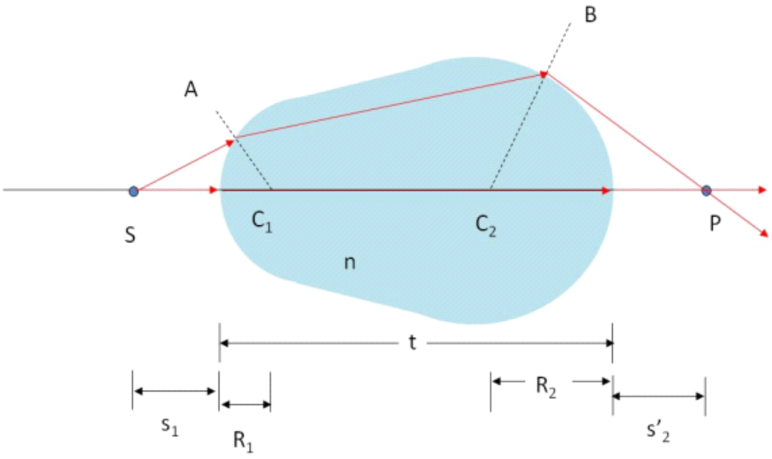
where

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{16.3}$$

but to appreciate what it means, lets find out where it comes from.

Derivation of the lens equation

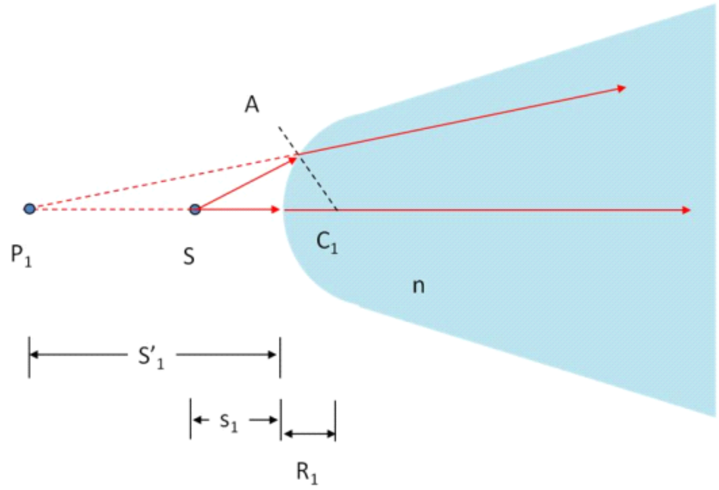
Consider the optical element in the figure below. Notice that our object is a dot, so our image will also be a dot. This is not as boring as it sounds if we consider any object can be considered as a collection of dots.



Light enters at a spherical surface on the left hand side. We use a point object located at S on the principal axis. We could put out dot object anywhere, but let's put it on the axis and trace two rays. The ray along the principal axis crosses each spherical surface at right angles, and therefore traves straight through the optic (this makes on ray very easy to trace!). The second ray hits the first spherical surface at point A. It is refracted and travels to point B. It is again refracted and travels toward the principal axis, crossing at P. The image location is the intersection of these rays, so we have an image at P.

Lets study the surfaces separately

Surface 1:



Let's treat surface 1 as though surface two did not exist. The light would bend at point A and head off into the lens material. This is just our semi-infinite bump problem so we know that

$$\frac{n_1}{s}+\frac{n_2}{s'}=\frac{(n_2-n_1)}{R}$$

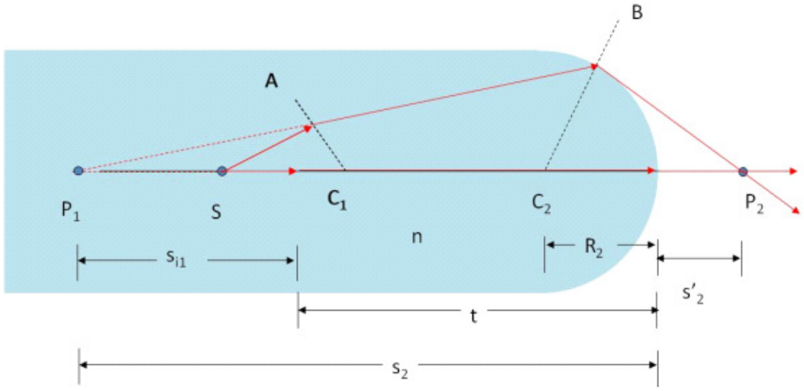
We can consider $n_1=1$ and $n_2=n$ for a air-glass interface and noting that s_1' is negative. Then

$$\frac{1}{s_1}-\frac{n}{s_1'}=\frac{(n-1)}{R_1} \tag{16.4}$$

Note that our rays are *not* converging in the glass. We can find the image formed by this side of our lens by tracing the diverging rays backward as we did for the fish tank. The image formed from the first side of the lens is virtual.

Surface 2

Now consider the second surface.



The second surface sees light diverging as though it came from a semi-infinite piece of glass with the object at P_1. It's not true, the light came from s. But the second surface can't tell the difference. It only knows light is incident in a particular pattern, and that pattern appears to come from point P_1. So we can treat the situation at surface 2 as though the object is the virtual image formed by surface 1. So

$$s_2 = s_1' + t$$

We again use our refractive equation

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{(n_2 - n_1)}{R}$$

but we identify $n_1 = n$ and $n_2 = 1$. We have for surface 2

$$\frac{n}{s_2} + \frac{1}{s_2'} = \frac{(1-n)}{R_2} \tag{16.5}$$

or

$$\frac{n}{s_1' + t} + \frac{1}{s_2'} = \frac{(1-n)}{R_2} \tag{16.6}$$

Now we take our thin lens approximation. Let $t \to 0$. Then equations (16.4) and (16.6) become

$$\frac{1}{s_1} - \frac{n}{s_1'} = \frac{(n-1)}{R_1}$$

$$\frac{n}{s_1} + \frac{1}{s_2'} = \frac{(1-n)}{R_2}$$

Adding these two equations yields

$$\frac{1}{s_1} - \frac{n}{s_1'} + \frac{n}{s_1'} + \frac{1}{s_2'} = \frac{(n-1)}{R_1} + \frac{(1-n)}{R_2}$$

or

$$\frac{1}{s_1} + \frac{1}{s_2'} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

This equation is very useful. It reminds us of the mirror equation (well, a little) . If we again let $s_1 = \infty$ (put the object at ∞ so the rays enter surface 1 parallel) we find

$$\frac{1}{s_2'} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

The spot where the rays gather if the object is infinitely far away is the focal point, f. so for parallel rays we can identify $s_2' = f$ as the focal length of the optic

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

which is known as the *lens makers' equation.*

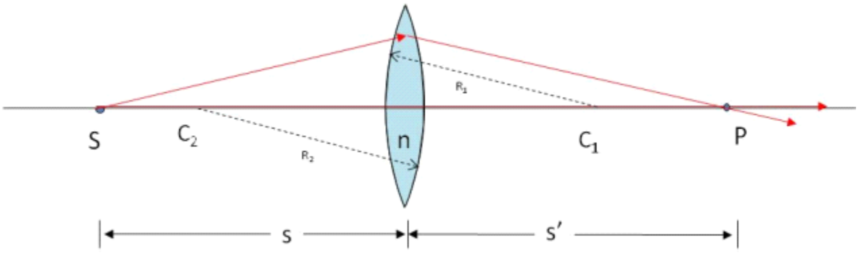
Then we have a relationship between the object distance in front of the lens, and the final image in back of the lens:

$$\begin{aligned} \frac{1}{s_1} + \frac{1}{s_2'} &= (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ &= \frac{1}{f} \end{aligned}$$

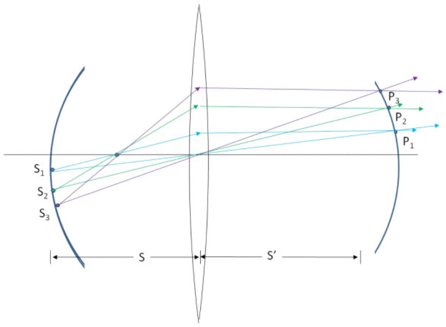
We can drop the subscripts (which we can do now that we let $t = 0$ since the internal distances for the inside points are not important).

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

This is called the *thin lens equation*. The resulting approximate geometry is shown below.



Of course any real object is made of lots of points, and not all of the points are on an axis. But each point will be imaged to a corresponding point on the image. Here is an example with three points in the object (the s_i points) and where their images are (the p_i points).



but it would work for millions of points. Our simple analysis explains the formation of actual images and not just point images. Question 123.16.2

Sign Convention

We need to add to our sign convention table a second radius, and the focal length.

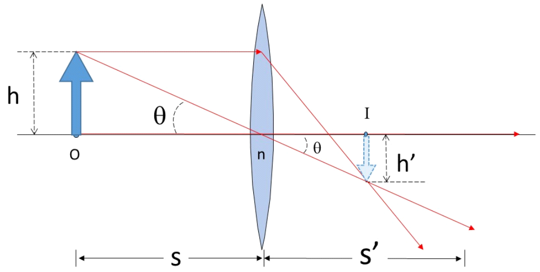
Quantity	**Positive if**	**Negative if**
Object location (s)	Object is in front of surface	Object is in back of surface (virtual object)
Image location (s')	Image is in back of surface (real image)	Image is in front of surface (virtual image)
Image height (h')	Image is upright	Image is inverted
Radius $(R_1$ and $R_2)$	Center of curvature is in back of surface	Center of curvature is in front of surface
Focal length (f)	Converging lens	Diverging lens

Magnification

We defined the magnification earlier as a comparison of the image height to the object height.

$$m = \frac{h'}{h} \tag{16.7}$$

But we can see that we could write this another way using simple trigonometry.



Note that

$$\tan\theta = \frac{h}{s}$$
$$\tan\theta = \frac{h'}{s'}$$

then

$$\frac{h}{s} = \frac{h'}{s'}$$

or

$$\frac{h'}{h} = \frac{s'}{s}$$

so we can write the magnification for a converging lens as minus the ratio of the image distance over the object distance.

$$m = -\frac{s'}{s}$$

The minus sign comes from our sign convention because the image will be upside down.

We found the thin lens formula using converging lenses, but it works for diverging lenses as well, so long as the thin lens approximation is valid.

17 Images Formed by Mirrors and Combinations of Lenses

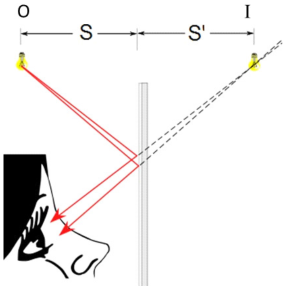
Fundamental Concepts

- Curved Mirrors can form images
- We can write the magnification in terms of the object and image distances
$$m = -\frac{s'}{s}$$
- The mirror formula for imaging is almost the same as the thin lens formula. The focal length part is quite different.
$$\begin{aligned} \frac{1}{s} + \frac{1}{s'} &= \frac{1}{f} \\ f &= \frac{R}{2} \end{aligned}$$
- The magnification of a two-lens system is just the product of the magnifications of the individual lenses $m_{\text{combined}} = m_1 m_2$
- If the two lenses in a two-lens system are placed so the distance between them is essentially zero, then the focal length of the two-lens system is given by
$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$
where f_1 and f_2 are the focal lengths of the individual lenses
- If the two lenses in a two-lens system are not places so the distance between them is essentially zero, the previous equation is not valid!

Imaging with Mirrors

Question 123.32.1

All of us have looked in a mirror at some time. We know what to expect. We see an image of ourselves. To study mirrors mathematically, we need to establish a sign convention and some standard notation.



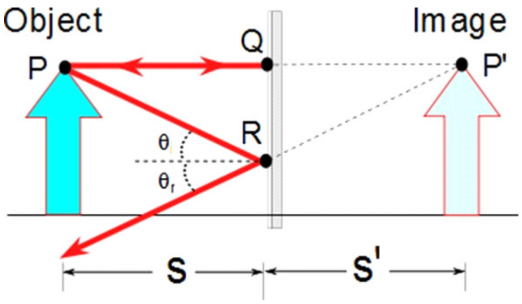
In the figure above, we have a person observing an object O in a mirror. The object is located at a distance s from the mirror. Just like for lenses, we will call this the *object distance*. The image appears to be located at a point I beyond the mirror a distance s'. Like before, we will call this the *image distance*.

How Images are Formed

This isn't our first look at mirrors, so let's review a bit. Images are located at a point from which rays of light diverge or at a point from which rays of light appear to diverge. This makes sense if you recall how our brain processes the light signals that we see. Our eyes intercept rays of light diverging from an object. Our brain processes those rays as though they traveled only in straight lines. So if we can create a situation that makes rays diverge in the same way the object did, we will have an image of the object.

Virtual Images

We already know that mirrors often create what we call *virtual* images because the image appears to be created from diverging rays from behind the mirror. If we look behind the mirror, no rays exist there (if they did exist, they would not make it through the mirror!).

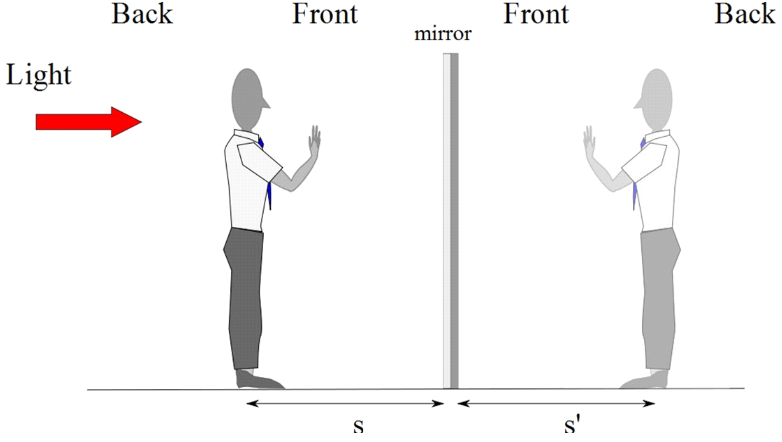


Let's look at a simple image as shown in the figure above. The object is an arrow. We could trace all the rays that diverge from this object and build a very nice representation of the arrow (like ray tracing-based computer graphics–the way movies like *Toy Story* are made) but that would take time and computation power. For lenses, we only really needed to use two rays, and to remember what the object looked like. The same is true for mirrors.

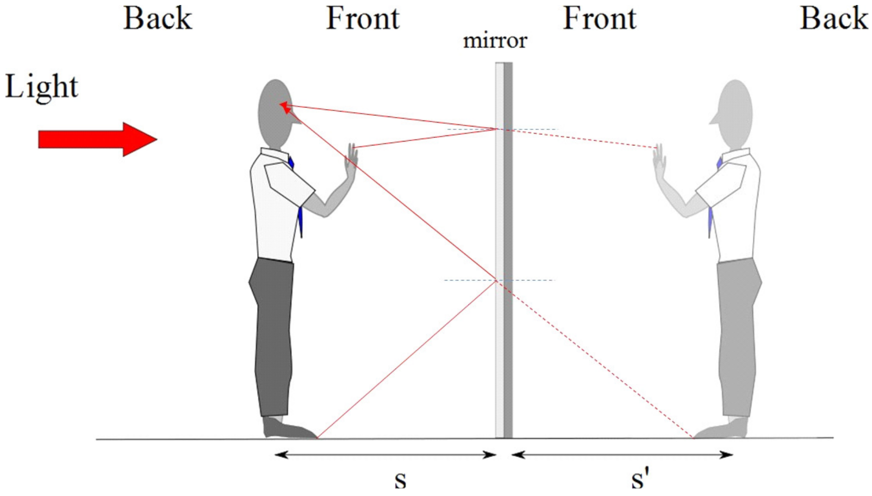
We pick one ray from the top of the arrow that travels straight to the mirror. This ray will travel a distance s and bounce back. We pick a second ray from point P that travels the path PR. This ray bounces off the mirror at an angle θ. So it appears that the tip of the arrow is at position P' and the rays from the tip appear to travel the paths $P'P$ and $P'R$. All this was review from our study of the law of reflection. Now let's go a little bit deeper into image formation by mirrors.

Mirror reversal

Look into a mirror. Raise your right hand.



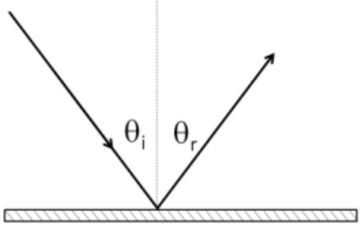
Your image raises what appears to be a left hand. It looks like a mirror switches the left and right sides of the image. But it might be better to note that the image raised hand is on the same side of the room as your real raised hand. Think of lying sideways on the ground in front of the mirror and raise a hand. Note that your image feet are on the same side of the room as your real feet. And note that as you put one hand up, the image hand also goes up. This is not exactly left-right reversal. What is happening?



The light from your raised hand would hit the mirror, reflect, and then be absorbed by your eye. Your brain tells you the light traveled in a straight line. So it looks like you have a raised hand in the mirror. But what has happened is that the light has traveled from your hand, to the mirror and back to your eye. This is not really left-right reversal. We need a name to describe what is really happening. The name is odd. To see why we use this name, notice that from your perspective (moving left to right across the figure) you first have the back part of your hand, then the front of your hand and then the front of the image hand, then the back of the image hand. We call this back-front reversal. It comes from the reversal in the order of parts of the object, like the front and back of your hand. So we say that a flat mirror performs a front-back reversal, not a left-right reversal.

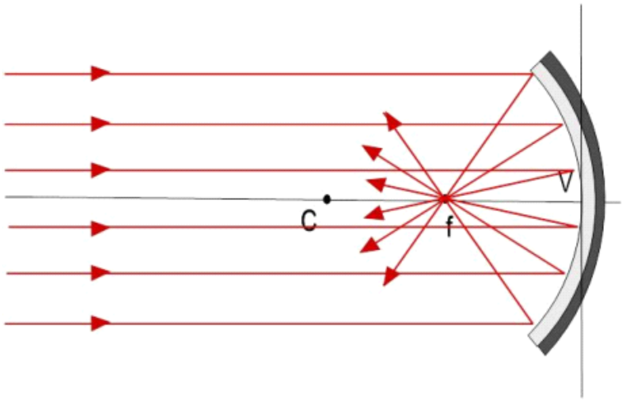
Concave Mirrors

Concave mirrors can form images. I'm sure you know that many telescopes are made with mirrors. We should see how this works. We recall the law of reflection



$$\theta_i = \theta_r$$

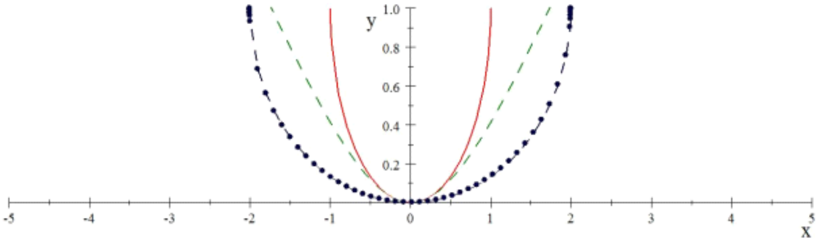
Armed with this, we can see what would happen if we curved the mirror surface. Each ray has a different normal due to the curvature of the mirror. The result is that parallel rays all meet at a spot on the axis.



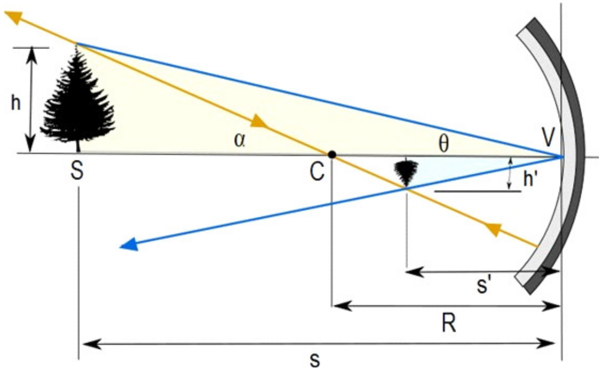
If we place something at this location, we could start a fire! We have just re-invented the solar cooker! Look at the point where all the rays meet. This is just like the lens situation where parallel rays met at a point after passing an optical element. This point must be a focal point.

Paraxial Approximation for Mirrors

The correct shape of a mirror is more like a parabola, but parabolas are hard to machine or build. Spherical shapes are relatively easy. So we often see spherical mirrors just like we often see spherical lenses. This will work so long as we allow only rays that make small angles with respect to the principal axis. We can see why this works if we plot a sphere and a parabola (and a hyperbola). For small deviations from the center, the shape of the functions all look alike.



We would expect the reflections to be similar under these circumstances, so, if we meet the criteria for the paraxial approximation, our spherical mirrors should work. Note that when you need the entire mirror, say, in a communications antenna, you must do better than a spherical approximation to the correct shape for your mirror. Your satellite dish is likely not a spherical section.



Like with our flat mirror, we will measure distances from the mirror surface (from point V in the figure). We can find the image location, s', by again taking two rays. We could use any of billions of rays. But just like for lenses, let's try to pick rays that are easy to draw. One convenient ray is the ray that passes through the center of curvature, C. This ray will strike the mirror surface at right angles and bounce back along the same path. The incident angle will be zero, so the reflected angle must be zero by the law of reflection. That is the yellow ray in the figure.

Another convenient ray is the ray from the tip to point V. This ray will bounce back with angle θ. Right at point V the mirror surface is perpendicular to the optic axis. This makes the bounce of the blue ray just like the a bounce from a flat mirror! That is easy to draw. Where these two reflected rays cross, we will find the image of the tip of our arrow. Knowing the shape of the arrow and that the bottom is on the axis, we can fill in the rest of the image.

We can calculate the magnification for this case. We use the gold triangle to determine that

$$\tan\theta = \frac{h}{s}$$

and the blue triangle to determine that

$$\tan\theta = \frac{-h'}{s'}$$

We want to indicate that the image is inverted by making it's sign negative. So we have arbitrarily added the negative sign to make the equation fit our sign convention. This gives us $\tan\theta$ instead of $-\tan\theta$ when h' is inverted (when h', itself, is negative). This is just a convention. But we will use it. So our magnification would be

$$m = \frac{h'}{h}$$

But we can write

$$\begin{aligned} h &= s\tan\theta \\ h' &= -s'\tan\theta \end{aligned}$$

so the magnification could be written as

$$m = \frac{h'}{h} = \frac{-s'\tan\theta}{s\tan\theta} = -\frac{s'}{s}$$

This is the same definition for magnification that we found for lenses.

Mirror Equation

We can further exploit this geometry to get a relationship between s, s', and R. Notice that

$$\tan\alpha = \frac{h}{s-R}$$

and that

$$\tan\alpha = \frac{-h'}{R-s'}$$

Then

$$\frac{h}{s-R} = \frac{-h'}{R-s'}$$

or

$$\frac{R-s'}{s-R} = -\frac{h'}{h}$$

We can use our magnification definition to replace h'/h

$$\frac{R-s'}{s-R} = \frac{s'}{s}$$

we perform some algebra

$$\begin{aligned}
(R-s')\,s &= s'\,(s-R)\\
-s's+Rs &= ps'-Rs'\\
Rs+Rs' &= ss'+s's\\
\frac{Rs'}{Rss'}+\frac{Rs}{Rss'} &= \frac{2ss'}{Rss'}\\
\frac{1}{s}+\frac{1}{s'} &= \frac{2}{R}
\end{aligned}$$

This is called the *mirror equation*. Notice how similar this is to the thin lens equation! In fact, if

$$\frac{1}{f}=\frac{2}{R}$$

would be exactly the same equation.

Focal Point

Now that we know the mirror equation, let's let s be very large (for example, let s be the distance to the Sun). Then

$$\begin{aligned}
\frac{1}{\infty}+\frac{1}{s'} &\approx \frac{2}{R}\\
0+\frac{1}{s'} &\approx \frac{2}{R}\\
\frac{1}{s'} &\approx \frac{2}{R}
\end{aligned}$$

or

$$s'\approx\frac{R}{2}$$

This is a special image point. But we really already know what to call this special place where parallel rays come together. We call it the *focal point* and the distance from the mirror to the point f is called the *focal length*. We see that, indeed

$$f=\frac{R}{2} \tag{17.1}$$

so we can write the mirror equation as

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f} \tag{17.2}$$

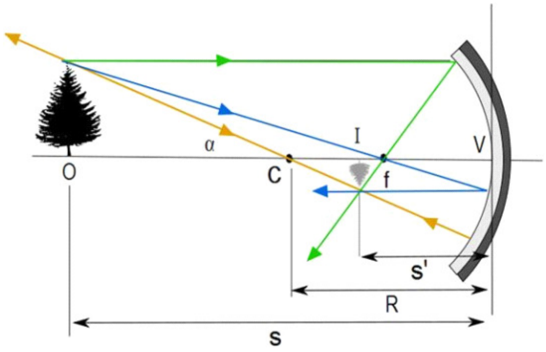
For a mirror, the value of f does not depend on the mirror material because in optical mirrors there is no glass in front of the metallic surface (this is not true for lenses optics).

Ray Diagrams for Mirrors

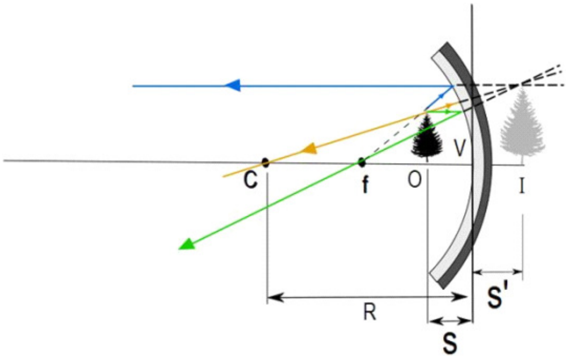
We have been drawing diagrams to find where images are formed for lenses, we should do the same for mirrors. We use a similar set of three rays. These rays are defined as follows:

Principal rays for a concave mirror:

1. Ray 1 is drawn from the top of the object such that its reflected ray must pass through f.
2. Ray 2 is drawn from the top of the object through the focal point to reflect parallel to the principal axis.
3. Ray 3 is drawn from the top of the object through the center of curvature. This ray will be incident on the mirror surface at a right angle and will be reflected back on itself.



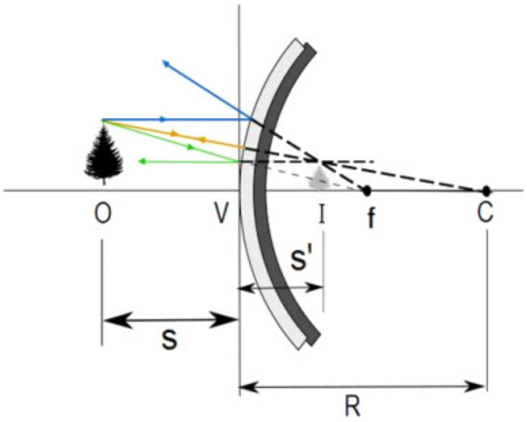
We can do the same for an object closer than a focal length



We also may have a mirror that curves, but curves the other way.

Principal rays for a convex mirror:

1. Ray 1 is drawn from the top of the object such that its reflected ray appears to have come from F.
2. Ray 2 is drawn from the top of the object to reflect parallel to the principal axis.
3. Ray 3 is drawn from the top of the object so that it appears to have come from the center of curvature. This ray will be incident on the mirror surface at a right angle and will be reflected back on itself.



We should tabulate our sign convention for mirrors like we did for lenses

Quantity	**Positive if**	**Negative if**
Object location (s)	Object is in front of surface (real object)	Object is in back of surface (virtual object)
Image location (s')	Image is in front of surface (real image)	Image is in back of surface (virtual image)
Image height (h')	Image is upright	Image is inverted
Radius $(R_1$ and $R_2)$	Center of curvature is in front of surface	Center of curvature is in back of surface
Focal length (f)	In front of surface	In back of surface

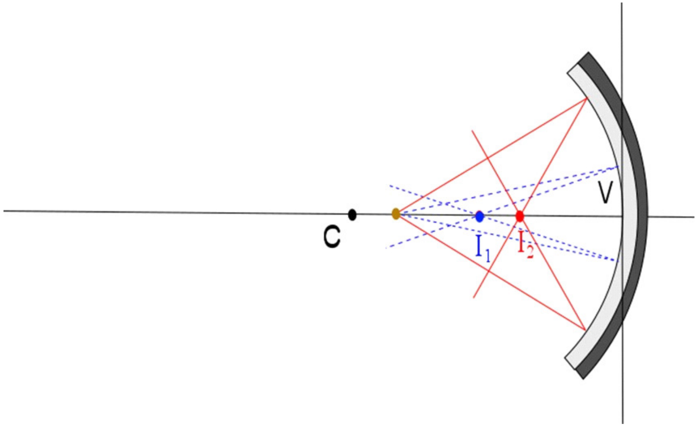
Aberrations

It's time to ask what happens when we don't use paraxial rays (when the small angle approximation is not valid) but our equations are based on using only small angles or paraxial rays. You might guess that we will have problems in our imagery. and we know that imagery problems are given a special name, “aberrations.”

Our first aberration came from lenses refracting different colors of light to form different images at different image locations. We called this “chromatic aberration” because it involved a problem with colors. But now we have a problem because we made the mirror the wring shape. It should be a parabola, but instead it is a section of a sphere. We will call the problem this creates, *spherical aberration.*

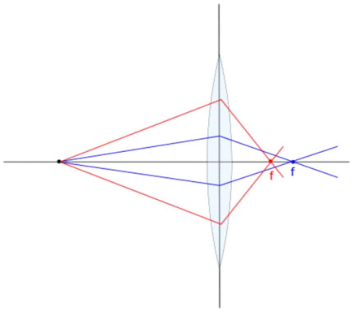
You might wonder why we would use the wrong shape if it causes problems in our imagery. The reason is that spherical shapes are easier to make than parabolas or hyperbole, or other shapes. So optics manufacturers have been using spherical optics for centuries. But we saw that our thin lens and thin mirror equations only work if we have paraxial rays. So, we should ask, if we used a different shape, would we be good without the paraxial restriction? The answer is yes, but for lenses we need hyperbolic shapes, and for mirrors we need parabolic shapes. A hyperbolic shaped lens is more than ten times the cost of a spherical shaped lens. So often we continue to use the spherical shapes even though they are wrong.

Spherical aberration has a similar effect to chromatic aberration. We end up with different images of our object at different image locations. But this time the color of the light does not matter. What matters is how the light enters the mirror. If we let rays converge from any direction from our spherical mirror we find that the rays do not form a single image.



In the figure above, notice that some rays (given red lines but remember color doesn’t matter here, just the angle) converge to one focal point (the red one) and other rays (given blue lines) converge to a different Instead, they converge on a volume near where the image should be. Rays from larger angles converge at different distances than rays from small angles. This problem is known as *spherical aberration*. Most of the time, we will point our optics so the object is near the principal axis, so we can make the paraxial approximation that fixes this problem. Question 223.16.5

The same problem happens with lenses



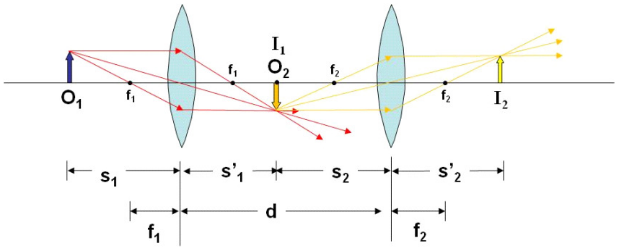
Again non-paraxial and paraxial rays focus at different spots because we have the wrong shape for our lens. Spherical aberration was made famous as the main problem with the Hubble Telescope.

There are many aberrations that come from making lenses that are easy to manufacture, but that are not the perfect shape. We won't study these in this class. If you are curious, we cover these in PH375.

Combinations of Lenses

We found that to correct chromatic aberration we used two lenses. Together they are called an "achromat" and all good cameras use achromats to fix chromatic aberration. We can do something similar to correct for spherical aberration. But in each case, we are combining two lenses (or even more!). How do we predict what the lens system will do? It turns out that we have all we need to know to combine lenses already.

To combine lenses, we do the same thing we did for the two surfaces of a thin lens. We form the image from the first lens as though the second lens is not there. Then we use the image from the first lens as the object for the second lens. Suppose we take two lenses of focal lengths f_1 and f_2 and place them a distance d apart.



Because this system would use a magnified image as the object for lens 2, the final magnification is the product of the two lens magnifications

$$m_{\text{combined}} = m_1 m_2 \tag{17.3}$$

For the first lens we have

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \tag{17.4}$$

where s_1' is our first lens image distance. We can solve for s_1'

$$s_1' = \frac{s_1 f_1}{s_1 - f_1} \tag{17.5}$$

We then take as the second object distance

$$s_2 = d - s_1'$$

we use the lens formula again.

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2}$$

and again find the image distance

$$s_2' = \frac{s_2 f_2}{s_2 - f_2}$$

but we can use our value of s_2 to find

$$\begin{aligned} s_2' &= \frac{(d - s_1')\, f_2}{(d - s_1') - f_2} \\ &= \frac{(d - s_1')\, f_2}{d - s_1' - f_2} \end{aligned}$$

We have and expression relating the image distances, d and f_2. But we would really like to have an expression that relates s_1 and s_2'. Lets use

$$s_1' = \frac{s_1 f_1}{s_1 - f_1}$$

and substitute it into our expression for s_2'

$$s_2' = \frac{\left(d - \frac{s_1 f_1}{s_1 - f_1}\right) f_2}{d - \frac{s_1 f_1}{s_1 - f_1} - f_2}$$

This looks messy, but we can do some simplification

$$s_2' = \frac{df_2 - \frac{s_1 f_1 f_2}{s_1 - f_1}}{d - f_2 - \frac{s_1 f_1}{s_1 - f_1}} \tag{17.6}$$

Well, it is still a little messy, but we have achieved our goal. We have s_2' in therms of the focal lengths, d, and s_1.

Suppose we let $d \rightarrow 0$. Then

$$\begin{aligned} s_2' &= \frac{-\frac{s_1 f_1 f_2}{s_1 - f_1}}{-f_2 - \frac{s_1 f_1}{s_1 - f_1}} \\ &= \frac{\frac{s_1 f_1 f_2}{s_1 - f_1}}{\frac{f_2(s_1 - f_1)}{s_1 - f_1} + \frac{s_1 f_1}{s_1 - f_1}} \\ &= \frac{s_1 f_1 f_2}{f_2 s_1 - f_2 f_1 + s_1 f_1} \\ &= \frac{s_1 f_1 f_2}{s_1 (f_2 + f_1) - f_2 f_1} \end{aligned}$$

So

$$s_2' = \frac{s_1 f_1 f_2}{s_1 (f_2 + f_1) - f_2 f_1}$$

Lets undo the math that brought us s_2' in the first place

$$\begin{aligned} \frac{1}{s_2'} &= \frac{s_1 (f_2 + f_1) - f_2 f_1}{s_1 f_1 f_2} \\ &= \frac{s_1 (f_2 + f_1)}{s_1 f_1 f_2} - \frac{f_2 f_1}{s_1 f_1 f_2} \\ &= \frac{(f_2 + f_1)}{f_1 f_2} - \frac{1}{s_1} \end{aligned}$$

or

$$\frac{1}{s_2'} + \frac{1}{s_1} = \frac{(f_2 + f_1)}{f_1 f_2}$$

Which looks very like the lens formula with

$$\frac{1}{f} = \frac{(f_2 + f_1)}{f_1 f_2}$$

If we unwind this expression, we find

$$\frac{1}{f} = \frac{f_2}{f_1 f_2} + \frac{f_1}{f_1 f_2}$$

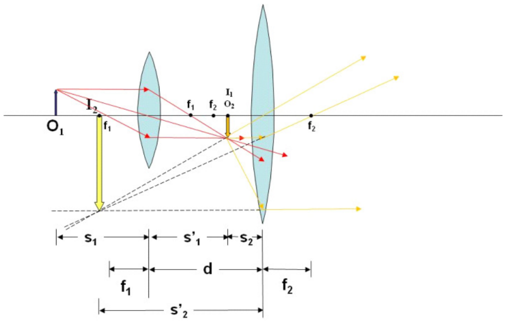
$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \tag{17.7}$$

This is how we combine thin lenses. We see that the two lenses are equivalent to a single lens with focal length f as long as they are close together.

Of course, we had to place our lenses right next to each other for this to work. This is not the case for a telescope or microscope. We should look at such a case. There is no need for more math. We can go back to equation (17.6).

$$s_2' = \frac{df_2 - \frac{s_1 f_1 f_2}{s_1 - f_1}}{d - f_2 - \frac{s_1 f_1}{s_1 - f_1}}$$

But let's look at a case using ray diagrams. For this case, let's take two lenses, and let's have the first lens make a real image. Once again, let's have that image be the object for the second lens. But this time, let's move the second lens so that the image from the first lens (object for the second lens) is closer to the second lens than f_2. If that is the case, the second lens works like a magnifier. The final image is enlarged.



In our next lecture, we will take on to common optical systems that have more than one lens and see how they work.

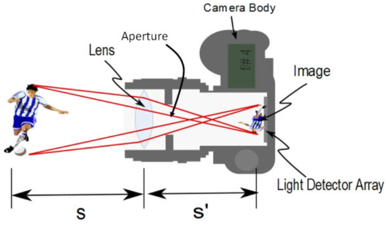
18 The Camera and Eyes

Fundamental Concepts

- Cameras and other imaging systems use a strange term called and f-number to tell what the intensity of the image will be
$$I \propto \frac{1}{(f/\#)^2}$$
where $f/\#$ is the symbol for f-number and the f-number is given by
$$f/\# \equiv \frac{f}{D}$$
- Medical people measure the power of a lens in diopters. A diopter is one over the focal length with the focal length measured in meters.

The Camera

in 1900 George Eastman introduced the Brownie Camera. This event has changed society dramatically. The idea behind a camera is very simple.



The camera has a lens (often a compound lens like the ones we have just discussed) and a screen for projecting a real image created by the lens.

Let's take an example camera. Say we wish to take a picture of Aunt Sally at a family reunion. Aunt Sally is about $1.5\,\mathrm{m}$ tall. She is standing about $5\,\mathrm{m}$ away. Then to fit the image of Aunt Sally on our $35\,\mathrm{mm}$ detector, we must have $h' = 0.035\,\mathrm{m}$ so our camera description would be

$$h = 1.5\,\mathrm{m}$$
$$h' = 0.035\,\mathrm{m}$$
$$s = 5\,\mathrm{m}$$
$$f = 0.058\,\mathrm{m}$$

We wish to find s' and m so we can design our camera. Let's do m first.

$$\begin{aligned} m &= \frac{h'}{h} = \frac{-0.035\,\mathrm{m}}{1.5\,\mathrm{m}} \\ &= -2.333\,3 \times 10^{-2} \end{aligned}$$

so our image is small and inverted. The small size we wanted. We want to be able to include a small sensor array to capture the image created by our lens. Bigger arrays cost more, so reasonably small is tood. Our experience with thing lenses tells us that we should expecte that the image would be inverted. A digital camera uses it's built-in computer to turn the image right side up for us on the display on the back of the camera.

Now let's find s'. From the thin lens formula we know

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f_{eq}}$$

where we are treating the complicated compound lens of the camera like one equivalent lens with a focal length f_{eq}. We can rearrange the thin lens formula to solve for s'

$$\begin{aligned} s' &= \frac{fs}{s-f} \\ &= 5.868\,1 \times 10^{-2}\,\mathrm{m} \\ &= 58.681\,\mathrm{mm} \end{aligned}$$

so our detector must be $58.681\,\mathrm{mm}$ from the lens. We build a camera such that the distance from the lens to the detector array is $58.681\,\mathrm{mm}$.

Now suppose we want to photograph a $1000\,\mathrm{m}$ tower from $2\,\mathrm{km}$ away. Then

$$\begin{aligned} m &= -\frac{0.035\,\mathrm{m}}{1000\,\mathrm{m}} \\ &= -3.5 \times 10^{-5} \end{aligned}$$

and

$$\begin{aligned} s' &= \frac{(0.058\,\mathrm{m})\,(2000\,\mathrm{m})}{2000\,\mathrm{m} - (0.058\,\mathrm{m})} \\ &= 5.800\,2 \times 10^{-2}\,\mathrm{m} \\ &= 58.002\,\mathrm{mm} \end{aligned}$$

It appears that our camera must allow the lens-sensor distance to change. This is why you need a focus adjustment on the lens of a good camera. Objects far away require a different s' value than objects that are close. Usually you twist the lens housing to make this adjustment (the lens housing has a threaded screw system that increases or decreases s' as you twist. Consumer cameras often have a motor that makes this adjustment for you. Even some cell phone cameras do this. You may see the lens move back and forth as someone takes a picture.

There are several things that govern whether a picture will be good. When you buy a quality manual lens, it will be marked in $f\#s$. The specification of an automatic lens will also be given in terms of $f/\#s$. To help us buy such lenses, we should understand what the terminology means.

Most things we want to take a photograph of are much farther than $58\,\mathrm{mm}$ from the camera. For such objects we can revisit the magnification.

$$m = -\frac{s'}{s}$$

but from the thin lens formula

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

If $s \gg f$ then we can say that $1/s \approx 0$ and so $s' \approx f$. Then

$$m \approx -\frac{f}{s}$$

and we see that the size of the image is directly proportional to the focal distance. If we change the focal distance, we can change the size of the image. This is how a zoom lens works. Usually a zoom lens is a compound lens, and the focal length is changed by increasing the distance between the component lenses. This is what your camera is doing when it zooms in and out when you push the telephoto button. Some of the image goes off the edges of the sensor, so what is actually detected is just the center of what the lens saw, but what you do see on the sensor is magnified.

Remember we studied intensity

$$I = \frac{P}{A}$$

Photographic film and digital focal plane arrays detect the intensity of light falling on them. We can see that the area of our image depends on our magnification, which depends on s' and for our distant objects it is proportional to f. The image area is proportional to h'^2 which is proportional to $s'^2 \approx f^2$. So we can say that the area is proportional to f^2. Then

$$I \propto \frac{P}{f^2}$$

The power entering the camera is proportional to the size of the aperture (hole the light goes through). If the camera has a circular opening, this area is proportional to the square of the diameter of the opening, D^2 so

$$I \propto \frac{D^2}{f^2}$$

This ratio is useful because it tells us how much intensity we get in terms of things we can easily know. Good cameras have changeable aperture sizes, and good lenses have changeable focal lengths. But by using the combination of these two terms, we can ensure we will get enough light (but not too much) when we take the picture.

It would be good to give this number a special name. But instead, we named the ratio

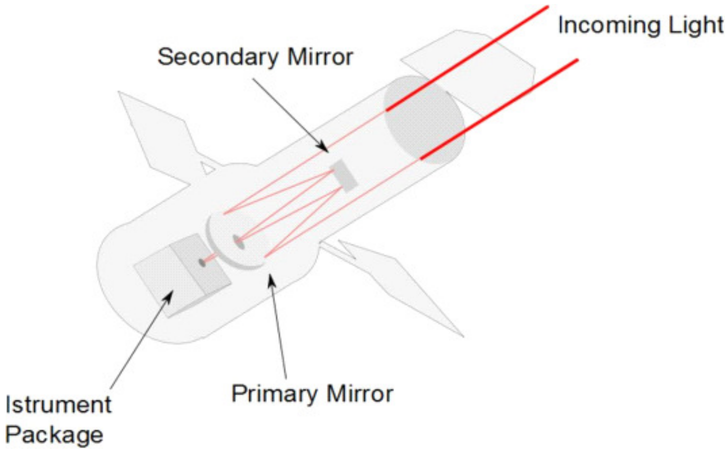
$$f/\# \equiv \frac{f}{D} \tag{18.1}$$

It is called the $f/\#$ (pronounced f-number) so

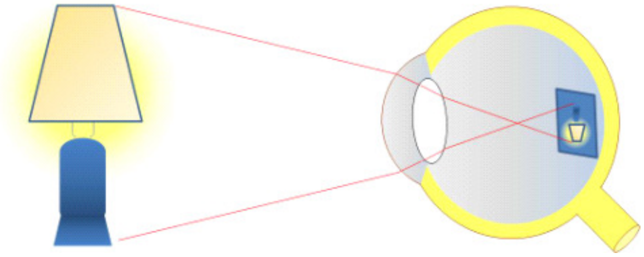
$$I \propto \frac{1}{(f/\#)^2} \tag{18.2}$$

So good cameras have adjustable lens systems marked in $f/\#'s$. Typical values are $f/2.8$, $f/4$, $f/5.6$, $f/8$, $f/11$, and $f/16$.

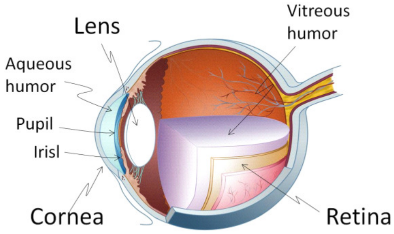
This terminology is use for telescope design as well. The Hubble telescope is an $f/24$ Ritchey-Chretien Cassegrainian system with a $2.4\,\mathrm{m}$ diameter aperture. The effective focal length is $57.6\,\mathrm{m}$.



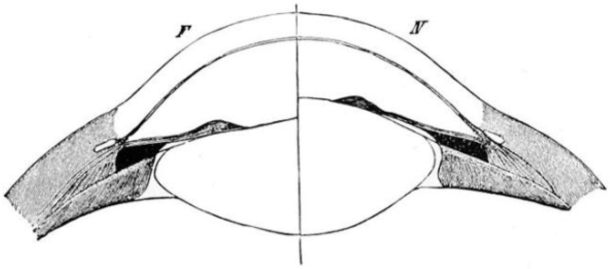
The Eye



The figure above shows the parts of the eye. The eye is like a camera in its operation, but is much more complex. It is truly a marvel. The parts that concern us are the cornea, crystalline lens, pupil, and the retina.



The Cornea-lens system refracts the light onto the retina, which detects the light. The lens is focused by a set of mussels that flatten the lens to change it's focal length. The focusing process is different from a standard camera. The camera moves the lens to achieve a different image distance. Our eye can't change the distance between the lens system and the retina. So our eye changes the shape of the lens, changing it's focal length.



The crystaline lens becomes thicker, and therefore more curved when the cilliary musscle flexes. Austin Flint, "The Eye as an Optical Instrument," *Popular Science Monthly,* Vol. 45, p203, 1894 (Image in the public domain)

The focusing system is called accommodation. This system becomes less effective at about the time you reach an age of 40 years because the lens becomes less flexible. The closest point that can be focused by accommodation is called the near point. It is about $25\,\mathrm{cm}$ on average. There is, of course, no such thing as an average person, all of us are a little bit different. You young students probably have a much shorter near point than $25\,\mathrm{cm}$. For those of us that are a little older, $25\,\mathrm{cm}$ or more is more likely.

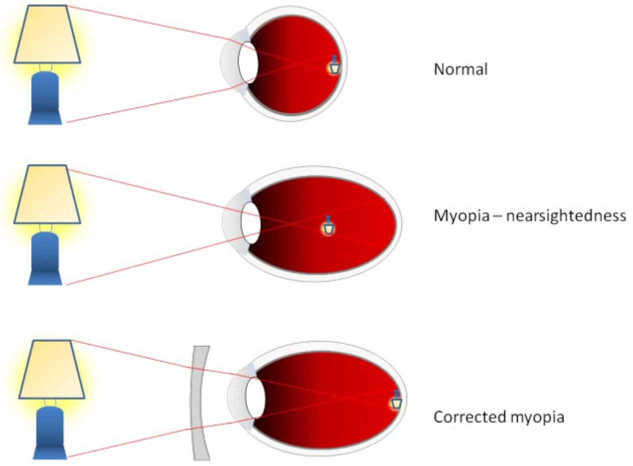
The farthest point that can be focused is a long way away. It is called the far point. Both the near and far points degrade with years leading to bifocal glasses (and much irritation because you can't see as well, but I'm really not complaining because at least I can see).

The iris changes the area of the pupil (the aperture of the eye). The pupil is, on average, about $7\,\mathrm{mm}$ in diameter. This acts like the aperture adjustment of a camera.

Nearsightedness

Question 223.18.3

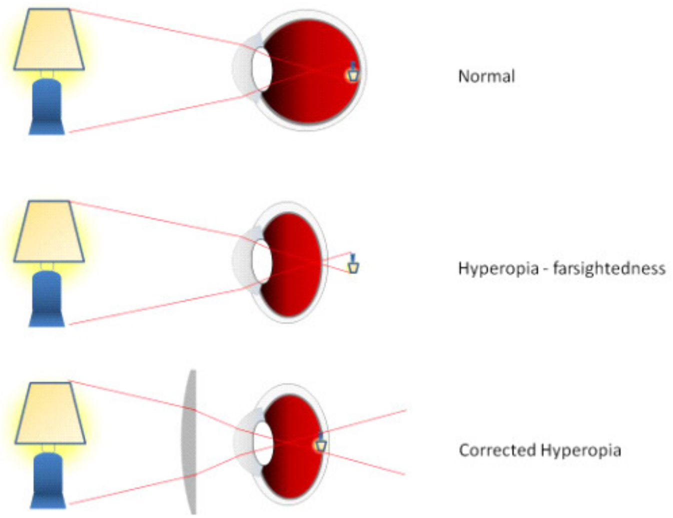
In some people the cornea-lens system focuses in front of the retina. Usually this is because the shape of their eyes is not spherical but is elongated along the optic axis of the eye. This is called nearsightedness or myopia.



So their perfectly good cornea-lens system makes a great image in the vitreous humor (the jellylike stuff that fills the eye) and not on the retina where it would be detected. From your experience with lenses you know this would produce a blurry image. And that is what nearsighted people see much of the time. We can help nearsighted people by effectively changing the cornea-lens system of the eye by adding another lens. We want a diverging lens that makes the light more spread apart so that the lens system of the eye can make it focus where the retina actually is. Alternately we could flatten the cornea, itself, with laser ablation.

Farsightedness

Sometimes the cornea-lens system focuses in back of the retina. This is usually because the eye grew too flat or "oblate." This is called farsightedness or hyperopia. Once again we can fix the problem by adding an additional lens. This time a converging lens.



The converging lens will make the effective focal length of the system shorter, so that it can form an image where the retina actually is.

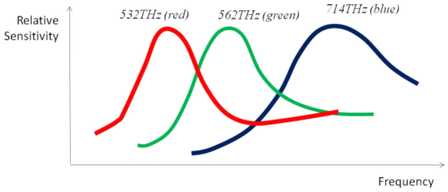
It would be convenient and very understandable if medical professionals prescribed eye glasses by telling us what focal length we needed to correct our vision. But that would not be the medical way! Eye glasses use a different unit of measure to describe how they bend light. The unit is the *diopter* and it is equal one over the focal length, but with the focal length measured in meters.

$$\text{diopter} = \frac{1}{f\,(\mathrm{m})} \tag{18.3}$$

This measurement is called the *power* of the lens. It is just the same as giving the focal length, but less clear for science students.

Color Perception

The eye detects different colors. The respecters called cones can detect red, green, and blue light.



The eye combines the red, green, and blue response to allow us to perceive many different colors.

Most digital cameras also have red, green, and, blue pixels to provide color to images. The detectors in digital cameras are often have much narrower frequency bands than the eye. Likewise, television displays and monitors have red, green, and blue pixels. By targeting the eye receptors, power need not be wasted in creating light that is not detected well by the eye. The difference in band-width can cause problems in color mixing. Yellow school busses (perceived as different amounts of green and red light) may be reddish or green if the bandwidths are chosen poorly.

The science of human visual perception of imagery is called *image science*. There are many applications for this field, from forensics to intelligence gathering.

19 Optical Systems that Magnify

Fundamental Concepts

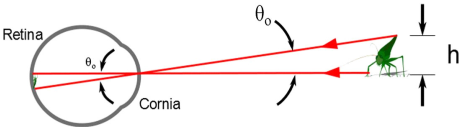
- Angular magnification is a comparison of how big an image looks with and without an optical system.
- Telescopes and Microscopes are double lens systems
- Resolution is a name we give to the fundamental blurriness in geometrical optical systems due to the wave nature of light.

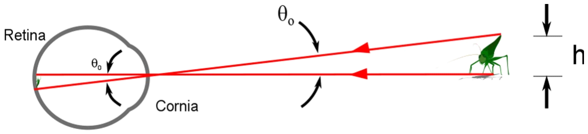
Angular Magnification

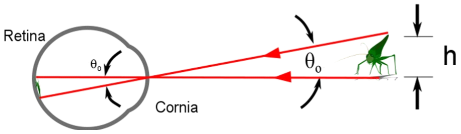
We already encountered the simple magnifier when we studied ray diagrams. But by this point in our study of optics you are probably wondering about our definition of magnification. If you are an Idahoan and are out hunting, when you look through your binoculars or scope you don't want to know how big the image is compared the actual dear, you want to know if you can see the deer better than you could with just your eyes. The magnification on your scope doesn't compare the image size to the object size, but it is comparison between two optical systems, one is just your eyes, the other is your eyes and the scope working together.

Let's use the simple magnifier that we know to define a new kind of magnification that does this comparison between two optical systems. We can use what we know about easy rays to draw to describe both optical systems. We usually use three principle rays, but for this analysis, let's just use one for each side of an object. Since the image is on the retina, we can see where these singular rays strike the retina and understand the size of the image on our eye light detection system. Let's choose to draw the rays that go straight through the middle of the lens of the eye (because they are the easiest ones!).

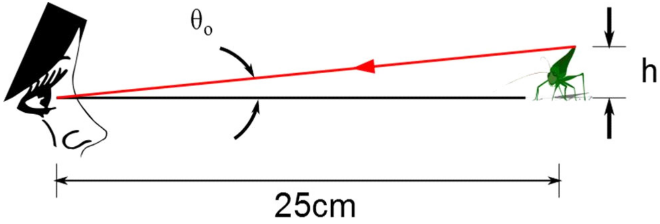
If we pick a ray from the top of our object that goes through the center of the lens, that ray won't seem to change direction at all. It will hit the retina to form the top of the image of the object. We can do the same for the bottom of the object. Then we can see from the next figure that the angle θ_o subtends[15] both the object and the image of the object. If you think about the previous figure, you will see that if the angle were to increase, so would the size of the image on the retina. We can increase this angle by, say, moving the object closer to our eyes.







When we get to our $25\,\mathrm{cm}$ we reach the limit of the eye for focusing.

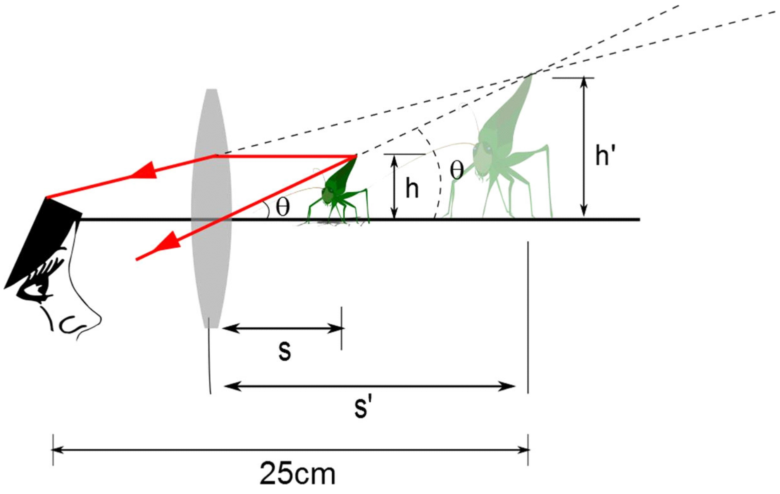


If we move the object any closer, it will appear blurry. We call this position, the closest point where we can place an object and still bring it into focus with our eye, the *near point.* Thus the maximum value of θ will be at the near point for unaided viewing. We will call this maximum unaided angle θ_o.

But suppose we want to see this object in more detail. We can use a magnifying glass.

---

[15] The angle that "subtends" and object is the one who's bounding lines just hit either side of the object.

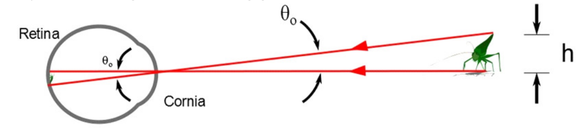


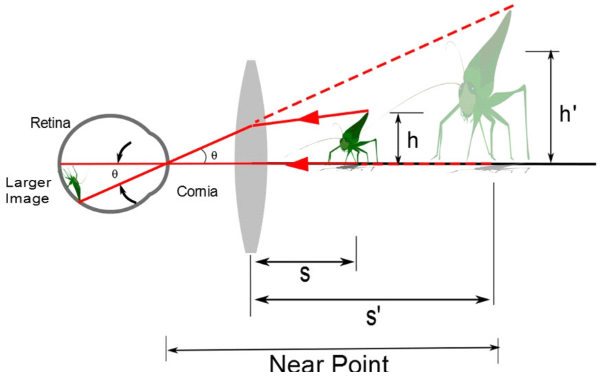
If we place the object closer to the magnifying glass than the focal distance$(s < f)$, then we have a virtual image with magnification

$$m = \frac{-s'}{s} \tag{19.1}$$

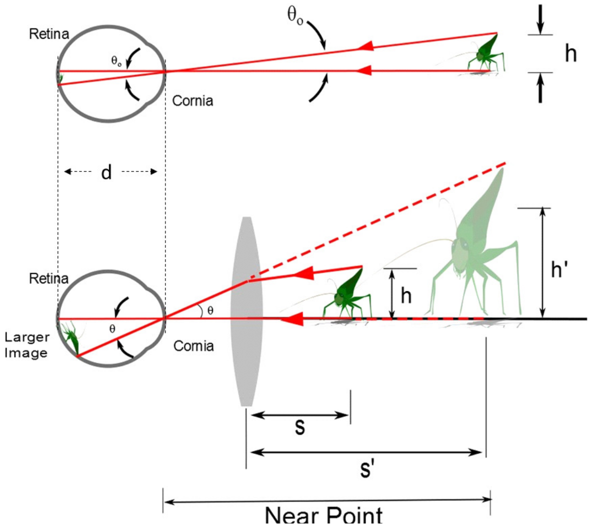
which is larger than one and positive (because s' is negative).

But what we really want to know is how much bigger the image looks with the lens than it did without the lens. By looking at what happens to the rays when they enter our eye, we can see why the image looks bigger.





The magnifying glass has bent the light, and the bent rays make a bigger angle, θ, so the image on our retina is bigger. Since the image fills more of our retina, we perceive the image as being bigger.



We need a mathematical formula that will tell us how much bigger the image on our retina will be. Notice from the figure that

$$\begin{aligned}\tan\theta_o &= \frac{h'_{eye}}{d}\\ \tan\theta &= \frac{h'_{lens-eye}}{d}\end{aligned}$$

so

$$\begin{aligned}h'_{eye} &= d\tan\theta_o\\ h'_{lens-eye} &= d\tan\theta\end{aligned}$$

then if we compare the new, larger image on the retina formed with the lens-eye system to the one formed with just the eye, we get

$$\frac{h'_{lens-eye}}{h'_{eye}} = \frac{d\tan\theta}{d\tan\theta_o} = \frac{\tan\theta}{\tan\theta_o}$$

and if we once again use the small angle approximation

$$\frac{h'_{lens-eye}}{h'_{eye}} \approx \frac{\theta}{\theta_o}$$

this would tell us how much bigger our object looks when viewed with the magnifying glass compared to how it looked without the magnifying glass. This is just what we want! Let's give this a new symbol

$$M = \frac{\theta}{\theta_o} \tag{19.2}$$

Remember, this is really different than the magnification we have studied before. The magnification we have been using compared the size of the image with the size of the object. We call M the *angular magnification.*

So, the angular magnification compares how big the object seems to be with and without a lens or lens system. It is really a comparison between the size of the real image on the retina formed with just our eye, and the one formed with the magnifier.

If the virtual image formed is farther than the near point of the eye, ($s' >$ ˜$25\,\mathrm{cm}$) the image on our retina will be smaller than it would be at the near point because it is farther away. If the virtual image is closer than the near point, it will be fuzzy because the eye cannot focus closer than the near point. Thus, the value of M will be maximum when s' for the magnifying glass is at the near point of the eye. We can find where to place the image so that we get maximum magnification. Taking just the magnifier, and placing the image at about $-25\,\mathrm{cm}$,

$$\begin{aligned}\frac{1}{s}+\frac{1}{s'} &= \frac{1}{f}\\ \frac{1}{s}+\frac{1}{-25\,\mathrm{cm}} &= \frac{1}{f}\end{aligned}$$

and so

$$\frac{1}{s}=\frac{-25\,\mathrm{cm}-f}{-f\,(25\,\mathrm{cm})}$$

or

$$s=\frac{(25\,\mathrm{cm})\,f}{25\,\mathrm{cm}+f} \tag{19.3}$$

Using small angle approximations, we can write

$$\tan\theta_o=\frac{h}{25\,\mathrm{cm}}\approx\theta_o$$

and

$$\tan\theta=\frac{h}{s}\approx\theta$$

then the maximum angular magnification is

$$\begin{aligned} m_{\max} &= \frac{\theta}{\theta_o}=\frac{\frac{h}{s}}{\frac{h}{25\,\mathrm{cm}}}\\ &= \frac{25\,\mathrm{cm}}{\frac{25\,\mathrm{cm}f}{25\,\mathrm{cm}+f}}\\ &= \frac{25\,\mathrm{cm}+f}{f}\\ &= 1+\frac{25\,\mathrm{cm}}{f}\end{aligned}$$

We can also find the minimum magnification by letting s be at f. This gives

$$\theta = \frac{h}{f}$$

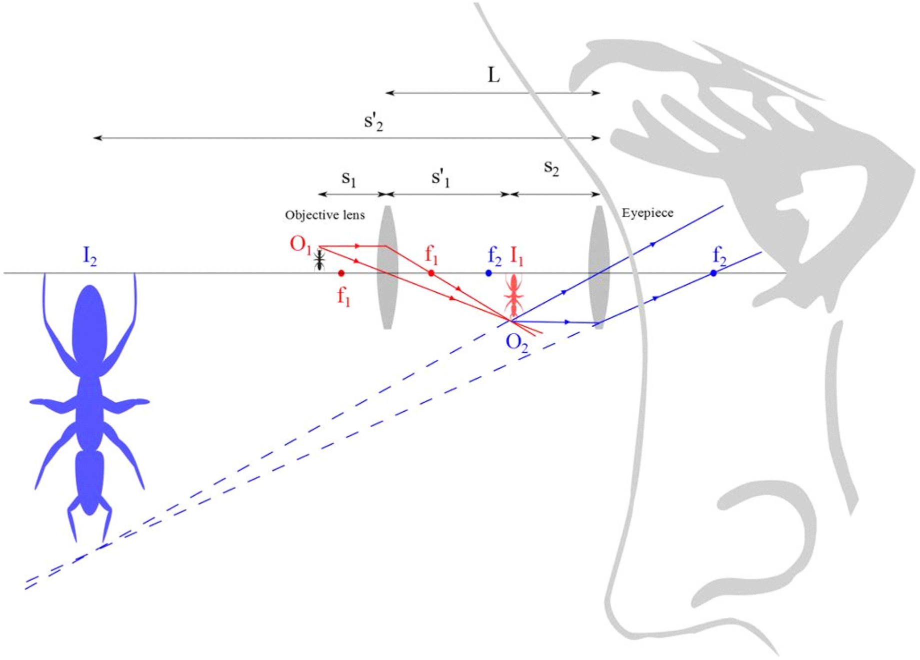
$$\begin{aligned} m_{\min} &= \frac{\theta}{\theta_o} = \frac{\frac{h}{f}}{\frac{h}{25\,\mathrm{cm}}} \\ &= \frac{25\,\mathrm{cm}}{f} \end{aligned}$$

We use the idea of a simple magnifier in combination with other lenses to make the magnification happen in telescopes, microscopes, and other instruments that magnify. So all of these systems can be described using the idea of an angular magnification.

The Microscope

To see things that are very small, we add another lens to our simple magnifier. We will place this lens near the object and call the lens the *objective* because it is next to the object. We will keep a simple magnifier and place it near the eye. This lens will be called the *eyepiece* because it is near your eye.

The objective will have a very short focal length. The eyepiece will have a longer focal length (a few centimeters).



We separate the lenses by a distance L where

$$\begin{aligned} L &> f_o \\ L &> f_e \end{aligned}$$

We place the object just outside the focal point of the objective. The image formed by the objective lens is then real and inverted. We use this image as the object for the eyepiece. The image formed is upright and virtual, but it looks upside down because the object for the eyepiece (first image for the objective) is upside down.

Of course, we want to know the magnification of the system. We know what the eyepiece will do because it is being used as just a simple magnifier. Recall that magnifications are a factor. Think from our basic equation

$$m = \frac{h'}{h}$$

tells us that

$$h' = mh$$

or in words, m is the factor by which h' is bigger (or smaller) than h. So magnifications are factors. For example, $h\prime$ could be 10 time bigger. That means if we want to know the total magnification of the system we start with the magnification of the eyepiece M_e, say, 20 times bigger than it would look with just our eye, but we have to account for eyepiece's object (the image from the first lens) being bigger than the actual object. If you think about it, it makes sense that if the objective makes the object look 10 times bigger, and the eyepiece makes the image look 20 times bigger than if you looked at it with your eye, the system makes it look 200 times bigger. The combined magnification for the two-lens system is

$$m = m_o M_e \tag{19.4}$$

The minimum magnification of the eye piece will be roughly

$$M_{e_{\min}} \approx \frac{25\,\mathrm{cm}}{f_e}$$

and the maximum will be

$$M_{e_{\max}} = 1 + \frac{25\,\mathrm{cm}}{f_e}$$

But remember the object for the eyepiece is the image from the first lens. And that image is larger than the object by an amount

$$m_1 = m_o = \frac{-s_o'}{s_o}$$

and let's estimate how big the magnification due to the first lens will be. Because

$s_o \approx f_o$. and $s'_o \approx L$ (roughly)

$$m_o = \frac{-s'_0}{s_0} \approx -\frac{L}{f_o}$$

The combined magnification for the two-lens system is about

$$m_{system} = m_o M_e = -\frac{L}{f_o}\frac{25\,\mathrm{cm}}{f_e} \tag{19.5}$$

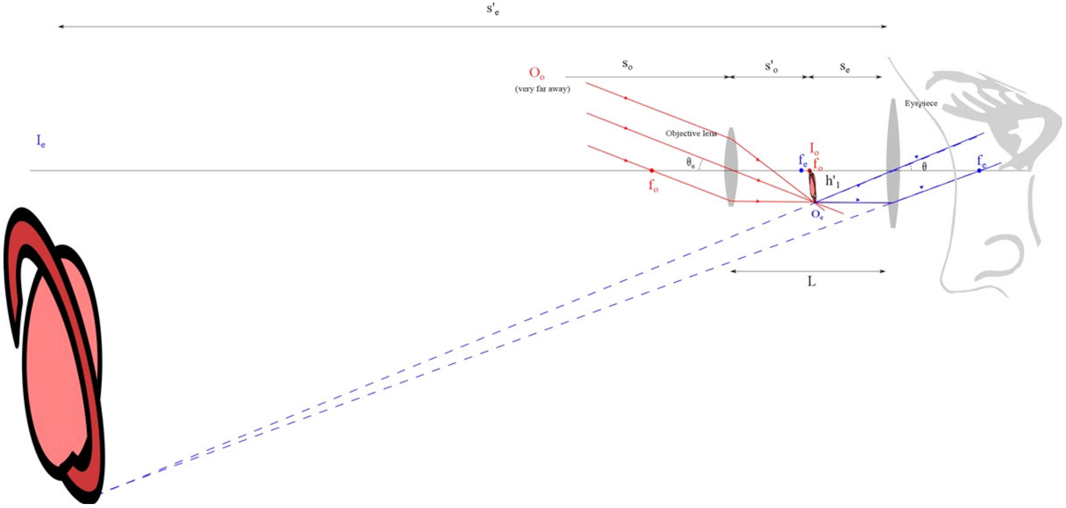
this is the minimum magnification (because we used the minimum magnification formula for the eyepiece).

Telescopes

There are two main types of telescopes *refracting* and *reflecting*. We will study refracting telescopes first.

Refracting Telescopes

Like the microscope, we combine two lenses and call one the objective and the other the eyepiece. The eyepiece again plays the role of a simple magnifier, magnifying the image produced by the objective.



We again form a real, inverted image with the objective. We are now looking at distant objects, so the image distance $s'_o \approx f_o$. Once again, we use the image from the objective as the object for the eyepiece. The eye piece forms an upright virtual image (that looks inverted because the object for the eyepiece is the image from the objective, and the real image from the objective is inverted). The largest magnification is when the rays exit the eyepiece parallel to the principal axis. Then the image from the eyepiece is formed at near infinity (but it is very big, so it is easy to see). This gives a lens separation of $f_o + f_e$ which will be roughly the length of the telescope tube.

The angular magnification will be

$$M = \frac{\theta}{\theta_o} \tag{19.6}$$

where θ_o is the angle subtended by the object at the objective (see figure above) and θ is subtended by the final image at the viewer's eye. Consider s_o is very large. We see from the figure that

$$\tan \theta_o = -\frac{h_o'}{f_o} \tag{19.7}$$

and with s_o large we can use small angles.

$$\theta_o = -\frac{h_o'}{f_o} \tag{19.8}$$

The angle θ will be the angle formed by rays bent by the lens of the eyepiece. This angle will be the same as the angle formed by a ray traveling from the tip of the first image and traveling parallel to the principal axis. This ray is bent by the objective to pass through $f_{e.}$ Then

$$\tan \theta = \frac{h_o'}{f_e} \approx \theta \tag{19.9}$$

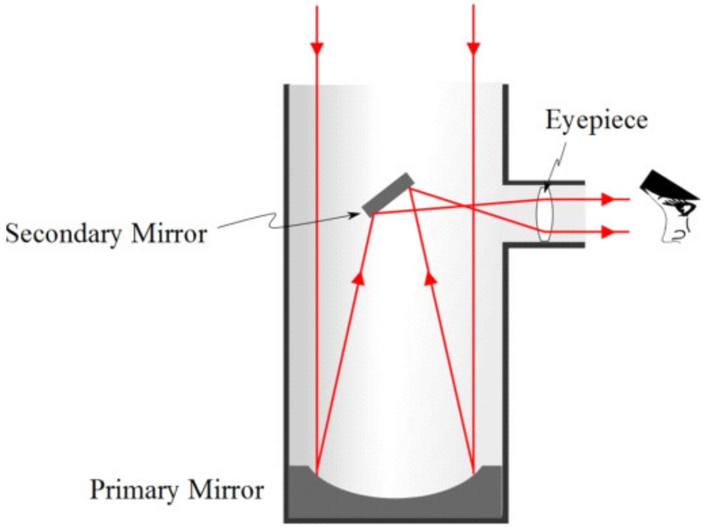
so

The magnification is then

$$M = \frac{\theta}{\theta_o} = \frac{\frac{h'}{f_e}}{-\frac{h'}{f_o}} = -\frac{f_o}{f_e} \tag{19.10}$$

Reflecting Telescopes

Reflecting telescopes use a series of mirrors to replace the objective lens. Usually, there is an eyepiece that is refractive (although there need not be, radio frequency telescopes rarely have refractive pieces).



There are two reasons to build reflective telescopes. The first is that reflective optics do not suffer from chromatic aberration. The second is that large mirrors are much easier to make and mount than refractive optics. The Keck Observatory in Hawaii has a $10\ \mathrm{m}$ reflective system. The largest refractive system is a $1\ \mathrm{m}$ system. The Hubble telescope has a $2.5\ \mathrm{m}$ aperture.

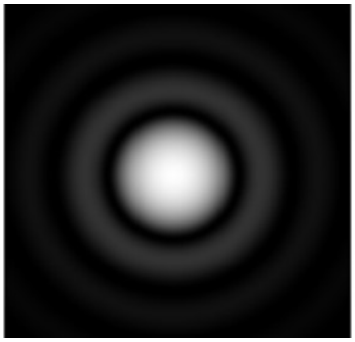
The telescope pictured in the figure is a Newtonian, named after Newton, who designed this focus mechanism. Many other designs exist. Popular designs for space applications include the cassigrain telescope.

The rough design of a reflective telescope can be worked out using refractive pieces, then the rough details of the reflective optics can be formed.

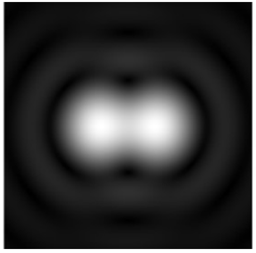
Resolution

We have emphasized that an extended object can be viewed as a collection of point objects. Then the image is formed from the collection if images of those point objects. We thought about the design of cameras and telescopes and other optical systems using ray optics. It would be great if optical systems could form images with infinite precision, but it turns out they can’t. And it is the fact that light acts as a wave prevents this from being true! Our wave nature of light comes back to complicate our simple optical designs!

Because light is really a wave, the images of the point objects won't be points, themselves. They will be little circular central maxima with dim concentric ring patterns.



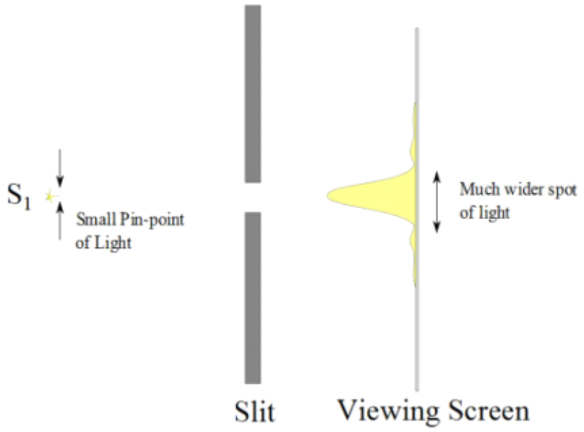
If we have more than one point of light, we will get two such patterns.



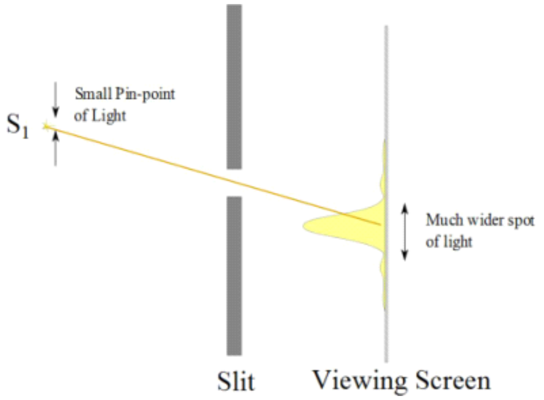
and our image of an extended object (like Aunt Sally) is made up of many, many points all reflecting light into our camera lens. We want each of those points of light to create corresponding points on our image. But instead they are creating small circles of light, and form the above figures we can see that those circles will overlap. All this makes our images fuzzy.

The quality of our image depends on how poorly a point object is imaged. If each point object makes a large circle of light on the screen or detector array, we get a very confusing image (it will look blurry to us). Let's review why this will happen so we can know how to minimize the effect.

We already know that if we take light and pass it through a single slit, we get an intensity pattern that has a central bright region.

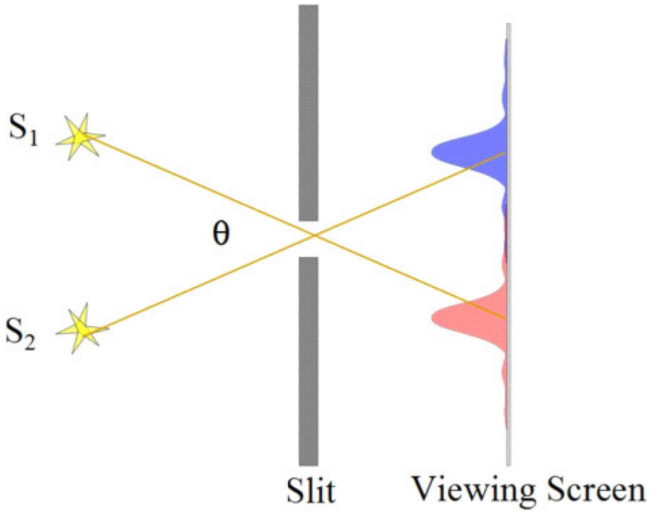


Remember that normal objects will be made up of many small points of light (either due to reflection or glowing) and each of these will form such an intensity pattern on a screen. Here is a bright point source that is not on the axis, an we see that it too makes a bright spot on the screen (and smaller bright spots or rings, depending on the shape of the aperture)

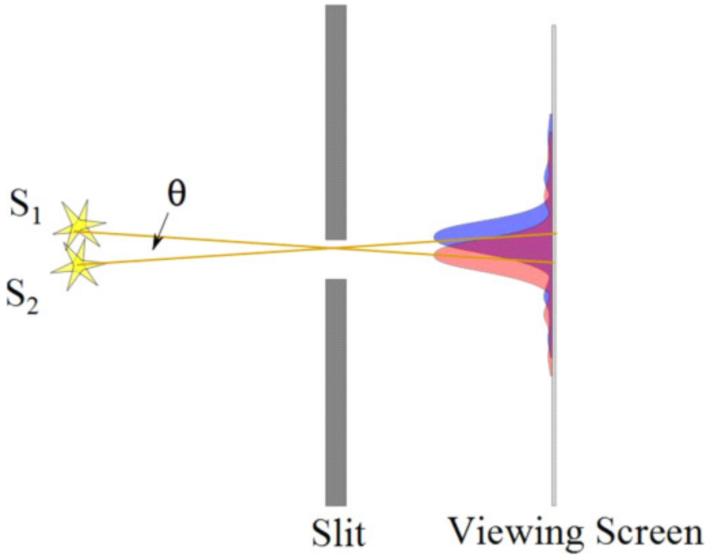


So our images will be made up of many central bright spots, each of which represents a point of light from the object. These central bright spots may overlap, (and their secondary maxima certainly will overlap).

Let's take a simple case of two points of light, S_1 and S_2. If we take a single slit and pass light from two distant point sources through the slit, we do not get two sharp images of the point sources. Instead, we get two diffraction patterns.



If these patterns are formed sufficiently far from each other, it is easy to tell they were formed from two distinct objects. Each point became a small blur, but that is really not so bad. We can still tell that the two blurs came from different sources. If our pixel size is about the same size of the blur, we may not even notice the blurriness in the digital imagery.



But if the patterns are formed close to each other, it gets hard to tell whether they were formed from two objects or one bright object. We now have a problem. Suppose you are trying to look at a star and see if it has a planet. But all you can see is a blur. You can't tell if there is one source of light or two.

Long ago an early researcher titled Lord Rayleigh developed a test to determine if you can distinguish between two diffraction patterns. When the central maximum of one point's image falls on the first minimum of anther point's image, the images are said to be just resolved. This test is known as *Rayleigh's criterion*.

We can find the required separation for a slit. Remember that

$$\sin(\theta) = m\frac{\lambda}{a} \quad m = \pm 1, \pm 2, \pm 3 \ldots \tag{19.11}$$

gives the minima (dark spots) for a single slit. We want the first minimum, so $m = 1$

$$\sin(\theta) = \frac{\lambda}{a} \tag{19.12}$$

Remember that

$$\tan\theta = \frac{y}{L}$$

where L is the distance from the slit to the viewing screen, and y is the vertical location of the dark spot. If we place the second image maximum so it is just at this location, the two images will be just barely resolvable. In the small angle approximation, $\sin(\theta) \approx \theta$ so

$$\theta_{\text{min}} = \frac{\lambda}{a} \tag{19.13}$$

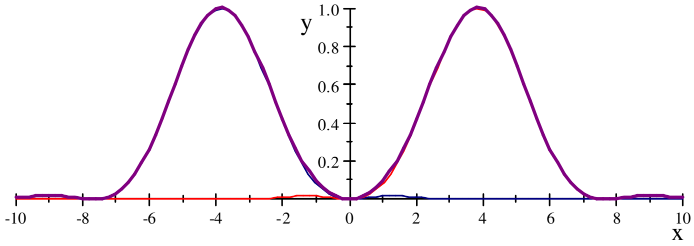
Now you may be saying to yourself that you don't often take pictures through single illuminated slits, so this is nice, but not really very interesting. But suppose, instead, that we image a circular aperture. We won't go through all the math (there are Bessel functions involved) but the criterion becomes

$$\theta_{\text{min}} = 1.22\frac{\lambda}{D} \tag{19.14}$$

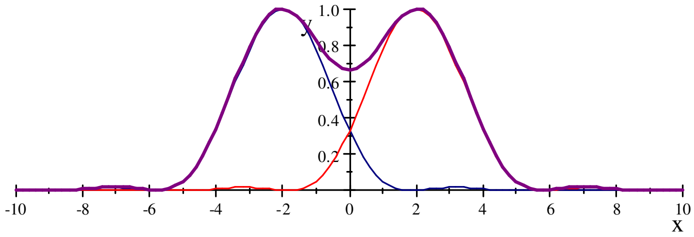
where D is the aperture diameter.

Still, you might think, "I don't like pictures taken through small circles any better than through small slits!" Yet, in fact, you do. Most cameras have circular apertures. The light that passes into your phone camera must pass though the circular lens. And, of course, your eyes have circular apertures.

The Rayleigh criteria tells you, based on your camera aperture size, how a point source will be imaged on the film or sensor array. If we consider extended sources (like your favorite car or Aunt Sally) to be collections of many point sources, then we have a way to tell what features will be clearly resolved on the image and what features will not (like you may not be able to see the lettering on the car to tell what model it is, or you may not be able to distinguish between the gem stones in Aunt Sally's necklace because the image is too blurry to see these features clearly).

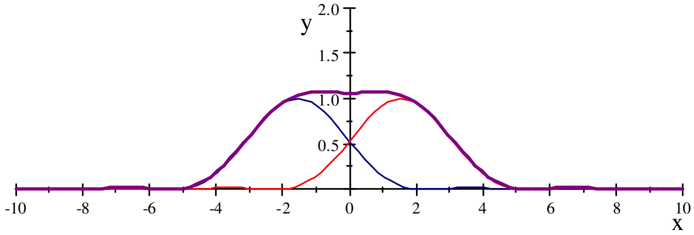


Pattern from two resolved circular slits.



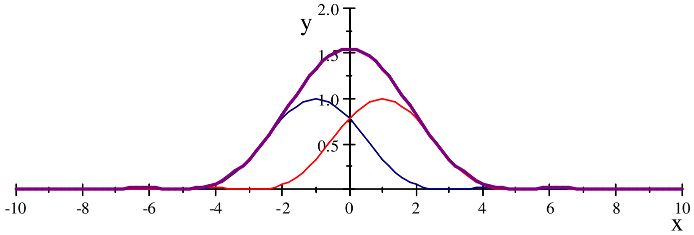
Rayleigh Criteria: Pattern from two circular soruces where the sources are close enough that the maximum from one pattern is placed on the minimum of the other. Lord Rayleigh gave this as the criteria for just barily being resolved.

Astronomers sometimes use Sparrow's criteria for two sources being resolved. It is shown below.



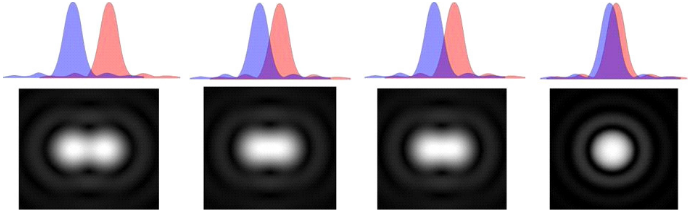
Sparrow Criteria: This is a less concervative resolution criteria than Rayleigh. When the intenisty pattern is flat on the top, there must be two sources. This criterial is used in astronomy.

Since astronomers just want to know if there are two stars or one, it is enough to see that the intensity pattern went flat at the center. That must mean that there are two stars. But if the stars are any closer, the flat center becomes a peak and the stars are unresolved.



Two circular sources unresolved

Here is what the easily resolved, Rayleigh resolved, Sparrow resolved, and unresolved cases look like.



This concludes our study of waves and optics!

You may be left feeling that we have really just gotten started, and you would be right. If you are a physics major (or curious and have elective credit) you will study waves more in PH295 and you may study more optics in PH375. Both are fantastic experiences. Even if you don't take a further class in these topics, you can study them on your own. There are many wonderful books on Optics.

Our goal this semester was to study the motion of many things and wave motion was a partial fulfilment of that goal. But not all motions of many objects are as uniform as wave motion. What if, say, the air molecules in our room instead of moving in simple harmonic motion like in a wave, they had random velocities? This is type of motion that we hinted about when we talked about thermal energy in PH121. We will take on this kind of motion next in this course.

20 Fluids and Pressure

We are changing topics radically. PH121 taught us how individual objects move. So far we have learned that wave motion is an organized motion of many objects. But often many objects move in less organized ways. We will take on less organized motion for the last part of this course. Let's review some basic properties of matter. Matter is made of many atoms. So matter is an example of many objects that might just move.

Fundamental Concepts

- Compressibility of fluids
- Density of fluids
- Pressure is a force spread over an area
- Pressure increases with depth in a fluid

Fluids

You are probably aware that there are four states of matter

solid
liquid
gas
plasma

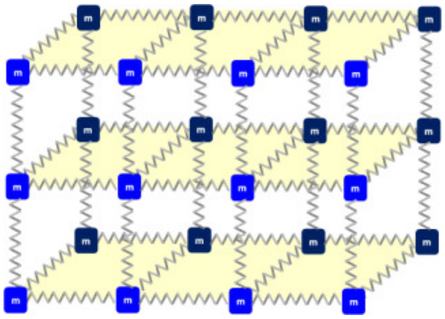
Believe it or not, plasma[16] is the most common, because stars are made of plasma. Planets are sometimes made of solids, liquids, or gases, but the great glowing stars are plasma. (I am ignoring the mysterious form of "dark matter" because so far we don't know what it is). Plasma is a heated gas that is ionized. We will mostly ignore this state, because unless you are dealing with neon signs, fluorescent lights, or the like, you don't encounter plasmas in every day experience.

---

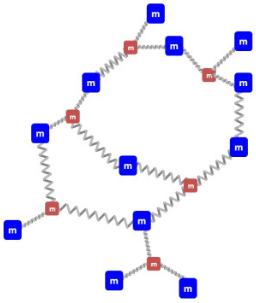
[16] This is not the kind of plasma that you donate!

Solids

We can view solids as having a set of forces that keep the molecules in place much as though they were attached using springs. Solids can have definite organization. If so, they are called crystals. You should observe the crystals around the Romney building if you have not already.

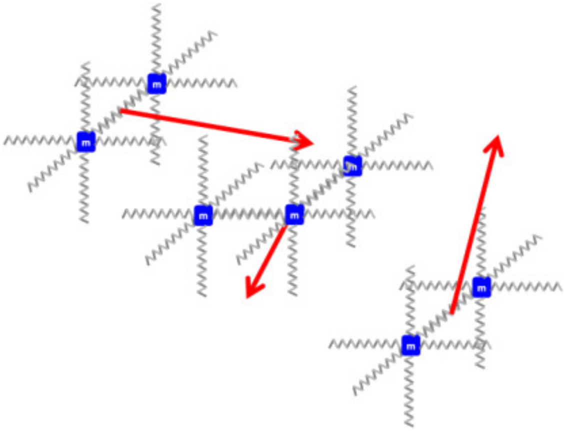


If the solid lacks definite order in its organization, it is called amorphous.



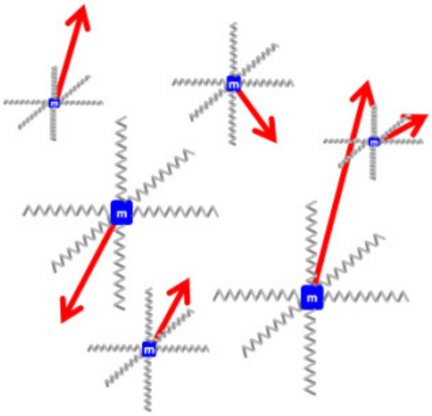
Liquids

The molecules in a liquid are less tightly bound than those in a solid. That is why they can flow. In the next figure, the atoms are bound by one spring-like force. But the atoms are not tied together in a tight set of bonds like a solid.



Gasses

The molecules in a gas are not bound to each other–not at all.



We have an intuitive feel for what a fluid is. But let’s make a more formal definition.

What is a fluid?

For the next few lectures we will study fluids, but what is a fluid?

A **Fluid** is a collection of molecules that are randomly arranged and are at most held together by weak cohesive forces and by forces exerted by the walls of a container.

Thus, liquids and gasses are definitely fluids. Solids are generally considered not fluids. But how about Jello™? or a combination of corn starch and water (sometimes called "ooblick")? These are non-Newtonian fluids. That is, things that are sort of solid and sort of not. But we will stick with things (at least at first) that are definitely fluids, and generally fluids with negligible friction. This is a little like in PH121 when we studied frictionless surfaces. How many surfaces are truly frictionless? Very few! you might guess that there are few fluids that have no friction, and you would be right. But just like with PH121, the assumption of frictionless fluids makes the math easier, and that is good when we are starting a now topic.

Pressure

Vollyball Demo

We have already seen pressure in this course. But it's been a while, so let's review. Consider a situation where we ask six of our class members to come up and press on the ball from all directions. Suppose further that we ask each person to exert a force on the ball. And suppose we ask the person to use the area of their hand to exert the force. The motion of the ball, and even it's shape depended on both the force (magnitude and direction) and the area involved in each push.

Noticing that each person is exerting a force but that the force is not acting on one point, but is spread out over an area, we recognize that each person is exerting a pressure on the ball

$$P \equiv \frac{F}{A} \tag{20.1}$$

Now consider a ball sitting in a room surrounded by air. The air is a fluid, so it's molecules are quite free to move around. Because there is some thermal energy in the room (even in Rexburg) the molecules will have some kinetic energy. So the air molecules will hit the ball. This will cause a force on the surface of the ball. And that force will be spread over the entire surface area of the ball. This is a force spread over an area. This is a pressure. We call this *air pressure*. This air force due to the colliding molecules is like having the hands on the pushing on the ball.

This force due to individual molecules is small and only lasts during the collision. But in the room we have many molecules, and many the molecules impact the ball. The molecules also impact the walls of the room. Suppose that every time a molecule bounces back from one wall it ends up headed back to the opposite wall bounces back again toward the first wall. If the molecules keep coming, there will be a force on the wall quite a bit of the time. At least, on average there is a force, anyway. This is the force that causes air pressure. The molecules impact the walls, and the ball, and us, and everything in the room all over the surface area of each object. The result is air pressure on every object in the room.

Likewise, the water pressure in a swimming pool is caused by moving water molecules. You should convince yourself that the reason the water stays in the pool is partly because the air molecules bounce against the water surface exerting a pressure on the water!

Working with the definition of pressure

We can work with equation 5.31 to define the force due to pressure

$$F \equiv PA$$

and can define a force due to the pressure at an element of area dA

$$dF \equiv PdA$$

Where there is a differential, expect that some time in the future we will integrate!

But before we go on, let's see what the units of pressure would be. We have a force divided by an area.

$$1 \frac{\mathrm{N}}{\mathrm{m}^2} = 1\,\mathrm{Pa}$$

The symbol is the Pa, and the unit is called the *pascal*. This is the name of a famous scientist.

Pressure Example:

Let's do a pressure problem together,

Problem statement:

A $50.0\,\mathrm{kg}$ woman balances on one heel of a pair of high heeled shoes. If the heel is circular and has a radius of $0.5000\,\mathrm{cm}$, what pressure does she exert on the floor?

Drawing

Variables

	Known	
M	Mass of woman	$M = 50\,\mathrm{kg}$
r	Radius of woman's heal	$r = 0.500\,\mathrm{cm}$
g	acceleration do to gravity	$g = 9.8\frac{\mathrm{m}}{\mathrm{s}^2}$
	Unknown	
F	Force	
x	Coordinate Axis	
z	Coordinate Axis	
A	Area of woman's heal	

Basic Equations

$$\begin{aligned} F &= ma \\ A &= \pi r^2 \\ P &= \frac{F}{A} \end{aligned}$$

Symbolic Solution

$$A = \pi r^2$$

$$F = ma = Mg$$
$$P = \frac{F}{A} = \frac{Mg}{\pi r^2}$$

Numerical Solution

$$\begin{aligned} P &= \frac{Mg}{\pi r^2} \\ &= \frac{50\,\mathrm{kg} * 9.8\frac{\mathrm{m}}{\mathrm{s}^2}}{\pi\left(0.500\,\mathrm{cm}\right)^2} \\ &= \frac{490.0\frac{\mathrm{m}}{\mathrm{s}^2}\,\mathrm{kg}}{0.785\,40\,\mathrm{cm}^2}\frac{\left(100\,\mathrm{cm}\right)^2}{\mathrm{m}^2} \\ &= \frac{490.0\frac{\mathrm{m}}{\mathrm{s}^2}\,\mathrm{kg}}{0.785\,40}\frac{100000}{\mathrm{m}^2} \\ &= 6.\,238\,9 \times 10^6\,\frac{\mathrm{kg\,m}}{\mathrm{s}^2\,\mathrm{m}^2} \\ &= 6.\,238\,9 \times 10^6\,\mathrm{Pa} \\ & P = 6.24\,\mathrm{MPa} \end{aligned}$$

Units Check

$$\frac{\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}}{\mathrm{cm}^2} = \frac{\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}}{\mathrm{cm}^2}\frac{\left(100\,\mathrm{cm}\right)^2}{\mathrm{m}^2} = 10000\frac{\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}}{\mathrm{m}^2} = 10000\,\mathrm{Pa}$$

Units

Check

Reasonableness

This seems like a large number, but I have had a high heeled person step on my toe, so I believe it!

Bed of Nails

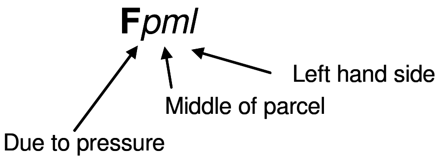
Variation of Pressure with Depth

Stacked Hymnbook

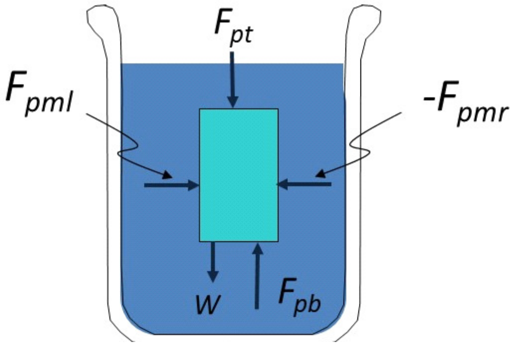
If you have been in a swimming pool, you probably have noticed that the pressure due to the water feels different at the surface than it does at the bottom of the deep end. Intuitively, we can say that the pressure at the bottom is larger than the pressure at the top. Let's see if we can show that this is true.

Take a glass of water or some other fluid. Let's look at just a part of the fluid (the darker section in figure **??**). Let's treat this "parcel of fluid" as a distinct body and look at the forces acting on it.

We need some way of labeling our forces. My way is kind of simple, but will work for now.



Consider Newton's second law. The sum of the forces must be equal to zero (Why?–think of the acceleration of the parcel of fluid)



$$\overrightarrow{F}_{net} = 0 = \mathbf{F}_{pt} + \mathbf{F}_{pb} + \mathbf{F}_{pmr} + \mathbf{F}_{pml} + \mathbf{W}$$

$$\begin{aligned} F_{net_x} &= 0 = 0 + 0 - F_{pmr} + F_{pml} - 0 \\ F_{net_y} &= 0 = -F_{pt} + F_{pb} - 0 + 0 - W \end{aligned}$$

The x-part tells us that

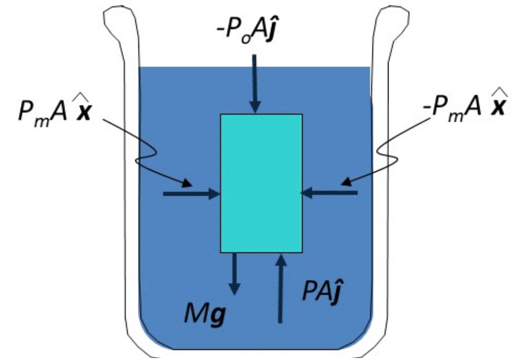
$$F_{pmr} = F_{pml}$$

which is not too much of a surprise. It seems reasonable that the side forces must be equal if the parcel doesn't accelerate. The y-part gives

$$F_{pb} - F_{pt} = W$$

The definition of pressure gives

$$\begin{aligned} P &= \frac{F}{A} \\ F &= PA \end{aligned}$$



So, for the top of our parcel of fluid, the force on the top must be

$$F_{pt} = P_tA$$

and for the bottom the force must be

$$F_{pb} = P_bA$$

Recalling that

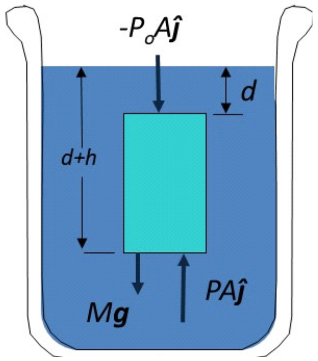
$$m = \rho \not{V} \tag{20.2}$$

where ρ is the density and $\not{V}$ is the volume, and substituting

$$\begin{aligned} F_{pb} - F_{pt} &= W \\ P_bA - P_tA &= -mg \\ (P_b - P_t)\,A &= -\rho V g \\ (P_b - P_t)\,A &= -\rho h A g \end{aligned}$$

where we have used the fact that the volume must be the area of the top or bottom, A, multiplied by the length, h, of our parcel.

$$\not{V} = Ah \tag{20.3}$$



We can solve this for the pressure at the bottom

$$P_b = P_t + \rho g h$$

Units Check

$$\mathrm{Pa} \Bumpeq \mathrm{Pa} + \frac{\mathrm{kg}}{\mathrm{m}^3}\frac{\mathrm{m}}{\mathrm{s}^2}\,\mathrm{m}$$

the last term can be written as

$$\left(\frac{\mathrm{kg\,m}}{\mathrm{s}^2}\right)\frac{1}{\mathrm{m}^2} \Rightarrow \frac{\mathrm{N}}{\mathrm{m}^2} \Rightarrow \mathrm{Pa}$$

so the units check.

This is a profound statement! (and a new basic equation for us). The pressure is larger at the bottom of the pool than the top. And it is larger by the amount $\rho g h$ where ρ is the water density, g is the acceleration due to gravity, and h is how far down we go to get to the bottom.

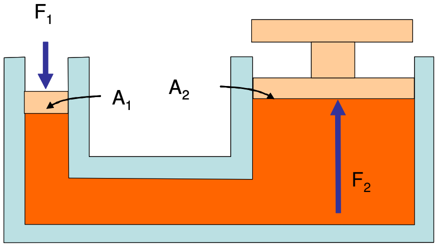
Note that we used a box shaped volume, but the result is general. To see this, view any arbitrary volume as consisting of little boxes. As we make the box size small, we approximate the actual volume (seem familiar from calculus?). Since it works for each box that makes up the volume individually, it will work for the whole volume (it takes a minute to think about this).

Pascal's Law

Let's step back and look at our new equation

$$P_b = P_t + \rho g h$$

This gives the pressure at some depth, knowing the pressure at the top. On what does P_b depend? We can see that it only depends on P_t, ρ, g, and h. Suppose I take a complex system like a hydraulic jack



Suppose the pressure on the left hand side at the top is $P_{t(left)}$. Then the pressure at the left hand side bottom would be

$$P_{b(left)} = P_{t(left)} + \rho g h_{left}$$

What is the pressure along the bottom of the jack system? Well we don't know a value, but we know it is the same pressure all along the bottom. It can only depend on h (since P_t, ρ, g are the same and h is the same simply because we are looking at a location along the bottom). So

$$P_{b(left)} = P_{b(right)}$$

then the pressure on the top right must be

$$P_{t(right)} = P_{b(right)} - \rho g h_{right}$$

We could write this as

$$\begin{aligned} P_{t(right)} &= P_{b(left)} - \rho g h_{right} \\ &= P_{t(left)} + \rho g h_{left} - \rho g h_{right} \end{aligned}$$

Now suppose I increase the pressure at the top at the left by pushing down (increasing the force) on the piston on the left side. Then

$$P_{t(left)} \rightarrow P_{t(left)} + \Delta P \tag{20.4}$$

The pressure at the bottom of the left side will change too

$$\begin{aligned} P_{b(left)} &= \left(P_{t(left)} + \Delta P\right) + \rho g h \\ &= \Delta P + P_{t(left)} + \rho g h \end{aligned}$$

And since the pressure at the bottom left changes, and the pressure at the bottom right must be the same as the pressure at the bottom left, the pressure must change all along the whole bottom.

$$\begin{aligned} P_{b(right)} &= P_{b(left)} \\ &= P_{t(left)} + \Delta P + \rho g h_{left} \end{aligned}$$

and from what we did before we can see that the pressure at the top right must be

$$\begin{aligned}
P_{t(right)} &= P_{b(right)} - \rho g h_{right} \\
&= (P_{t(left)} + \Delta P + \rho g h_{left}) - \rho g h_{right} \\
&= (P_{t(left)} + \Delta P + \rho g h_{left}) - \rho g h_{right} \\
&= (P_{t(left)} + \rho g h_{left} - \rho g h_{right}) + \Delta P
\end{aligned}$$

which is the pressure we had on the right side before the change plus the exact same change in pressure that we applied on the left side! We can see that we have changed the pressure at the top right as well, and by the same amount. This is an amazingly simple result! If I change the pressure at one point in the fluid, I automatically change the pressure in the rest of the fluid. This result is so useful it has a name.

Pascal's Law: **a change in the pressure applied to a fluid is transmitted undiminished to every point of the fluid and to the walls of the container.**

There is an assumption that we have made in stating Pascal's law. We have assumed that the fluid is incompressible. Pascal's law won't work for air, because we can compress air. But it will work for oil or water because these fluids are (mostly) incompressible.

So now there is a pressure change at piston 2. Since we changed the pressure P_1 from P_t to $P_t + \Delta P$, and we know that this pressure change will be transmitted to all parts of the fluid, so we can tell that at piston 2 we will change the pressure from it's original pressure, P_{2i} to $P_{2i} + \Delta P$.

The jack works because the areas of the two pistons are different. A_1 is small, and A_2 is large. Then an applied force

$$F_1 = \Delta P A_1$$

will cause a force

$$F_2 = \Delta P A_2$$

on the second piston Because $A_2 > A_1$ then

$$F_2 > F_1$$

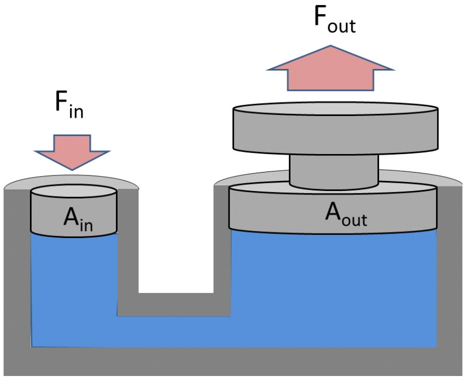
In many cases we can sum up pressure over an area because the pressure will be the same at every point within the area. Our hydraulic jack is one of these cases. In our next lecture we will use Pascal's principle and our new pressure as a function of depth equation in some important problems.

21 Using and Measuring Pressure and Buoyancy

Fundamental Concepts

1. Details of a Hydraulic Press (Pascal's Law)
2. Barometers
3. Manometers
4. Buoyancy

Hydraulic Press



Let's start were we left off last lecture, with a hydraulic jack. We usually have a constant force acting on the jack area.

$$F = PA \tag{21.1}$$

To begin with we usually have atmospheric pressure pushing on the jack piston,

$$F_{atm} = P_{atm}A_{in} \tag{21.2}$$

but we wish to add a force F_{in} to this. So on the input side, we have, using Newton's second law

$$\Sigma F = F_{atm} + F_{in} = P_{atm}A_{in} + \Delta P_{in}A_{in} \tag{21.3}$$

where

$$F_{in} = \Delta P_{in}A_{in} \tag{21.4}$$

is due to our push on the jack input. We will ignore the atmospheric pressure for now, since we only care about a change in pressure for this problem. By changing the pressure on the input side we have changed the pressure by ΔP_{in}. On the output side (the lifting side) we still have

$$F = PA \tag{21.5}$$

but we expect our ΔP_{in} to be transmitted throughout the entire fluid. Then we can just call it ΔP (no in). So the output will have an amount of force added to it. On this side

$$F_{out} = \Delta P_{out}A_{out} = \Delta P A_{out} \tag{21.6}$$

since ΔP in both equations is the same when the two sides are at the same elevation, then

$$\frac{F_{out}}{A_{out}} = \frac{F_{in}}{A_{in}} \tag{21.7}$$

So how does your car's hydraulic jack work?

$$F_{out} = F_{in}\frac{A_{out}}{A_{in}} \tag{21.8}$$

If $A_{out} > A_{in}$ a much smaller F_{in} can produce a large F_{out} (you already knew that, didn't you!).

It is important to note that we have assumed our jack fluid is not compressible. So the volume of fluid leaving the cylinder at the input side must be the same as the volume of fluid entering the output side. Since $A_{out} > A_{in,}$ it is clear that the output piston will travel a much smaller distance than the input piston. This is why you have to pump quite a lot on the input side of your jack to move a car a relatively small distance.

Pressure Measurements

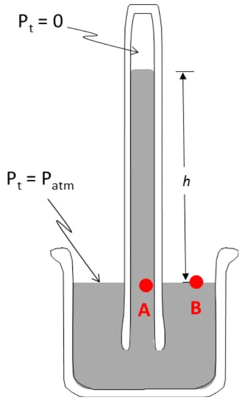
Now that we understand pressure and how it changes in fluids, we can study several pressure measurement devices.

The Barometer

Bring a Barometer

Test Tube Barometer

The barometer is a common pressure measuring device. It consists of



1. a long tube closed at one end
2. a dish
3. Mercury or another fluid

The dish and the tube are filled with the fluid. The tube is inverted in the dish. The pressure at the top of the tube is essentially zero. This is because the weight of the fluid pulls the mass of fluid downward. If the fluid is mercury, it is massive enough to leave a vacuum behind.

The pressure at point A and point B must be the same (or fluid would flow until $P_A = P_B$).

It would be useful to find out how high up the mercury (or other fluid) will stay

$$P_b = P_t + \rho g h$$

At the top we have zero pressure, so $P_t = 0$. Note, this is not atmospheric pressure, **it is vacuum**. Then for this special case

$$P_b = 0 + \rho g h \tag{21.9}$$

And we can solve for the height of the fluid

$$h = \frac{P_b}{\rho g} \tag{21.10}$$

Let's consider what P_b must be. The air outside the barometer pushes down on the fluid. When we first construct a barometer, we fill the tall tube with fluid and then turn it upside down in the dish. The fluid will fall until the force due to air pressure matches the weight of the column of fluid. Then the pressure at B must be atmospheric pressure. Since the pressure at point A equals the pressure at point B, we can use this as a convenient P_b. So

$$P_b = P_{atm}$$

and

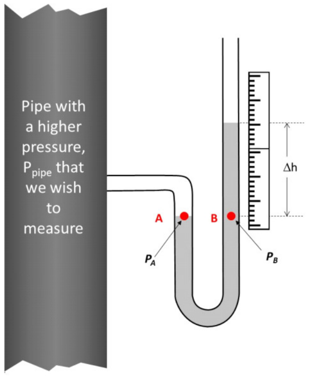
$$h = \frac{P_{atm}}{\rho g} \tag{21.11}$$

Thus as the atmospheric air pressure changes, the height of the mercury changes. We often hear of atmospheric pressure given in mm of Hg. This is why. Question 123.5.1

Manometer

Question 123.5.2

The manometer finds an unknown pressure



The atmospheric pressure P_{atm} is applied on one end of the tube. In our example it is on the right side, where the tube is open at the top. The pressure to be tested is applied to the other side, in our case, the left side. The pressure at point A must equal the pressure at point B. Why? (think of their heights and $\rho g h$). The pressure at point A is the pressure to be measured. Again

$$P_A = P_t + \rho g h \tag{21.12}$$

where we take P_A to be the test pressure, and solve for the height h.

Buoyant Forces, Archimedes' Principle

Question 123.5.3

Let's investigate the net force on our block of water that we have used before. We found the net force must be zero, or it would be accelerating.

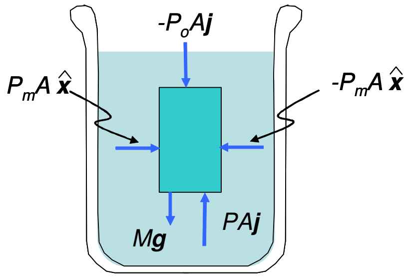
$$\begin{aligned}\Sigma F_x &= ma_x = 0 = P_m A_{side} - P_m A_{side} \\ \Sigma F_y &= ma_y = 0 = -P_{atm} A_{top} + P_{bottom} A_{bottom} - W\end{aligned}$$

But suppose I ignore the weight of the parcel of water, $W = mg$. Then there would be a net force due to all other forces in the $\hat{\jmath}$ direction. That force must be matching the downward force due to gravity.

$$W = P_{bottom} A_{bottom} - P_{atm} A_{top}$$

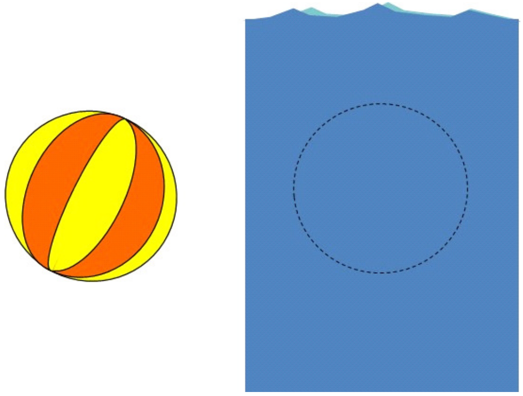
We could give this net force due to all the water pressure forces a symbol, say "B."

$$W = P_{bottom} A_{bottom} - P_{atm} A_{top} = B$$



Then B would be equal to the weight of the parcel of water!

Now take a beach ball sized parcel (piece) of water



The parcel is in equilibrium just as before. We know there is a force on the parcel of water due to gravity. It is downward. Just as before, the ball-shaped parcel of water is not sinking so again there is no net force. Then if we add up all the forces due to the water pressure, only, we must have a net pressure force that is upward to balance the downward force due to gravity. Again this is what we called "B." And again B is equal to the weight of the parcel of water.

The idea of a net force due to pressure due to a fluid is so useful we give it its own name We will call the net force due to pressure from the fluid the *Buoyant force*.

Note that this is **not** the net force, it is just the sum of the forces due to the water pressure. It does not include gravity or any tension or spring or any other forces. It just contains the pressure forces due to the fluid.

Note also that the buoyant force is not some new kind of force. It is a name we give to the sum of the pressure forces due acting on an object in the fluid. Like we call a group of soldiers a battalion or a group of cows a heard, the sum of the group of pressure forces due to a fluid is given the name "buoyant force." But herds and battalions are not new things, they are groups of cows and soldiers. A Buoyant force is not a new force, it is the vector sum of a group of pressure forces.

Because the parcel is not sinking, we can determine that in this special case

$$B = W_{\text{fluid parcel}} \qquad \text{Floating} \tag{21.13}$$

for this situation that we call floating (these are magnitudes, are the directions the same?). The term on the right is the weight of the water or fluid.

Suppose we replace this amount of water with the beach ball. The weight of the beach ball is different than the weight of the water parcel. But will any of the pressure forces be different? The two volumes are exactly the same!

It turns out that the buoyant force will be exactly the same for the beach ball as it would for the beach ball-shaped parcel of water. We can calculate the buoyant force by thinking of replacing the actual object we have with an equal volume parcel of water. The buoyant force will be equal to the weight of that equal volume of the fluid. We could say that to replace the water with the beach ball we have *displaced* a beach ball volume's worth of water. The water that was at the location of the beach ball is now somewhere else, so it is displaced. Then the buoyant force would be equal to the weight of the water that was displaced.

$$B = W_{\text{displaced fluid}} \tag{21.14}$$

Of course, the weight of the beach ball is far less than the weight of the water displaced

$$W_{ball} < W_{\text{displaced fluid}}$$

and this is why the beach ball accelerates upward.

It turns out that this idea of the buoyant force being equal to the weight of an equal volume of fluid is general! This concept of the Buoyant force being equal to the weight of the water that fits inside our volume is called *Archimedes' principle*.

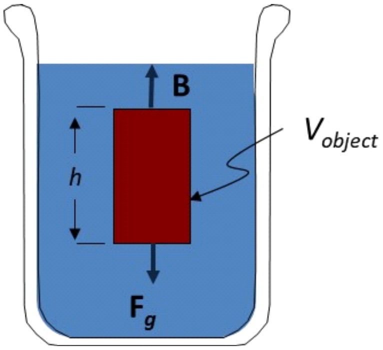
The magnitude of the buoyant force equals the weight of the fluid displaced by the object.

Question 123.5.4

Buoyant Force: Two Cases

We can gain insight into this concept of a buoyant force by considering more floating objects. Let's consider an object that is floating submerged in the fluid (like a fish or submarine) and one that is floating at the surface and partially extends out of the fluid (like a floating log or boat).

Totally submerged object



Consider a solid object, a block of some material, placed in water. The weight of the solid object is Mg.

The magnitude of the buoyant force is equal to the weight of an equal volume of water. If we imagine replacing the block with a block-sized volume of water, then

$$\begin{aligned} B &= m_{fluid} g \\ &= \rho_{fluid} V_{object} g \\ &= \rho_{fluid} g V_{block} \end{aligned}$$

Think Archimedes: The weight of an equal volume of water was displaced by the block. So we have the volume of the block as part of our equation for the buoyant force.

If the object has mass M then we can write the mass of the block as

$$\begin{aligned} M &= \rho_{object} V_{object} \\ &= \rho_{block} V_{block} \end{aligned}$$

and the weight of the block is then

$$\begin{aligned} W &= Mg \\ &= \rho_{object} V_{object} g \\ &= \rho_{block} V_{block} g \end{aligned}$$

The net force is then

$$\begin{aligned} F_{net} &= B - W = \rho_{fluid} g V_{object} - \rho_{object} V_{object} g \\ &= \rho_{fluid} g V_{block} - \rho_{block} V_{block} g \\ &= g V_{block} (\rho_{fluid} - \rho_{block}) \end{aligned}$$

Consider this last equation. If the density of the block is large, larger than the density of the fluid, then $(\rho_{fluid} - \rho_{block})$ is negative, and the force will be negative. The block will accelerate downward. If the density of the block is smaller than the density of the fluid, then $(\rho_{fluid} - \rho_{block})$ will be positive and the block will accelerate upward. If the density of the block is just the same as the density of the fluid, the block will float in place (or at least sink or float upward at a constant rate. A submarine adjusts it's density to dive or to surface.

We did this calculation for our block, but it could have been any object. So in general

$$F_{net} = g V_{object} (\rho_{fluid} - \rho_{object}) \tag{21.15}$$

If the object has a density that is less than the fluid, the object will be accelerated upward. And if the density of the object is greater than the density of the fluid the object will be accelerated downward.

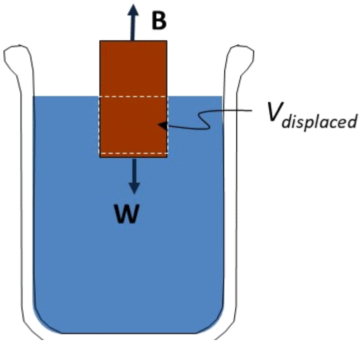
bb in a test tube demo

Note that the motion of a totally submerged object is determined only by the relative density! In a way, this could be confusing because we did not allow our object to change volume as it moved. If we had an object that could change size, then it changes density as it changes size. A balloon full of air released from the bottom of a swimming pool does change volume as it rises. So in that case we would need to know ΔV_{object} and how the density, ρ_{object}, changes as the balloon moves.

Partially submerged object

Question 123.5.5

Now let's consider a partially submerged object



Let's assume $\rho_{object} < \rho_{fluid}$ (why? think of $\rho_{fluid} - \rho_{object}$). We assume static equilibrium, which means we are observing this floating object when it is just sitting still relative to the objects around it, not accelerating.

Since $a = 0$ then

$$\Sigma F = ma = 0$$

and

$$\Sigma F = B - W$$

so for an object floating on the surface

$$B = -W$$

Lets call the volume of the fluid displaced by the object $\not{V}_{fluid}$. Now this is not the same as $\not{V}_{object}$ for this case! Part of the object is sticking out of the fluid, so $\not{V}_{fluid} < \not{V}_{object}$. But, it is still true that the weight of the displaced volume of fluid gives the buoyant force

$$B = \rho_{fluid} g \not{V}_{fluid}$$

Then returning to our force equation

$$\begin{aligned} B &= -W \\ \rho_{fluid} g \not{V}_{fluid} &= -\rho_{object} \not{V}_{object} g \end{aligned}$$

or

$$\frac{\not{V}_{fluid}}{\not{V}_{object}} = \frac{\rho_{object}}{\rho_{fluid}} \tag{21.16}$$

The fraction of the volume of a floating object that is below the fluid surface is equal to the ratio of the density of the object to that of the fluid. Question 123.5.5

In the case of an iceberg, not all of the ice is visible above the surface. The density of ice is about $920\frac{\mathrm{kg}}{\mathrm{m}^3}$ and that of sea water is about $1000\frac{\mathrm{kg}}{\mathrm{m}^3}$. So for ice

$$\frac{\not{V}_{fluid}}{\not{V}_{object}} = \frac{920\frac{\mathrm{kg}}{\mathrm{m}^3}}{1030\frac{\mathrm{kg}}{\mathrm{m}^3}} = 0.893\,2$$

$$\not{V}_{fluid} = 0.89\not{V}_{object}$$

that is the fluid displaced is 89% of the ice. Just about 10% of the ice sticks up out of the water. I

Iceberg with both beautiful blue-green submerged portion and a reflection of the surface ice and snow. Approximately 90 per cent of the iceberg is submerged. Antarctica, Palmer Peninsula, Northern area. (Dr. Mike Goebel, NOAA NMFS SWFSC, NOAA NMFS SWFSC Antarctic Marine Living Resources (AMLR) Program.)

22

Conservation of Energy for Fluid Flow

So far, we have only dealt with fluids in equilibrium. The topic of fluid dynamics is a complicated mathematical field that requires differential equations to do with any exactness. But with some simplifying assumptions, we can get a feel for how fluids flow.

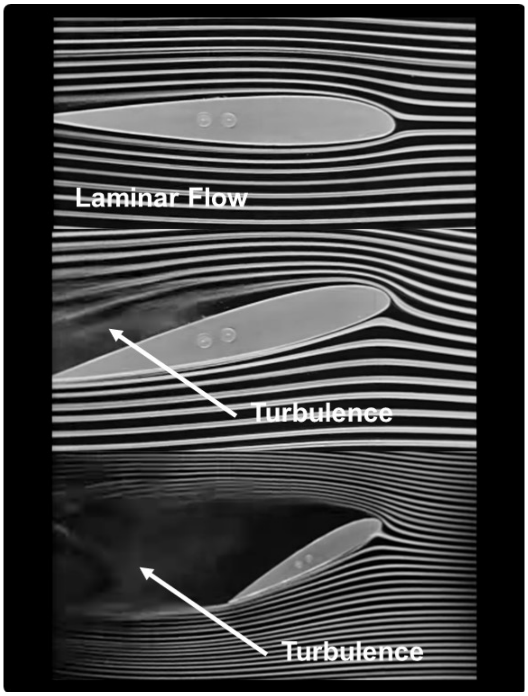
Fundamental Concepts

1. Continuity
2. Bernoulli's Equation

Moving Fluids

To start with, let's make some definitions.

New Definitions:



Question 123.6.1

Question 123.6.2

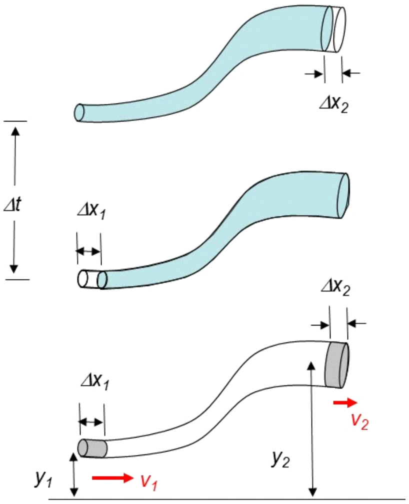
Question 123.6.3

1. Laminar: Steady flow, if each particle of the fluid follows a smooth path, such that the paths of the different particles never cross each other.
2. Turbulent: non-laminar flow characterized by whirlpool or eddy regions
3. Viscosity: term describing the internal friction of a fluid (Honey is more viscus than water)
4. Irrotational: flow having no angular momentum
5. Streamline: a line indicating the path taken by a particle under steady flow

Sadly, we will limit our study to laminar, nonviscus, irrotational, incompressible fluids. This is a little bit like saying we will only study frictionless surfaces in PH121! But to go beyond non-viscus laminar flow we need the math of partial differential equations, which is not a prereq for this class. So we will have to content ourselves with this restriction. For those of you who are interested (or are ME majors) you can take our junior level fluid dynamic course taught by the Mechanical Engineering Department.

Equation of continuity

Question 123.6.4



Let's consider a pipe that changes in size and position. We allow an ideal (non-compressable) fluid to flow through the pipe in laminar flow. The segment of fluid is the shaded part in the top part of figure **??**. In a time Δt the fluid at the left hand side moves a distance

$$\Delta x_1 = v_1 \Delta t$$

If A_1 is the area of the pipe at the left hand side (LHS) then the mass contained in the shaded region (bottom left part of figure **??**) is

$$m_1 = \rho A_1 \Delta x_1 = \rho A_1 v_1 \Delta t$$

where ρ is the density of the fluid. Likewise, for the RHS region in the bottom of figure **??**

$$m_2 = \rho A_2 \Delta x_2 = \rho A_2 v_2 \Delta t$$

Unless we create, destroy, or pool mass, the mass that crosses A_2 must equal the mass that crosses A_1

$$\begin{aligned} m_1 &= m_2 \\ \rho A_1 v_1 \Delta t &= \rho A_2 v_2 \Delta t \end{aligned}$$

or

$$A_1 v_1 = A_2 v_2 \tag{22.1}$$

This is the *equation of continuity for fluids* (remember that we assumed ideal fluids).

Since we assumed that no mass was created, destroyed, or pooled as the fluid flowed, this is what we might call "conservation of mass." And as usual, when something is conserved in a problem, we can use it to solve problems. With this equation, knowing the diameter of a pipe and how it changes tells us about the speed of the fluid in the pipe and how it changes. Of course, there could be situations where mass is not conserved (say, a leaky pipe). So conservation of mass is an assumption that we have to check. But for now let's assume that the equation of continuity holds for our problems.

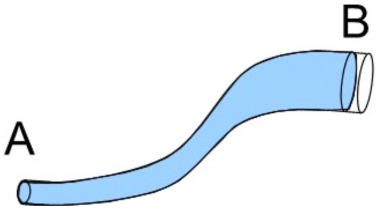
Bernoulli's Equation

It would be great if we could find the pressure as well as the speed of the fluid inside our non-uniform pipe. We would need an equation that included both pressure and speed. Let's try to find such and equation!

In PH121 we developed to ways of looking at motion problems. One was the force picture where we had to deal with the vector nature of forces and motion. The other was the energy picture where we could solve problems without considering directions (but we gave up being able to find directions). Usually the energy picture made solving problems easier. And except for the flow direction, we don't really want to know the direction of the motion of each molecule in our fluid, so giving up direction doesn't seem too bad. So, the work energy theorem is something we know from PH121. Let's start there,

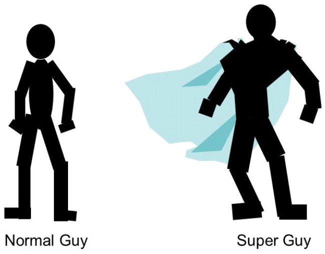
$$W = \Delta K + \Delta U$$

We still need to use forces to find the work done on the parcel of water, but if we choose our axes carefully, we can keep the work equation in one dimension.

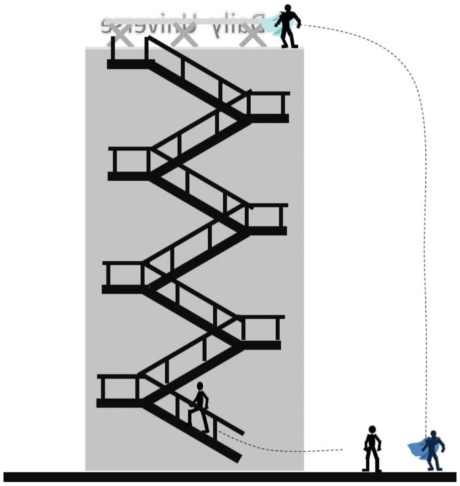


Work done in Δt

Let's review a little of what we learned in PH121. Suppose we have two guys, Normal Guy and Super Guy.



And suppose they want to get to the top of a tall building. Normal guy has to take the stairs, but Super guy can jump right up to the top.

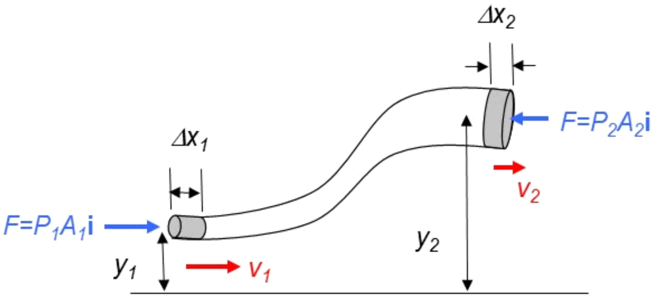


Who has done more work?

We know the change in potential energy is the same for both. And we know that

$$W = -\Delta U$$

so the amount of work is the same for both cases! Now let's return to our fluid flow.



On the left hand side (LHS) of our pipe we have a force due to pressure from the fluid in the rest of the LHS of the pipe. That fluid has molecules that will bang in to our small marked parcel of fluid. The force from the LHS fluid in the rest of the pipe will be

$$F_1 = P_1 A_1 \widehat{\mathbf{i}} \tag{22.2}$$

On the right hand side (RHS) there is also more fluid in the rest of the RHS of the pipe. Our fluid will bang into that fluid, and since the fluid is non-compressible, the rest of the fluid in the RHS will push back. The force will be

$$F_2 = -P_2 A_2 \widehat{\mathbf{i}} \tag{22.3}$$

From PH121 we remember that works is

$$w = \int \mathbf{F} \cdot \mathbf{dx}$$

and since our force is not changing this would be simply

$$w = \mathbf{F} \cdot \boldsymbol{\Delta} \mathbf{x} \tag{22.4}$$

We need to find the work done in moving our particular parcel of fluid from the left hand side to the right hand side. As the parcel moves in the $\hat{\imath}$ direction the amount of work will slowly change (like Normal Guy going up the stairs) but really the amount of work only depends on the beginning and ending conditions. So we could ignore all the details of moving the parcel and envision the parcel jumping from the left up to the right side of the pipe (like in the Super Guy case).

Then for the LHS

$$W_1 = P_1 A_1 \Delta x_1$$

and for the RHS

$$W_2 = -P_2 A_2 \Delta x_2$$

The total work will be the sum of these two works. But before we add them together, let's make some substitutions that will make our final formula for the total work more meaningful.

Lets define the volume of our fluid segment

$$V\!\!\!- = A_1 \Delta x_1 \tag{22.5}$$

Note that this volume is NOT the volume of the entire segment. It is just the part that is Δx_1 long. Then from the same arguments that lead to the equation of continuity, we see that the volume of the marked part of the fluid must not change as it flows Question 123.6.5

$$V\!\!\!- = A_1 \Delta x_1 = A_2 \Delta x_2$$

Question 123.6.6

The we can write the work equations as

$$W_1 = P_1 V$$

and

$$W_2 = -P_2 \not{V}$$

So finally, the total work done is

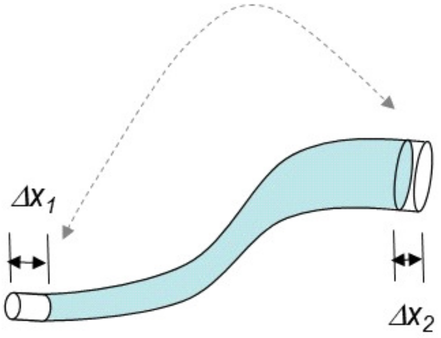
$$\begin{aligned} W &= W_1 + W_2 \qquad (22.6) \\ &= P_1 \not{V} - P_2 \not{V} \\ &= (P_1 - P_2)\not{V} \end{aligned}$$

so, from the work-energy theorem

$$\begin{aligned} W &= \Delta K + \Delta U \\ (P_1 - P_2)\not{V} &= \Delta K + \Delta U \end{aligned}$$

We have an expression for the left hand side of our work-energy theorem! Now for the right hand side $\Delta K + \Delta U$

Kinetic Energy



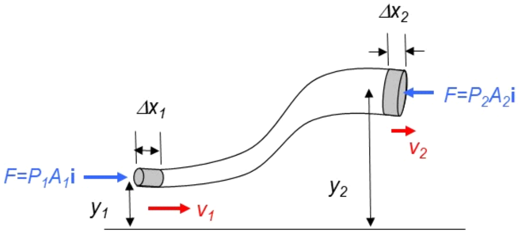
If we think about the energy of the segment, we will realize that after time Δt, it is as though most of the fluid did not move. We can't tell one part of the fluid from another. The shaded region in figure **??** is occupied before and after Δt. We can treat this problem as if we moved fluid from region 1 to region 2 (Super Guy vs. Normal Guy again!) and left the rest of the segment alone!

So, ignoring the rest of the fluid, we can see that if the velocity of our parcel of water changed, then the kinetic energy of the parcel must change as well

$$\Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{22.7}$$

(the mass is the same as we found before).

Potential Energy



Let's look at the change in potential energy for the parcel of fluid. Once again we can consider just the beginning and ending case (think of Super Guy and Normal Guy once again). We work as though we moved a mass $m = m_1 = m_2$ from y_1 to y_2.

$$\Delta U = mgy_2 - mgy_1 \tag{22.8}$$

Since our pipe went upward as the parcel of water flows, we gained potential energy. But if the pipe went downward we would loose potential energy.

Total Work done on the system

Let's now assemble our work-energy theorem. We had found the work so far,

$$(P_1 - P_2)\cancel{V} = \Delta K + \Delta U$$

but now we know ΔK and ΔU We have calculated W, ΔK, and ΔU for our parcel of water, so let's substitute in what we have found.

$$((P_1 - P_2)\cancel{V}) = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + (mgy_2 - mgy_1) \tag{22.9}$$

This equation describes our fluid flow much like the equation of continuity, but now it has pressure involved! We have achieved our goal. But we can make this equation a little easier to understand if we play a trick by dividing by V

$$((P_1 - P_2)) = \left(\frac{1}{2}\frac{m}{\cancel{V}}v_2^2 - \frac{1}{2}\frac{m}{\cancel{V}}v_1^2\right) + \left(\frac{m}{\cancel{V}}gy_2 - \frac{m}{\cancel{V}}gy_1\right) \tag{22.10}$$

and recalling that

$$\rho = \frac{m}{\cancel{V}}$$

then

$$(P_1 - P_2) = \left(\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2\right) + (\rho g y_2 - \rho g y_1)$$

Rearranging gives

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 \tag{22.11}$$

which completes our goal. We have an equation that contains the velocity *and* the pressure of the fluid.

Sometimes you see this written as

$$P + \frac{1}{2}\rho v^2 + \rho g y = \text{constant} \tag{22.12}$$

which really should excite us. This is another conservation equation. The thing that is being conserved is $P + \frac{1}{2}\rho v^2 + \rho g y$. We have not named this quantity, but whatever it is, it is not changing as the fluid flows. Our experience with conservation equations tells us that this new conservation law will help us solve problems. This equation is so useful we give it a name. It is called *Bernoulli's equation* (after the scientist that came up with it).

Question 123.6.7

Question 123.6.8

Applications of Fluid Dynamics

Ping Pong Ball Demo

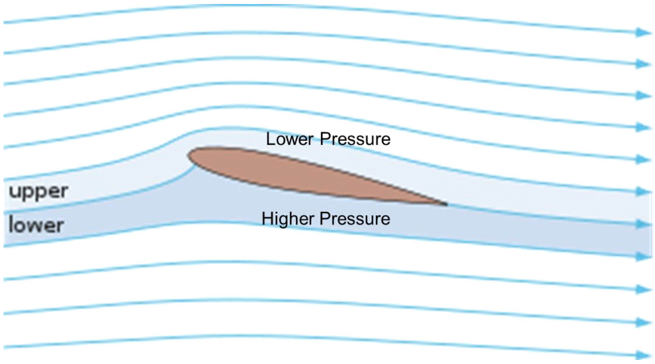
Question 123.6.9

Question 123.6.10

An object moving through a fluid experiences a force due to any effect that causes the fluid to change its direction as it flows past the object.

Some influencing factors:

1. Shape
2. orientation
3. spinning motion
4. Texture
5. Airfoil



Pressure on an airfoil. (Michael Belisle, public domain)

The air that travels over the top part of the wing is deflected (changes direction) as it flows past the wing. There is an upward force (lift) due to this effect.

6. @@@@@@@@@@@@@@@ Not done yet.

23 Elasticity-State Variables

We dealt with liquids, let's briefly think about solids in our preparation for studying thermodynamics.

Fundamental Concepts

1. Elasticity
2. States of Matter
3. Density

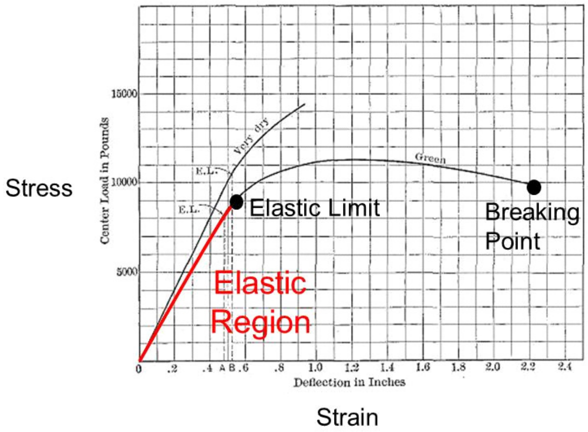
Deformation of Solids

We are working our way toward thermodynamics–the study of how many objects move due to thermal energy. We will need to know a little bit more about solids. Specifically we need to know how solids bend, stretch, and break because all of these things can happen due to thermal energy.

Stretching, bending, and breaking all require forces to be applied to our objects. But in our study of pressure, we know the area involved in applying the force matters. We called a force spread out over an area a pressure. This works for fluids. But it turns out that pressure is only one kind of force spread out over an area. For solids the general form of a spread out force is called a *stress*. Let's start with some definitions.

1. Stress: Force per unit area causing deformation
2. Strain is a measure of the amount of deformation

If we don't allow the stress to get too big, the strain is proportional to the stress. The next figure is a stress vs. strain plot for bending pine beams.



Stress-strain diagrams of two longleaf pine beams. E.L. = elastic limit. The areas of the triangles 0(EL)A and 0(EL)B represent the elastic resilience of the dry and green beams, respectively (Public Domain Image from Samuel J. Record, The Mechanical Properties of Wood Including a Discussion of the Factors Affecting the Mechanical Properties, and Methods of Timber Testing, http://www.gutenberg.org/ebooks/12299)

Clearly at some point, the stress is to great, and a solid pine beam will break. All of this is due to the same type of forces that create normal and tension forces. That is, the bonding of the atoms that make up the solid. A strain is present because the stress (applied spread out force) works against the bonding forces. If the bonds are weak, there is more deformation. At the breaking point the bonds are ripped apart.

Liquids will not experience strains other than pressure, because they can flow. Fluid bonds don't support a particular shape, so we can't reasonably talk about a deformation of a fluid.[17]

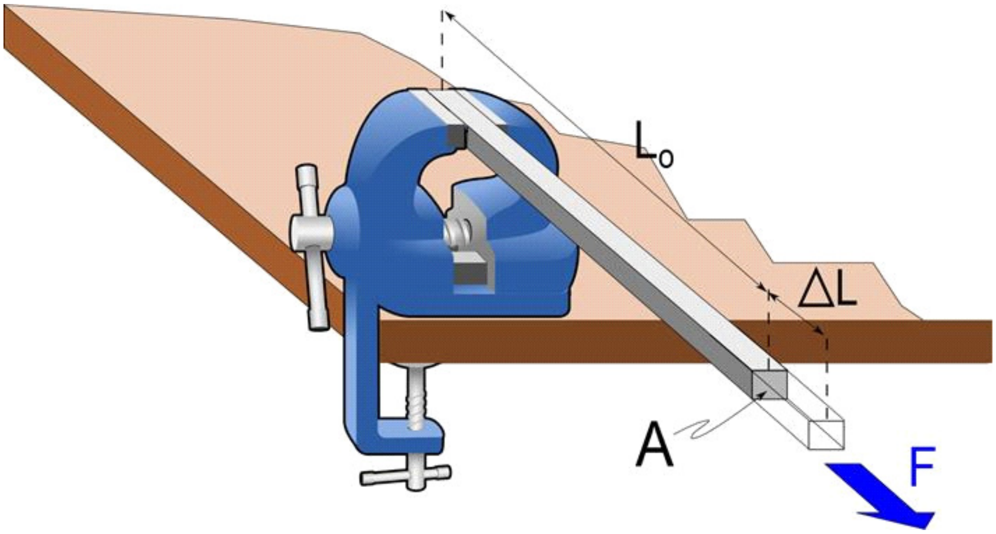
Young's Modulus

A modulus is a constant that describes how a solid can be deformed. Think of it like the spring constant k that tells us how hard it is to stretch or compress a spring.

Suppose we pull on a rod with a force $\overrightarrow{\mathbf{F}}$.

---

[17] There are some fluids, the non-Newtonian fluids, for which this is not true.



The rod will be stressed. We call this *tensile stress* which comes from a pull . This pull will stretch the solid's molecular bonds (like a tension force).

We write the stress as

$$\frac{F}{A} \tag{23.1}$$

which looks like a pressure, but this time we mean that we are pulling on the beam so that the pull is spread over the cross sectional area (marked A in the figure above). That is not much like a pressure!

By pulling on the beam, the beam will stretch. That stretching is a strain. We write the strain as the percent change in shape. In this case, the percent change in length

$$\frac{\Delta L}{L_o} \tag{23.2}$$

If the stress is not too great, then we can write a linear equation that relates the stress to the strain.

$$\frac{F}{A} = Y\frac{\Delta L}{L_o} \tag{23.3}$$

this linear equation only works in the elastic region of a stress vs. strain curve for the particular object. For example, if we pull on a pine beam we get the stress vs. strain curve in figure**??**. Our equation 23.3 only works for the red part marked "Elastic Region" on the curve. But this is just the region that we want to use to build buildings and airplanes and supercollider support structures, etc. We don't want to build buildings with the stress on the boards causing the boards to bend and break! So this equation is useful.

The constant of proportionality, Y, is called *Young's Modulus*. If we write this as

$$F = \frac{YA}{L_o}\Delta L \tag{23.4}$$

$$F = k\Delta L \tag{23.5}$$

it looks very much like Hook's law. Our restoring force or our "pull" is the rod pulling back with a force like $F = -k\left(x - x_o\right)$.

Young's modulus depends on the microscopic properties of the solid (which we will not study in detail, but these are our stretchy/squishy forces between atoms that make normal and tension forces). We will look up Young's modulus for each material in tables like the following:

Material[18,19]	Young's Modulus $\left(\mathrm{N/\,m^2}\right)$
Aluminum*	7×10^{10}
Titanium*	$11. \times 10^{10}$
Steel*	$20. \times 10^{10}$
Carbon Steel*	$21. \times 10^{10}$
Lead†	1.6×10^{10}
Brass†	$10. \times 10^{10}$
Concrete†	2.0×10^{10}
Nylon†	0.5×10^{10}
Bone† (arm or leg)	1.5×10^{10}
Pine Wood† (parallel to grain) (perpendicular to grain)	1.0×10^{10} 0.1×10^{10}

Let's take an example. Suppose we have a steel piano wire. One end of the piano wire is fixed in place, and the other is connected to a peg that can be rotated so the piano wire is tightened. This is what you do when you tune a piano, you turn the peg and tighten the wire. Suppose we tighten the wire so that there will be $980\,\mathrm{N}$ of tension on the wire. The wire is $1.6\,\mathrm{m}$ long and has a diameter of $2\,\mathrm{mm}$. How much will the wire stretch as we tighten the peg? We will need to know that the Young's modulus for steel is $20 \times 10^{10}\,\mathrm{N/\,m^3}$.

---

[18] *Ledbetter, H. M., Physical Properties Data Compilations Relevant to Energy Storage, US National Bureaus of Standards, 1982. Different alloys have different properties, so for any real work see the original tables in the original publication.

[19] †Average values from numbers given in various text books. These numbers should be taken as example values and more exact numbers found for any real work.

Here is a summary of what we know

$$\begin{aligned} T &= 980\,\mathrm{N} \\ L_o &= 1.6\,\mathrm{m} \\ d &= 2\,\mathrm{mm} \\ Y &= 200 \times 10^9\,\mathrm{N/\,m^2} \end{aligned}$$

Let's start with our stress/strain basic equation

$$\frac{F}{A} = Y\frac{\Delta L}{L_o}$$

and solve for ΔL

$$\frac{F}{A}\frac{L_o}{Y} = \Delta L$$

the force is our tension, and the cross sectional area will be

$$A = \pi\frac{d^2}{4}$$

$$\frac{T}{\pi\frac{d^2}{4}}\frac{L_o}{Y} = \Delta L$$

$$\frac{4T}{\pi d^2}\frac{L_o}{Y} = \Delta L$$

then

$$\begin{aligned} \Delta L &= \frac{4\,(980\,\mathrm{N})}{\pi\,(2\,\mathrm{mm})^2}\frac{(1.6\,\mathrm{m})}{(200 \times 10^9\,\mathrm{N/\,m^2})} = \\ &= 2.4955 \times 10^{-3}\,\mathrm{m} \end{aligned}$$

so the wire stretched about a quarter of a centimeter.

Of course, if we pull too hard the material will stretch, and finally break. The stress that can break the bonds holding the material together is called the *ultimate strength*.

Material[20,21]	Tensile Ultimate Strength $\left(\mathrm{N/\,m^2}\right)$
Aluminum*	4.2×10^8
Titanium*	9.9×10^8
Steel*	17.0×10^8
Carbon Steel*	6.5×10^8
Bone† (arm or leg)	1.3×10^6

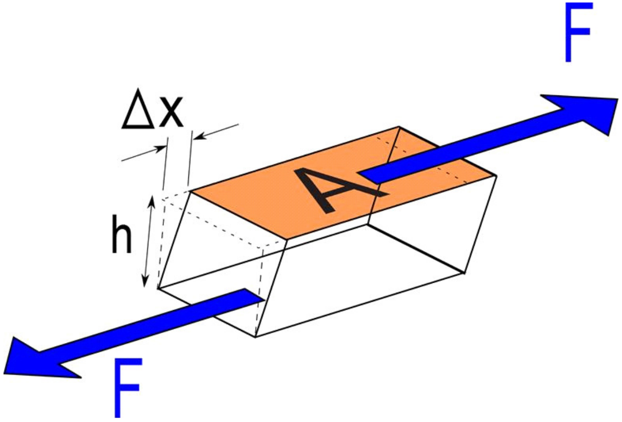
---

[20] *Ledbetter, H. M., Physical Properties Data Compilations Relevant to Energy Storage, US National Bureaus of Standards, 1982. Different alloys have different properties, so for any real work see the original tables in the original publication.

[21] †Average values from numbers given in various text books. These numbers should be taken as example values and more exact numbers found for any real work.

Shear Modulus

A shear stress is also a force acting on an area, but it is a force parallel to the surface of the object. This is like pushing on the top of Jello™. The Jello will deform.



Again the stress is

$$\frac{F}{A} \tag{23.6}$$

But the area is very different that the area we used in tensile stress. Now we will use the area of the top of the Jello. The strain is the percentage change in the x position of the top surface, relative to the height of the Jello

$$\frac{\Delta x}{h} \tag{23.7}$$

Or, in other words, Δx is how far the top surface moves, and h is the height of the object. The strain is the ratio of these two quantities.

The stress and strain are related by

$$\frac{F}{A} = S\frac{\Delta x}{h} \tag{23.8}$$

where S is the shear modulus. Of course, this equation is also only good if we are in the linear region of the stress vs. strain curve. At some point you will just sheer off the top of the jello[22]. But so long as we are in the linear region, our equation works. Again, to find S for a particular material, we should look it in a table.

---

[22] For example, if you are scooping off the layer of Jello that has shredded carrots in it.

Material[23]	Sheer Modulus $\left(\mathrm{N/\,m^2}\right)$
Steel§	80×10^9
Cast Iron§	40×10^9
Brass§	135×10^9
Aluminum§	25×10^9
Bone§ (arm or leg)	80×10^9
Jello*	0 to 8×10^4

Bulk Modulus

This one is harder. It tells us how much a solid can be squashed or, compressed. Let's define

$$\Delta P = \frac{\Delta F}{A} \tag{23.9}$$

where now A is the outer surface area of the solid. If we were dealing with a fluid, this would be pressure. Often fluids are very hard to compress (we assumed they were not at all compressible). Many solids are easy to compress. Think of Styrofoam or any foam rubber. For solids $\Delta F/A$ where the force is all around the solid and the area is the surface area is a stress. The strain is the percentage change in volume (change amount over the original volume)

$$\frac{\Delta V}{V} \tag{23.10}$$

and the relationship is

$$\Delta P = -B\frac{\Delta V}{V} \tag{23.11}$$

B is the *bulk modulus*, which again, we look up in tables.

Material[24]	Bulk Modulus $\left(\mathrm{N/\,m^2}\right)$
Steel‡	140×10^9
Cast Iron‡	90×10^9
Brass‡	80×10^9
Aluminum‡	70×10^9
Water‡	2×10^9
Ethyl Alcohol‡	1×10^9
Mercury‡	2.5×10^9
Air‡	1.01×10^5

The minus sign may be surprising. But in our definition of bulk modulus, we are

---

[23] §Average values from numbers given in various text books. These numbers should be taken as example values and more exact numbers found for any real work. *Depends on how much water and what temperatre the Jello and what agents you put in it for strength. See for example A. Bigi *et al.* Biomatrials 22 (2001) 763-768.

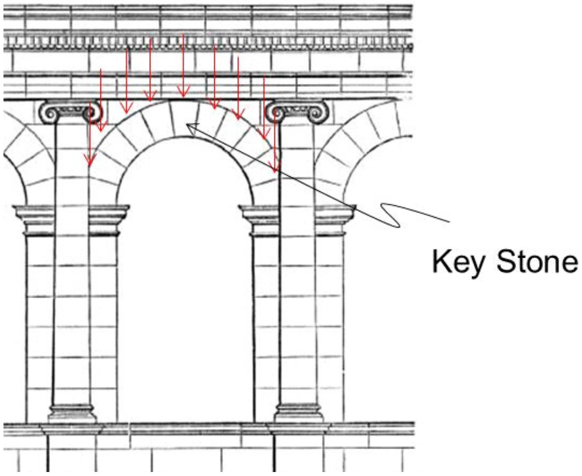
[24] ‡Average values from numbers given in various text books. These numbers should be taken as example values and more exact numbers found for any real work.

thinking of compression. So usually $\Delta \rlap{/}V$ is negative. The minus sigh means that for compression this formula gives positive values. Also notice that there is a ΔP or ΔF in our formula. Usually we start compressing a solid while it is experiencing air pressure. So we don't start from zero stress. We change the force from the force due to air pressure to some larger compressive force.

Ultimate Strength of Materials

Arch Demo

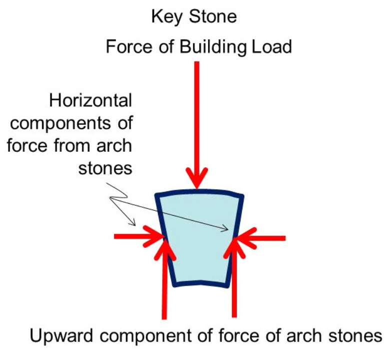
Many materials are stronger under compression than under tension. That is, it is harder to squash them than to pull them apart.



Thomas Roger Smith John Slate, Architecture Classic and Early Christian, London, William Clows and Sons, Limited, London, 1882)

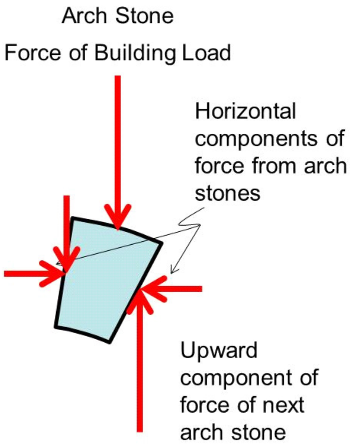
Arches are made based on this principle. By placing bocks in an arch, the weight of the wall presses down on the arch. The stones of the arch are compressed into each other. The force is extended into the walls next to the archway.

The block at the top holds the arch together and distributes the weight from above to the other blocks in the arch.

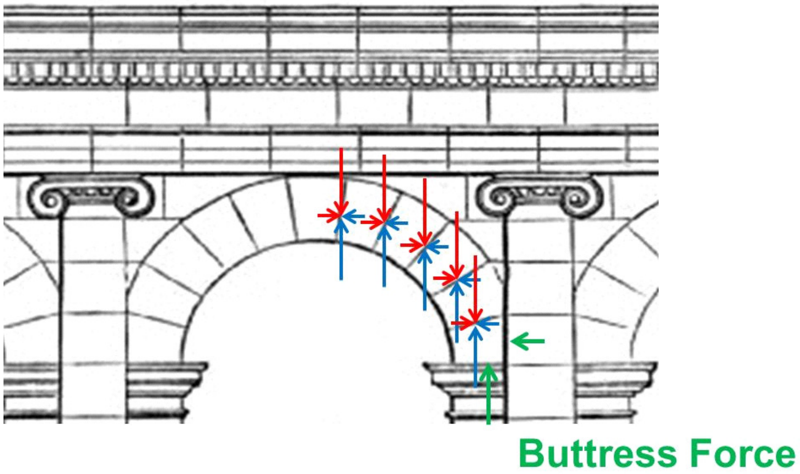


It is called the *key stone*. This is what Joseph Smith was talking about when he said the *Book of Mormon* is the "keystone of our religion." It holds the rest together! In the figure, the vertical and horizontal components of the forces due to the building load and due to the surrounding arch stones are shown. These must balance if the arch is to be stable.

If we look at the blocks next to the key stone we see that the blocks to the side and below must support this block.



The key stone pushes with force components downward and to the right. The next block must push up and to the left to balance the force due to the keystone and the building load on this block.

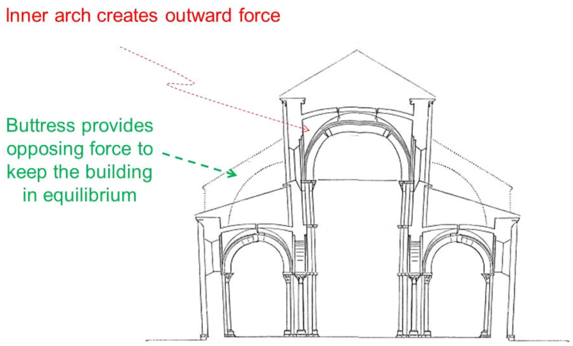


This distribution of force along the facing areas of the arch stones continues all the way through the arch. In practice, it looks like this

But notice that we must provide a buttressing force from the rest of the structure on the bottom most stone. If you don't have wall to provide this force, you might be in trouble. Suppose you want to build an arched ceiling for a cathedral.

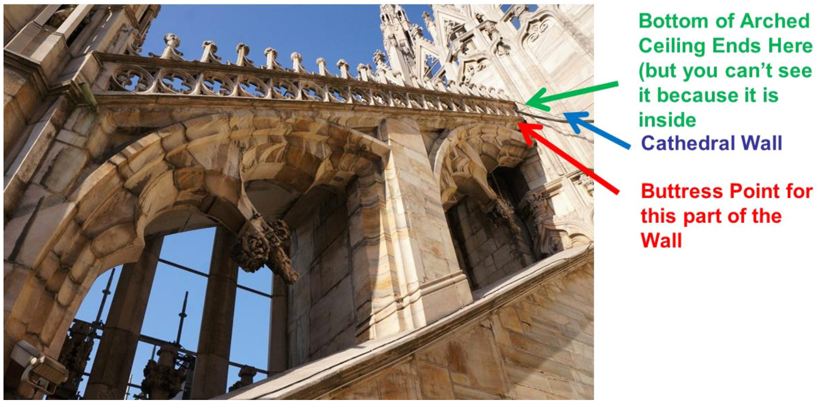
Arched Ceiling of the Cathedral of Milan

You would need to build a large solid mass outside the building to provide the buttressing force. But this might look bulky and ugly. So you could provide the buttressing force with a series of “flying buttresses” or structures that can provide the buttressing force at points along the length of the building. The buttress is there to provide an opposing force on the outside of the wall where there would be no next arch.



Flying Buttress (Dictionary of French Architecture from 11th to 16th Century (1856) by Eugène Viollet-le-Duc (1814-1879), Image public domain)

Here is an example of a buttress from the Duomo in Milan, Italy.



“Flying Buttress” Supporting the Arched Ceiling of the Cathedral of Milan

At some point, though, if we put too much force on an object, it will break. Our normal (atom squashing) or tension (bond stretching) forces can’t push back or stretch enough and the atoms are torn from each other. This is described by the *ultimate strength* parameters given in the next table. Here is a table for ultimate strength for some materials.

Material[25]	Ultimate Tensile strength $\left(\mathrm{N/\,m^2}\right)$	Ultimate Sheer Strength $\left(\mathrm{N/\,m^2}\right)$	Ultimate Compressive Strength $\left(\mathrm{N/\,m^2}\right)$
Steel†	500×10^6	250×10^6	500×10^6
Cast Iron†	170×10^6	170×10^6	550×10^6
Brass†	250×10^6	200×10^6	250×10^6
Aluminum†	200×10^6	200×10^6	200×10^6
Nylon†	500×10^6		
Bone† (arm or leg)	130×10^6		170×10^6

Suppose we return to our piano wire. If we keep turning the peg, the wire will break (which is quite dangerous!). How much tension will break the wire? We need the ultimate tensile strength for steel, $T_u = 500 \times 10^6\,\mathrm{N/\,m^2}$. The ultimate stress that will break the wire is given by

$$\frac{F}{A} = T_u$$

Think about what this means, we would expect that a wire made of steel would break before a large beam made of steel would break. The force is not enough to describe the motion, we also need to know how spread out that force is over the cross sectional area

---

[25] †Average values from numbers given in various text books. These numbers should be taken as example values and more exact numbers found for any real work.

of the wire or beam. So to find the tension that will break the wire we need

$$F = T_u A$$

$$T = T_u \pi \frac{d^2}{4}$$

$$\begin{aligned} T &= \left(500 \times 10^6 \,\mathrm{N/m^2}\right) \pi \frac{(2\,\mathrm{mm})^2}{4} \\ &= 1570.8\,\mathrm{N} \\ &\approx 1600\,\mathrm{N} \end{aligned}$$

State Variables

Question 123.7.2

We learned before about solids, liquids, and gases. We learned that the degree of bondedness made the difference between these states.

We are starting a new topic, and we will find that physics did not develop as a cohesive whole. The new topic is called *Thermodynamics* and it developed independently of the theories of oscillation and waves and optics. Thus the language we use will be different, and sometimes old words will need new meanings.

The states of matter are often called *phases* in thermodynamics. This use of the word "phase" has nothing to do with the phase of an oscillator or wave. It is just the condition of being solid, liquid or gas.

To change from solid to liquid, or from liquid to gas, is called a *phase change*.

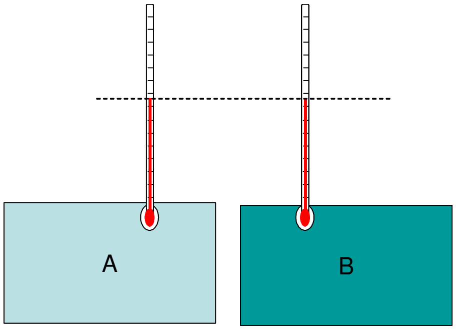
To describe that state of a sample of matter, we use a set of variables like mass, volume, pressure, density, and energy. We have used these throughout PH 121. Thermodynamics has it's own set of descriptive variables to add to these: moles, number density, and Temperature. These variables, along with our original set, are called *state variables* and together, they describe the conditions of a particular pice of matter.

Temperature and the Zeroth Law of Thermodynamics

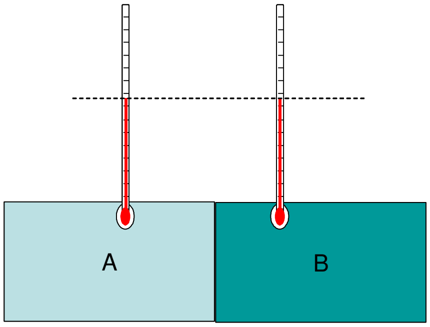
Question 123.7.3

Question 123.7.4

We need a basis for talking about thermal systems. We sort of have an intuition for temperature, so let's use our intuition for now.



In the figure we have two objects. I measure both objects temperature. I find that they are the same. Suppose I put the two objects in "contact."



Would you expect there to be energy transferred between the two objects? For most of us our intuition says "no." Given this intuition, lets make some more formal definitions:

- Thermal Contact: Two objects are in thermal contact if they can exchange energy due to a temperature difference
- Thermal Equilibrium : A state where two objects would not exchange energy by heat or electromagnetic radiation if they were placed in thermal contact
- Temperature: The property that determines if an object is in thermal equilibrium with other objects (two objects in thermal equilibrium have the same temperature)

Given all this, we can state a law of thermal dynamics: **If objects A and B are separately in thermal equilibrium with a third object C, then A and B are in thermal equilibrium with each other.**

Again this fits our intuition, if we have two containers of water that are both at the same temperature as a third container of water, we expect they all have the same temperature.

In a sense, this seems obvious, so obvious that it was always assumed without statement. But in retrospect, it is a fundamental concept that deserves it’s place as one of the laws of thermodynamics. By the time this was acknowledged, the names “first law” and “second law” were taken, so this became the *Zeroth law of Thermodynamics.*

24 Moles, Temperature, and Phase Changes

Our study of the motion of many things naturally starts with atoms. Our world is built of atoms, and there are a lot of them. So let's think about atoms (and leave collections of stars and planets in galaxies, say, for a later course)

Fundamental Concepts

1. Number Density
2. Atomic Mass
3. Atomic Mass Number
4. The Mole
5. Temperature
6. Kelvin Temperature Scale
7. Phase Changes and Phase Diagrams

Numbers of Atoms and Number Densities

Atoms are small. They are really incredibly small. A typical atom measures a few Angstroms across. An angstrom is $1 \times 10^{-10}\,\mathrm{m}$. That is unbelievably small. Let's try to get an understanding of this scale. Picture a meter stick. Then divide it up into 10000000000 equal units. That is how small an atoms is. Or, consider scaling our meter stick to be the size of the continental US, that is about $4.0877 \times 10^{6}\,\mathrm{m}$. Then for our US sized meter stick, on this scale an atom would still only be about half a millimeter across! About the size of a cake sprinkle.

So it is no surprise that in a typical sample of a solid we can have something like 10^{25}

atoms. We use the capital letter N to mean the number of atoms in a sample of matter.

Because these numbers are so large, it is sometimes convenient to talk about how many atoms there are in a small volume of a substance. This is often less than in the whole sample. We can say, for example, that there are 1000 atoms per cubic micrometer of material. We call such a ratio of number of atoms per volume a *number density.* In the SI system, the units for the number of atoms per unit volume are $1/\,\mathrm{m}^3$. So number densities can still be quite large! Number densities are more useful for samples where the density is uniform, but we can talk about substances who's number density changes. Unfortunately you will often see a small n used to mean number density. This over use of n is confusing, so sometimes we will write number density as $N/\cancel{V}$ so it is obvious what we mean.

Atomic Mass

You probably ran into atomic mass in High School. The idea of atomic mass is that each nucleon (proton or neutron) has about the same mass, and the electrons don't have much mass at all, so we can sort of count nucleons and say that an atom has a mass in units of nucleon masses. Carbon has 12 nucleons (six protons and six neutrons) so it would have a mass of $12\,\mathrm{u}$. The unit u is the atomic mas unit, the mass of one nucleon. Because of the electrons (and other effects) this is not an exact approach, and it is not good enough if we attempt to do nuclear physics. But it is close enough for us for now. So we can say that hydrogen $\left({}^1H\right)$ has an atomic mass of $1\,\mathrm{u}$. If we have a molecule, we can just add the atomic masses of each atom in the molecule together to find the molecule's mass. Note that to be more accurate, we can use a periodic table and find the atomic mass, or, if we need many digits of accuracy, we can go to a table of the nucleotides. We can convert these masses to SI units using

$$1\,\mathrm{u} = 1.66 \times 10^{-27}\,\mathrm{kg}$$

Moles

Chances are that you are familiar with the concept of a mole as well. But experience has shown me that the mole will be new to some people. So let's start with something we know, a dozen.

What is a dozen? Well, it is 12 of something. We could have a dozen doughnuts or a dozen eggs. or a dozen large white pickup trucks. It is just an amount. It is 12 of whatever we are counting.

A mole is like a dozen, but it is a BIG dozen. Is is 6.02×10^{23} of something. I could have 6.02×10^{23} doughnuts, or 6.02×10^{23} eggs. I would need quite a few hungry friends to eat it all. But a mole is just a number of something. The mole is not a very practical amount for everyday objects like doughtnuts or trucks. But think of the large number of atoms in a sample of matter. For this situation, a mole seems like a reasonable number to use.

Like we say I want to buy a dozen eggs, we could say we want to buy a mole of hydrogen atoms. Since atoms are so small, packaging them in big numbers makes sense.

This number, 6.02×10^{23}, is called *Avogadro's number*

$$N_A = 6.02 \times 10^{23}$$

To find the number of dozens of a something, we take the number of items and divide by 12.

$$n_{dozen} = \frac{N}{12}$$

To find the number of moles of a something, we take the number of items and divide by Avogadro's number

$$n_{moles} = \frac{N}{N_A}$$

usually we will use small n to be the number of moles of something. This could be a problem because n can also mean number density. So we will have to be careful to state what we mean with our variables.

Molar mass

We could give a mass for a dozen eggs. If each egg has the same mass (more or less) then the dozenal mass would be the mass of a dozen eggs and it would be twelve times the mass of one egg. The dozenal mass of a dozen trucks would be larger than the dozenal mass of eggs because the mass of a single truck is larger than the mass of an egg, and the mass of a dozen trucks is 12 times the mass of one truck.

We can also define a molar mass, which would be the mass of 6.02×10^{23} of something. Usually, we choose the units for this molar mass to be the mass *in grams* of a mole of something.

The molar mass is equal to the numerical value of the atomic or molecular mass. for example, the molar mass of He, with $m = 4\,\mathrm{u}$ is $M = 4\,\mathrm{g/\,mol}$. The molar mass of diatomic nitrogen would be $28\,\mathrm{g/\,mol}$ because each nitrogen atom has an atomic mass of $m = 14\,\mathrm{u}$ and we add the two atomic masses together for the molecular mass. With this definition of molar mass we can see that

$$n_{moles} = \frac{M\text{ (in grams)}}{M_{mol}}$$

Thermometers: Measuring Temperature

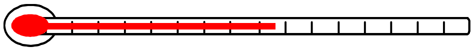
Question 123.8.10.1

We will find that several material properties change with temperature

1. volume of a liquid
2. the dimensions of a solid
3. the pressure of a gas at constant volume
4. the volume of a gas at constant pressure
5. The electric resistance of a conductor
6. The color of an object

We can use any of these to make a temperature measuring device.

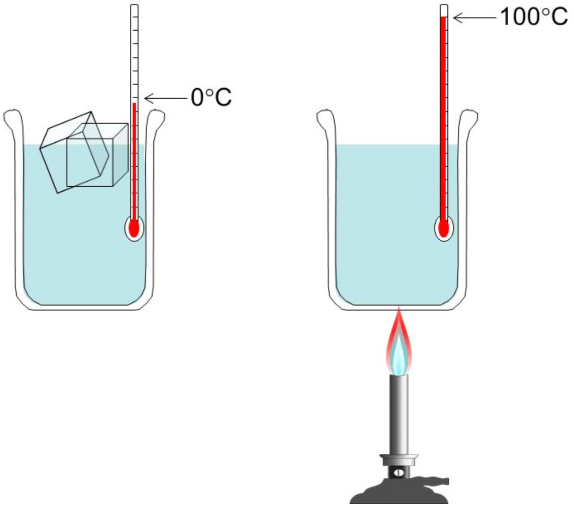
Think of old fashioned thermometers, how do they work?



Old fashioned mercury thermometers (and their glycerine replacements) use the first property, volume of a liquid increases with temperature. The liquid is placed in an evacuated tube, and the amount of the tube that is filled depends on the temperature. By placing a ruled background behind the tube or even by making rule marks on the outside of the tube, we can find out how much the liquid volume changed, and calculate the temperature change.

Celsius temperature scale

The most important thing to say about the Celsius scale is not to use it. (The same goes double for the Fahrenheit scale). But since it is common to see Celsius temperatures, we will talk about them.



This scale sets the zero point at the ice point of water and sets $100\,^{\circ}\mathrm{C}$ at the steam point of water. The scale uses 100 evenly spaced divisions between these points. This calibration does not work well if high accuracy is required. There are often large discrepancies when temperatures are measured far from the calibration points.

Worse yet, water only boils at $100\,^{\circ}\mathrm{C}$ at sea level, and does not always freeze at $0\,^{\circ}\mathrm{C}$.

Even worse, there is nothing really very 0ish about $0\,^{\circ}\mathrm{C}$. We picked it because it was a nice value for water. But we can certainly have lower temperatures than $0\,^{\circ}\mathrm{C}$.[26] This makes $0\,^{\circ}\mathrm{C}$ more like an origin of a coordinate system. We could say that the intersection of Main St. and Center St. is our origin, but there is noting really zeroish about that intersection. But it turns out there *is* a natural zero point for temperature ! A better temperature scale would start at the natural zero point. Let's see if we can make such a scale.

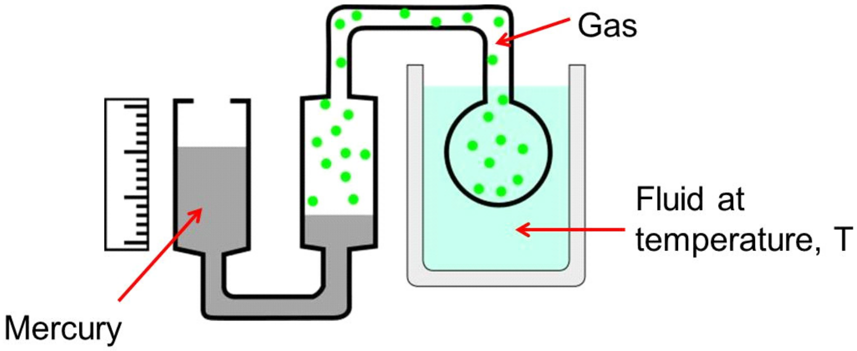
The Constant-Volume Gas Thermometer and the Absolute Temperature Scale

We wish to define a temperature scale that is based on a physical zero point. To do this

---

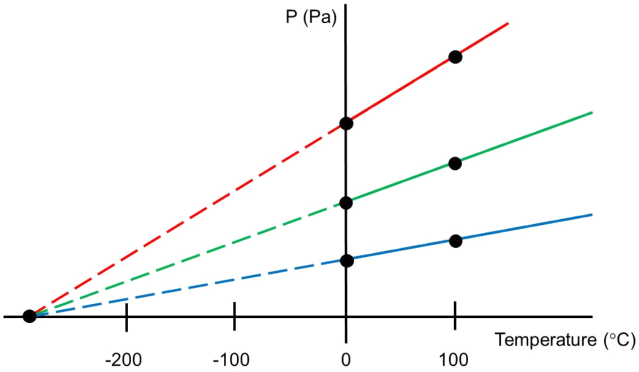
[26] We often do in Rexburg from about December to March.

we need a better thermometer. So here is one in the figure below.



This device uses the rise in pressure at a constant volume (the number 3 property from our list at the beginning of the lecture). It works by placing the gas bulb in thermal contact with the object who's temperature is to be measured. A scale (like a ruler) is placed in the apparatus such that the mercury in reservoir A is at the zero point. We could take water at its ice point, for example. Then the bulb is placed in thermal contact with another object, say water at its steam point. The mercury will move because the pressure increases in the bulb, and the gas tries to expand. We add mercury into reservoir B (just when you thought you were done with U-tubes!) until the level of reservoir A is again at the zero point. The height of the additional mercury we added is proportional to the change in pressure, as we now know. The Pressure for both measurements is calculated and plotted against the temperature and a linear curve is drawn between the two pressure-temperature points. This curve serves as a calibration for other temperatures.

Now we could fill the bulb with different gasses. If we do this, we get different sloped lines. It seems like we should have zero temperature if we have zero gas in the bulb. We have different lines for each gas, but if we then extend these lines back to where $P = 0$ (so that we have no gas), we find they intersect with $P = 0$ at $T = 273.15\,^{\circ}\mathrm{C}$. This is the basis we need for an absolute temperature scale!



We will take this common zero pressure point, $T = 273.15\,^{\circ}\mathrm{C}$ as our zero point for our improved temperature scale, but we need another point. We will choose the triple point of water (the one point where water, ice and steam exist in equilibrium; this happens at $0.01\,^{\circ}\mathrm{C}$ and $4.58\,\mathrm{mmHg} = 610.61\,\mathrm{Pa}$. We will call this point 273.15 on our new scale. Each unit of temperature will then be $1/273.15^{th}$ of the distance between our zero point and the water triple point. This way the degrees in this scale are the same size as the degrees in the Celsius scale (which is kind of nice). But this new scale that starts at the physical zero point we will call the Kelvin Temperature scale and the units are Kelvins (K). This is the temperature system we really should use.

Temperature scale conversions

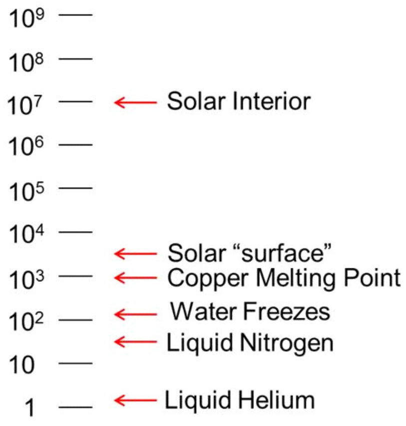
We will need some conversion factors so you can take temperatures from the Celsius and Fahrenheit scales to the Kelvin scale.

$$T_F = \frac{9}{5}T_C + 32\,^{\circ}\mathrm{F} \qquad (24.1)$$

$$\Delta T_C = \Delta T_K = \frac{5}{9}\Delta T_F \qquad (24.2)$$

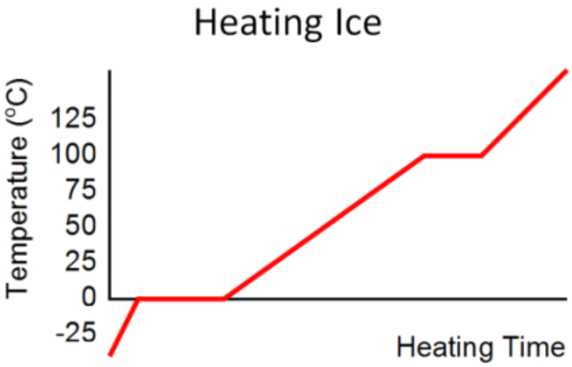
$$T_C = T_K - 273.15 \qquad (24.3)$$

Since the Kelvin scale is not the scale your doctor or your meteorologist uses, we should try to get some feeling for the Kelvin scale. Here are some interesting temperatures placed on the Kelvin scale.



Phase Changes and Phase Diagrams

Let's consider heating ice. You take a chunk of ice from outside on a cold February day in Rexburg, Idaho. The ice has a temperature of about $-25\,°\mathrm{C}$.[27] You place it in a pan on your stove and heat it up. Then, being a physics student, you plot the temperature as a function of time. Here is what you get.



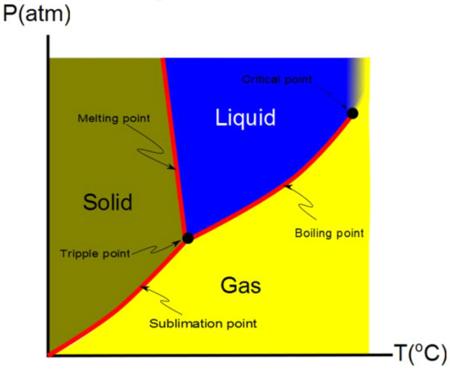
Notice the places where you are still heating, but the temperature does not change. These are phase changes. The first is melting, the second is boiling.

One of the things you will notice is that the melted ice does not boil at $100\,°\mathrm{C}$ here in Rexburg. This is because we are up so high that the air pressure is less than $1\,\mathrm{atm}$. In the next graph, I have plotted the melting, and boiling point of a substance as a function of pressure. The melting point and boiling point are not single points, but lines. This

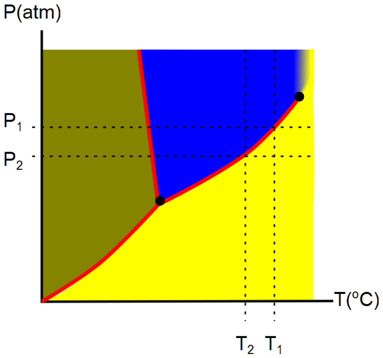
---

[27] OK, that is an exaggeration, it very rarely gets that cold in Rexburg.

is because melting and boiling points depend on pressure. There is also a line for the sublimation point. Along this line, a solid will change directly into a gas. Dry ice does this at normal room temperature and pressure.



If we observe the boiling boiling line. We see that as pressure goes down, the boiling temperature goes down. Look at T_1 and T_2 in the next figure, where $T_2 < T_1$. Note that $P_2 < P_1$.

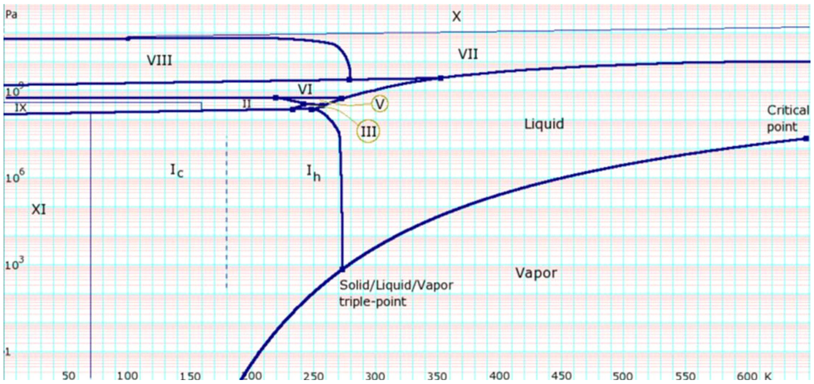


This agrees with our Rexburg boiling point experience. We can see from the graph that melting or boiling depends on pressure, and so does sublimation point.

The triple point is a special point where all three states are possible at the same pressure and temperature. Ice, steam, and water will all exist in equilibrium, for example, at the triple point of water.

The critical point is where the clear distinction between liquid and gas stops. beyond this boiling the material is in a fluid state, but not clearly liquid or gas.

Our diagram is rather schematic, and this is on purpose. An actual phase diagram for water is given in the next figure.



More Complete Phase Diagram for Water (Public Domain Image courtesy Karlhahn)

Along each of the lines, we can say that the material is in phase equilibrium, meaning that both phases from either side of the line are possible and equally probable. Only at the triple point are all three phases equally probable.

25 Expansion of Solids and Gasses (Ideal Gas Law)

We now have a better understanding of temperature, and we have new language to describe systems that have large numbers of objects. We have the zeroth law of thermodynamics. We are prepared to consider the next two laws. We will consider thermal expansion in this lecture. We will start with solids, but soon work our way to gasses. To model the thermal properties of gasses we will introduce and use the Ideal Gas Law.

Fundamental Concepts

- Thermal Expansion
- Ideal Gas Law
- Idea Gas Processes

Thermal Expansion

Ring and Ball Demo

We have not defined heat yet, but we kind of know what temperature is, so suppose we increase the temperature of a substance. We could take the mercury in a thermometer, for example. When the temperature is higher, the mercury takes up more space. It expands. This expansion is linear in temperature for many situations. When the expansion length is much less than the original dimensions of the object (L_i), we can consider the expansion to be linear in the change in temperature ΔT. Of course we could increase the temperature above this nice linear region, and we could get interesting effects like melting or exploding. But we will limit our discussion to a linear change in length with temperature for now.

Expansion Coefficient

What does it mean to have a linear change in length with temperature?

$$L_f = L_i + \alpha L_i \Delta T \tag{25.1}$$

It means that the final length of the object is equal to the initial length of the object plus a little bit more that depends on ΔT and not ΔT^2 or ΔT^3 or some other function of ΔT. You can see that this is the case for the equation above for thermal expansion. We can write it next to an equation for a straight line to show the linear nature of the equation

$$\begin{aligned} y &= b + mx \\ L_f &= L_i + \alpha L_i \Delta T \end{aligned}$$

If we plot L_f vs. ΔT this would make a straight line. The slope of the line would be αL_i so the amount of expansion depends on how big the object was to begin with. It also depends on what material we have. That is what the α tells us. Different substances expand at different rates even if they have the same initial length. So if you have a metal cavity of length L_i and fill it with a laser crystal of length L_i and heat them both up, you may find the crystal expands at a different rate than the metal faseners when the temperature rises–and may need to buy a replacement for the shattered crystal.[28]. The coefficient α is different for every substance, because every substance expands a little differently than other substances. We should give α a name. It is called the *average coefficient of linear expansion* where from equation (25.1) we can see that

$$\begin{aligned} \alpha &\equiv \frac{\frac{\Delta L}{L_i}}{\Delta T} \\ &= \frac{\Delta L}{L_i \Delta T} \end{aligned}$$

where ΔL is the change in length due to a temperature change ΔT. And of course we already know L_i is the original length of the sample.

As long as ΔT is "not too big" α is constant. We can write this as

$$L_f - L_i = \alpha L_i \left(T_f - T_i\right)$$

where the subscript f means final and i means initial as usual. The units of α are inverse temperature (unfortunately it is often in $^\circ\mathrm{C}$, but we would prefer to use K). Laboratories publish tables of average coefficients of linear expansion. A few are given

---

[28] It was a very sad day when this happened to me personally.

in the table below

Material	Average $\alpha\left(\mathrm{K}^{-1}\right)$
Aluminum	25×10^{-6}
Copper	17×10^{-6}
Gold	14×10^{-6}
Lead	29×10^{-6}
Steel	11×10^{-6}
Brass	18.7×10^{-6}
quartz	0.4×10^{-6}
Glass	9×10^{-6}
Concrete	12×10^{-6}

Suppose we increase the temperature of something that has a hole in it. Does the hole increase or decrease in size? A cavity in a piece of material expands in the same way as if the cavity were filled with the material. Let's see that this must be true.

Volume expansion

Don't do this in class

suppose we have a cube with sides of length L_i. What happens when we increase the cube's temperature? Each side will become larger. So what happened to the volume?

$$\begin{aligned}
\not{V}_i+\Delta\not{V} &= \left(L_i+\Delta L\right)\left(L_i+\Delta L\right)\left(L_i+\Delta L\right) \\
&= \left(L_i+\alpha L_i\Delta T\right)\left(L_i+\alpha L_i\Delta T\right)\left(L_i+\alpha L_i\Delta T\right) \\
&= L_i^3\left(1+\alpha\Delta T\right)\left(1+\alpha\Delta T\right)\left(1+\alpha\Delta T\right) \\
&= L_i^3\left(\Delta T^3\alpha^3+3\Delta T^2\alpha^2+3\Delta T\alpha+1\right) \\
&= \not{V}_i\left(\Delta T^3\alpha^3+3\Delta T^2\alpha^2+3\Delta T\alpha+1\right)
\end{aligned}$$

Now let's divide by $\not{V}_i$

$$\begin{aligned}
1+\frac{\Delta\not{V}}{\not{V}_i} &= \Delta T^3\alpha^3+3\Delta T^2\alpha^2+3\Delta T\alpha+1 \\
\frac{\Delta\not{V}}{\not{V}_i} &= \Delta T^3\alpha^3+3\Delta T^2\alpha^2+3\Delta T\alpha
\end{aligned}$$

and let's make an approximation. For normal conditions, $\alpha\Delta T\ll1$ (good for $T<373.15\,\mathrm{K}$ ($100\,^{\circ}\mathrm{C}$)). Then

$$\frac{\Delta\not{V}}{\not{V}_i}\approx3\alpha\Delta T \tag{25.2}$$

We can write this as

$$\begin{aligned}
\Delta\not{V} &\approx 3\alpha\not{V}_i\Delta T \qquad (25.3)\\
&= \beta\not{V}_i\Delta T
\end{aligned}$$

where β is called the *average coefficient of volume expansion.*

Question 123.9.1

We could do the same for a two dimensional metal plate (in the approximation that a plate is 2D!)

$$\Delta A = 2\alpha A_i \Delta T \tag{25.4}$$

We need to be careful that we don't use these linear expansion equations outside of their valid range. For example, suppose we build a satellite and launch it into a low Earth orbit (LEO). The space environment for a LEO orbit (sun side to shade side) has a very large ΔT.

$$\begin{aligned} T_{sun} &= 120\,^{\circ}\mathrm{C} = 393.15\,\mathrm{K} \\ T_{shade} &= -100\,^{\circ}\mathrm{C} = 173.15\,\mathrm{K} \end{aligned} \tag{25.5}$$

This ΔT range is too large for most materials. Special low α materials are used to allow satellite structures to survive

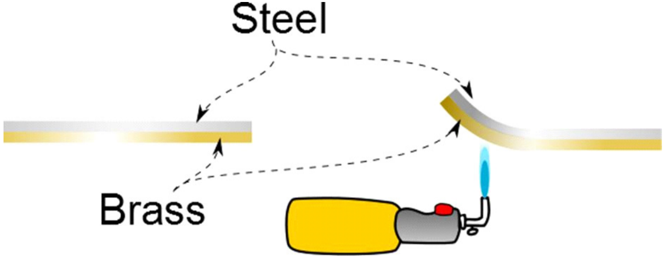
But how about on Earth? Even on Earth we can have extremes

$$\begin{aligned} T_{\max} &= 57.6\,^{\circ}\mathrm{C} = 330.75\,\mathrm{K} \\ T_{\min} &= -89\,^{\circ}\mathrm{C} = 184.15\,\mathrm{K} \end{aligned} \tag{25.6}$$

This ΔT range would be a challenge to cover with most materials.[29] So if you are going to build a device or instrument that will travel around the globe, you need to use special low α materials. Often making these materials is very expensive and the process is a tightly held industrial secret.

Different materials

Looking at table of average linear expansion coefficients we find see different α values for different materials. Sometimes α values are quite different. Suppose we take two materials with different α's and put them together.



The figure shows a bimetallic strip. What would happen if we heated this bimetallic

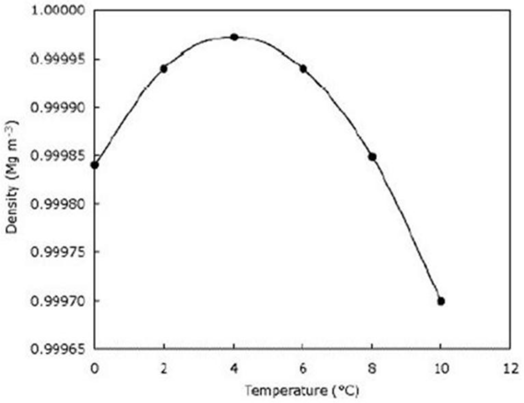
---

[29] Expeciall metal structures and laser crystals.

strip? The two metals will expand at different rates, forcing the bonded pair of metals to bend.

Water is weird (and the Lord knows what He is doing!)

Question 123.9.2



Density of water near $0\,^{\circ}\mathrm{C}$. (Public Domain image PD-1923, PD-Australia)

For most substances the volume increases with increasing temperature. Usually liquids increase in volume more than solids. But water is different. Look at the graph of density vs. Temperature.

Just as water melts, it starts to increase in density, but then it becomes less dense with increased temperature after about $277.15\,\mathrm{K}$ ($4\,^{\circ}\mathrm{C}$). We expect the behavior $277.15\,\mathrm{K}$ ($4\,^{\circ}\mathrm{C}$), but from $273.15\,\mathrm{K}$ ($0\,^{\circ}\mathrm{C}$) to $277.15\,\mathrm{K}$ ($4\,^{\circ}\mathrm{C}$) we see the volume decreases with ΔT. This is why a lake freezes from the top down. The colder water near $273.15\,\mathrm{K}$ ($0\,^{\circ}\mathrm{C}$) is less dense, and so before it freezes it floats to the surface. This is great, because if it became more dense, the water would freeze and sink to the bottom, leaving the water exposed ot the cold atmosphere to freeze more water. The lake would fill up from the bottom with solid ice, killing fish and water life.

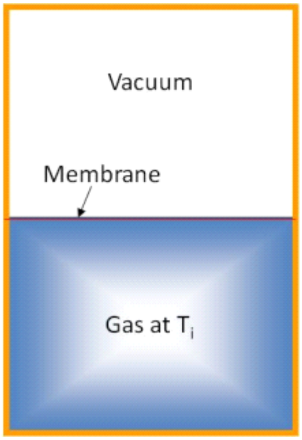
Lake Erie from Space. Notice the ice covering part of the lake. (Public Domain Image courtesy NASA Visible Earth, http://visibleearth.nasa.gov/view_rec.php?id=1286

Question 123.9.2

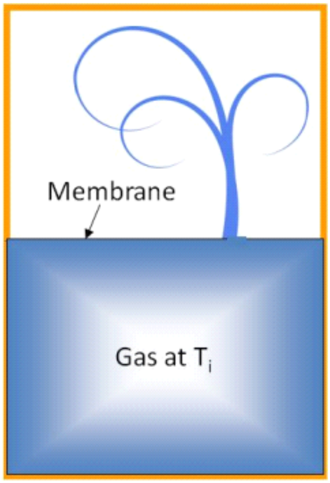
Question 123.9.3

Quazi-static processes

Consider the situation shown below



We have a gas in one side of a sealed container and a vacuum on the other side. A membrane separates the two. We have a very definite temperature in the gas, and also a definite temperature in the vacuum. But if I puncture the membrane,



gas goes swirling into the vacuum. The density will not be constant as the gas swirls. Where there is gas, we would say we have a temperature closer to T_i. But where there is no gas, we have a temperature nearer $0\,\mathrm{K}$. It takes a while for the gas to come to equilibrium. During this process we can't say that we have a definite temperature, or definite pressure for the system. Higher physics classes (and engineering classes) deal with the difficult situation that exists before equilibrium is achieved. For now, we will only look at situations that have reached an equilibrium, or are not too far from equilibrium. We call states that change slowly so they are never far from equilibrium, *quazi-static*. The situation we have described is called a *free expansion* because the gas is allowed to expand with no outside influences. A free expansion is *not* a quasi-static process. Let's think of how we might make a quasi-static process happen.

Reversible and Irreversible Processes

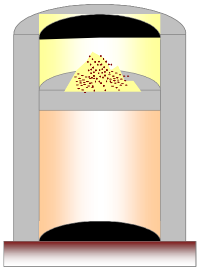
A reversible process is one in which every point along some path is an equilibrium state and one for which the system can be returned to its initial state along the same path.

All natural processes are known to be irreversible. Reversible processes are an idealization, like frictionless surfaces or massless strings. But some real processes are good approximations to reversible processes.

Our free expansion, of course, is not a reversible process. Imagine trying to get all the gas atoms back through the hole in the membrane! BYU Demo

We can do work on the gas to achieve this (say, put a piston in the system and apply a force). This would increase the temperature, so we would have to let energy transfer out of the system (by heat). This would allow the system to return to it's final state, but the surroundings will have changed. This is not a reversible process!

A quasi-static process can be nearly reversible.

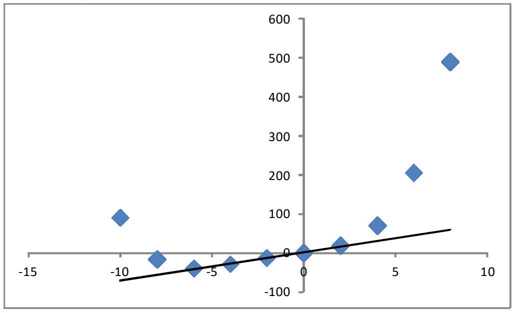


Imagine dropping sand on a frictionless piston to compress a gas. Then slowly removing the sand. The system stays almost in equilibrium as it changes, so that it returns to a new equilibrium right away. After all, you only moved one grain of sand at a time. This process is almost reversible. Of course, you still have to do work in moving the sand, so the outside environment is changed, so even this process is not truly reversible. But it is a good mental model for the sort of processes we will study in this class; slow, almost in equilibrium all the time.

Ideal Gases

Question 123.9.4-6

Let's consider the following data



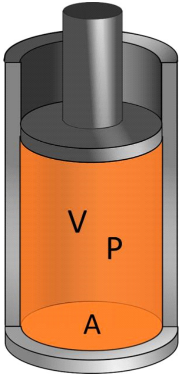
What can we say about the curve fit?

It might look useless. But look at the region from about $-6 < x < 5$. In this region, the curve fit is not too bad. Of course, the curve fit is not the right equation, and we know this from the regions $x < -6$ and $x > 4$. So our researcher that collected this data does not have an exact theory that explains the behavior of his/her system well. But could the curve fit be useful? If all you needed was an estimate of the value of the actual function at $x = -1$, then the curve fit might be enough to get your work done! This may not be terribly satisfying, but sometimes it is a useful way to work.

Another way to think about this is that if most of the time, under usual conditions, the phenomena we experience fall within the range $-6 < x < 5$ and the curve fit equation is much simpler, it might be good enough and convenient to use the curve fit. For example, we know that Newton's laws are not exact. We have to use General Relativity to be accurate. But in calculating our average speed on a trip to Idaho Falls, Newton's laws are plenty good enough, and *much* more convenient than their General Relativistic forms.

We will use the Ideal Gas Law to study the expansion of gasses as temperature changes. The Ideal Gas Law is an approximation very like our curve fit we have been considering. It works pretty well under normal conditions, and it is much easier to use than the exact law. It allows us to gain great insight, without the mathematical difficulty of the exact relationship.

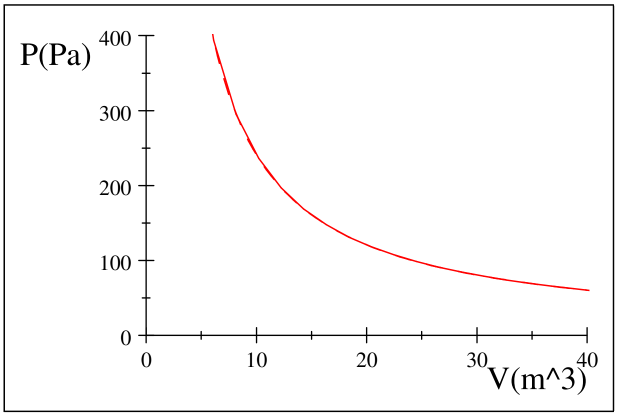
Given all this, we should realize that there is no such thing as and ideal gas. But in this approximation, the atoms are so weakly bound that there is no equilibrium position between atoms. Thus there is no definite volume defined for an ideal gas. This is generally true at standard pressure and temperature for real gasses. So we will let $\not{V}$ be a variable for gases. Let's look at where the ideal gas law came from. It is much like our hypothetical experiment. The experimentalists gave us some clues on how to relate $\not{V}$, P, and T.



What lead to the Ideal Gas Law

Boyle found that if a system is kept at a constant temperature then

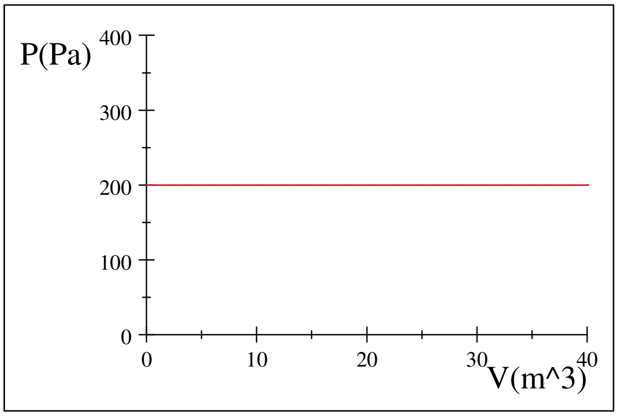
$$P \propto \frac{1}{\not{V}} \text{ for constant } T \tag{25.7}$$



That is, if you don't let the temperature of the gas change, then a change in pressures is inversely related to a change in the volume of the gas. The more pressure, the smaller the volume.

Two researchers, Charles and Gay-Lussac, found that if a system is kept at constant pressure then

$$\rlap{-}V \propto T \text{ for constant } P \tag{25.8}$$



The warmer the gas, the more it will expand.

Combining these two findings gives an equation that describes ideal gasses.

$$P\rlap{-}V = nRT \tag{25.9}$$

where n is the number of moles, and R is called the *universal gas constant*. This is the *ideal gas law*. The law came from researchers that could not access extreme pressures or temperatures, and so for normal conditions, it works just fine. But it will not work at all for extremely low or height pressures and temperatures.

The variables n, $\cancel{V}$, P, and T, are state variables for ideal gasses. Depending on the units for $\cancel{V}$, P, and T, the value for R may be different.

$$\begin{aligned} R &= 8.314\frac{\mathrm{J}}{\mathrm{mol\,K}} \qquad (25.10) \\ &= 0.08214\frac{\mathrm{atm\,l}}{\mathrm{mol\,K}} \end{aligned}$$

Alternate form of the Ideal Gas Law

We know we can describe the number of moles as

$$n = \frac{N}{N_A} \qquad (25.11)$$

where N is the number of molecules of gas that we have. Then we can write

$$P\cancel{V} = \frac{N}{N_A}RT \qquad (25.12)$$

if we define a new constant

$$k_B = \frac{R}{N_A} \qquad (25.13)$$

then

$$P\cancel{V} = Nk_BRT \qquad (25.14)$$

the constant k_B is called *Boltzmann's constant*

$$k_B = 1.38 \times 10^{-23}\frac{\mathrm{J}}{\mathrm{K}} \qquad (25.15)$$

Fire Syringe Demo

Ideal Gas Special Processes

In what we did above, I introduced a new kind of graph. I graphed P vs. $\cancel{V}$. In the ideal gas law, there are four state variables, n, P, $\cancel{V}$, and T. But if we have a sealed container so n can't change, we can describe the other three variables on one graph, so each point on the graph will show the possible states of the gas. This seems almost too good to be true, By plotting only P and $\cancel{V}$, how can we get T on the graph?

Well, think about the ideal gas law if n is constant

$$\frac{P\cancel{V}}{T} = nR$$

the right hand side is constant. Now suppose we also let T be constant.

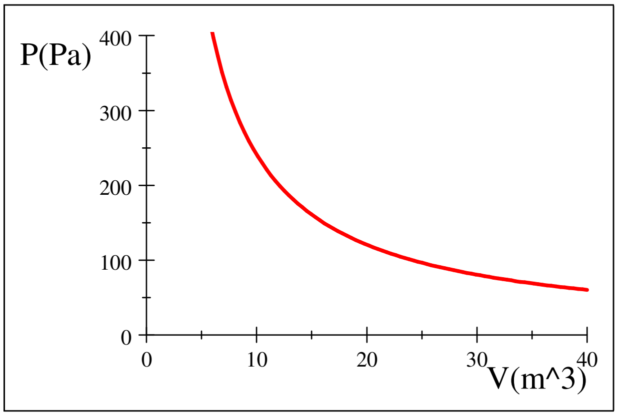
$$P = (nRT)\frac{1}{\cancel{V}}$$

everything in the parenthesis is constant so

$$P \propto \frac{1}{V}$$

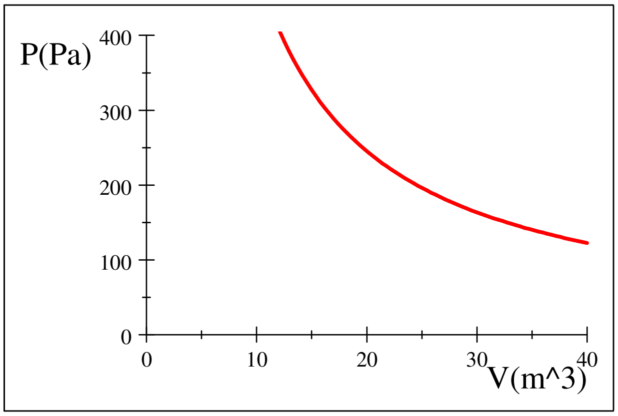
We can plot this for

$n = 1\,\mathrm{mol}$	number of moles
$R = 8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}$	Universal gas constant
$T = 290\,\mathrm{K}$	Temperature in Kelvins

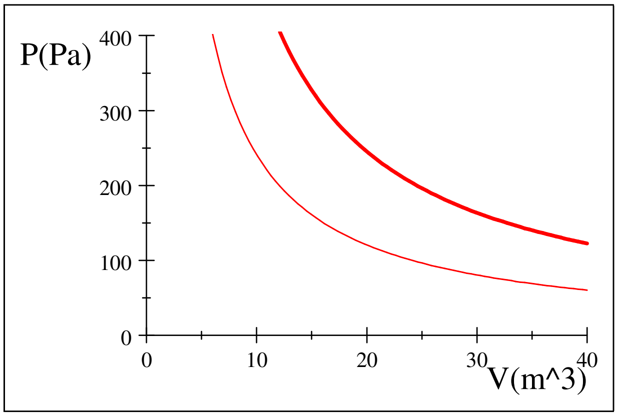


now let's plot it again for a different value of T

$n = 1\,\mathrm{mol}$	number of moles
$R = 8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}$	Universal gas constant
$T = 590\,\mathrm{K}$	Temperature in Kelvins



Notice that the curve looks like the same shape, but it has been shifted toward the upper right hand corner of the graph. If we place both the $T = 290$ and $T = 590$ graphs on the same plot, we can better see the shift.



The dark curve is our new pot. The light curve is the old plot for comparison. The curves look a lot alike. They are essentially the same shape. But their location (how far they are from the origin) depends on the temperature, with warmer temperatures moving to the upper right hand part off the graph. So our P vs. $\not{V}$ graph can tell us something about temperature.

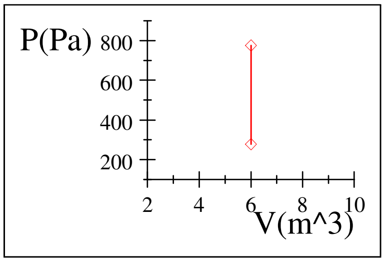
Knowing how to interpret a $P\not{V}$ diagram, we can now look at some simple processes that have special properties.

Constant volume process

Question 123.9.12

Question 123.9.13

Suppose we have a process that takes us between two states on a $P\not{V}$ diagram as shown in the next figure.



We start at one pressure, say $P = 275\,\mathrm{Pa}$, and end at another pressure, say, $P = 775\,\mathrm{Pa}$. In this process, notice that the volume does not change. Such a process is called *isochoric*. Would we predict that the temperature went up or down in taking this path? Well, we went from the lower to the upper part of the graph, so we expect that the temperature went up. Starting again with the ideal gas law

$$P = \left(\frac{nR}{\not{V}}\right) T$$

we can see that when $\not{V}$ is constant, P goes like T, so if P went up, so did the temperature.

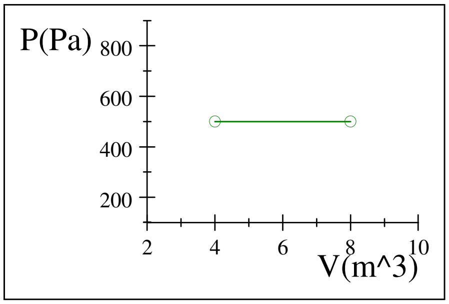
But how would create such a situation? Consider what would happen[30] if you placed an aerosol can in a fire. The aerosol can is rigid, so the volume can't change. But the fire will make the temperature change, and the pressure will change–that is–until the can explodes.

Constant Pressure process

Question 123.9.14

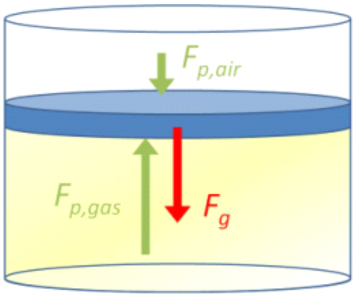
Question 123.9.13

Consider the next $P\rlap{\,-}V$ diagram.



We can see that in this process the pressure is not changing. If we go from left to right along this path, will the temperature rise? We know that going to the right means higher temperature on a $P\rlap{\,-}V$ diagram, so yes it will.

The process represented by this type of path on a $P\rlap{\,-}V$ diagram is called an *isobaric* process. How would we produce such a situation? Consider a piston in a cylinder. The cylinder has a gas inside, and air on the outside



The piston is free to move up or down. The forces acting on the piston are shown. The weight of the piston pulls it downward, the force due to pressure of the gas, $F_{p,gas}$

---

[30] Only consider this, because it would be irresponsible and dangerous (and illegal in some states) to do this!

pushes up, and the air pressure $F_{p,air}$ pushes down. The net force will be

$$\Sigma F_y = ma_y = F_{p,gas} - F_{p,air} - mg$$

where the mass of the piston is m. We can write this as

$$ma_y = P_{gas}A - P_{air}A - mg$$

If the piston is not accelerating , then

$$P_{gas} = P_{air} + \frac{mg}{A}$$

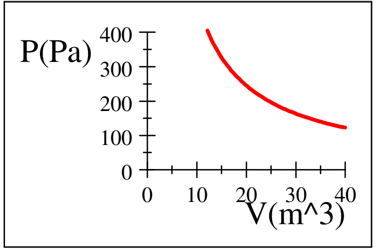
and we find that the pressure of the gas inside is slightly larger than the outside air pressure. What would happen if we heated the gas?

$$P\not{V} = NRT$$

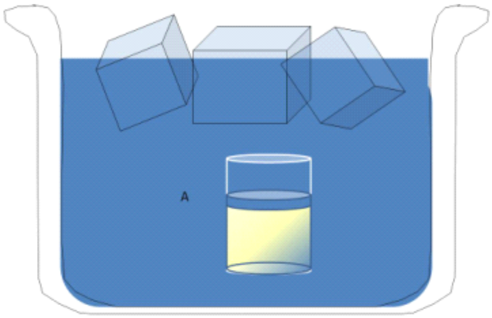
would suggest that P might change, but since the piston is free to move, what happens is that the force $F_{p,gas}$ increases when P_{gas} increases, and the piston accelerates upward. It stops when the forces are again balanced. But that means that P_{gas} will only change for a moment, and then we will achieve equilibrium at the same old value of P_{gas}. But something has changed, the volume has increased.

Constant Temperature Process

We have already met the constant temperature process.



How do we make such a path happen? Consider our piston again.



This time, let’s put it in a vat of ice water. We can compress the gas by pushing on the piston. But if the temperature is changed by compressing the gas, it is quickly changed back by the ice water.

26 Work in Thermodynamic Processes

Armed with the Idea Gas Law and our (somewhat) better understanding it is finally time to study the next law of thermodynamics. We will use the ideas of energy to do this. In PH121 we loved energy because the ideas of energy made problems easier to solve. The same is true here. We will start this lecture by thinking about the energy of large numbers of objects (atoms and molecules).

Fundamental Concepts

- ΔE_{int} is an invariant quantity in thermodynamics
- Work in thermodynamic processes can be described as the integral $w_{int} = \int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V}$
- Q is a transfer of transfer of internal energy, and it is called “heat”
- $\Delta E_{int} = Q + w_{int}$ is the first law of thermodynamics

Conservation of Energy: First Law of Thermodynamics

Question 123.10.1

Remember when we last studied energy in PH 121. We used the work energy theorem

$$\Delta K = w$$

and expanded the work side to show the different types of work explicitly. We have work done by conservative forces w_c and work done by non-concervitave forces (dissipative forces like friction) w_{nc}.

$$\Delta K = w_c + w_{nc}$$

and we could further divide up the work by categorizing the source of the work, say, work done by the force of gravity, w_g, or work done by a spring force w_s.

$$\Delta K = w_g + w_s + w_{nc}$$

and we could go on in more complicated systems listing all the different kinds of work.

We relabeled these different types of work as potential energies. For example,

$$\begin{aligned}\Delta U_g &= -w_g \\ \Delta U_s &= -w_s\end{aligned}$$

and rewrote the work energy theorem

$$\Delta K + \Delta U_g + \Delta U_s = w_{nc}$$

This gave us a way to think of mechanical energy in a system. But we realize that we have not covered all types of energy yet. Suppose we consider your car. You put gas in the car. This is a source of energy, *a chemical potential energy*, U_{chem}. So we should really include it.

$$\Delta K + \Delta U_g + \Delta U_s + \Delta U_{chem} = w_{nc}$$

and if your car is a DeLorean that has been modified to use nuclear energy to run, then you might have a nuclear potential energy

$$\Delta K + \Delta U_g + \Delta U_s + \Delta U_{chem} + \Delta U_{nuclear} = w_{nc}$$

and back in PH121 we called the non-conservative work $w_{nc} = \Delta E_{th}$ because friction like forces dissipate mechanical energy by turning them into thermal energy. Then

$$\Delta K + \Delta U_g + \Delta U_s + \Delta U_{chem} + \Delta U_{nuclear} = \Delta E_{th}$$

The quantities ΔU_{chem}, $\Delta U_{nuclear}$, and ΔE_{th} are different than the mechanical energies we studied before. They have to do with the internal workings of the material, itself. We will call an amount of energy that comes from the material, itself *internal energy.* If we think about it, we might expect that temperature has something to do with an internal energy of the object. And we would be right.

Work and Internal energy

Our potential energy terms are all derived from work.Long ago we found that the work done on an object may depend on the path the object takes. We found work using Question 123.10.2

Question 123.10.2.5

$$w = \int_{x_i}^{x_f} \overrightarrow{F} \cdot \overrightarrow{dr} \tag{26.1}$$

and the work was the area under the force vs. displacement curve. But this was work done moving an object as a whole. Now we can do work on an object by compressing it, for example. This is different from moving the whole object around. Let's call this new kind of work *internal work.* This new kind of work should be similar in form to regular mechanical work. We expect it to depend on the details of how the internal work is done.

For mechanical work we defined the potential energy, U, as

$$w = -\Delta U \tag{26.2}$$

Question 123.10.3

Using this concept, we showed that mechanical energy

$$E_{mech} = K + U \tag{26.3}$$

was often conserved, when $E_{thermal} = 0$ where E_{mech} was the total mechanic energy. Remember we solved a lot of problems back in PH121 by knowing that for a closed system, E_{mech} did not change. In other words, E_{mech} was independent of path we take on our F vs. x diagram. This makes E_{mech} much easier to work with. A path independent quantity makes problems easier.

We want another variable to work with that removes the difficulty of worrying about the details of how the motion happens (the path). But we can't use the kinetic energy for this. The kinetic energy is part of the mechanical work of the object as a whole, and does not affect the internal workings of the object.

Frictional forces do affect the object. Think of rubbing your hands together. There is mechanical work done by your muscles. There is kinetic energy while your hands move. But your hands heat up. This higher temperature is still there for a while after your hands stop moving. Eventually the temperature of your hands returns to normal. We expect that this is a transfer of energy, since dissipative forces take energy out of the system. We will call the movement of energy out of a system "*heat.*" We give it the symbol Q in this class.

Note that this is a very strange and specific use of the word "heat." For us heat is a transfer of energy. It is not the amount of energy in a body. We will call the amount of energy that is in the body the *internal energy* and give it the symbol E_{int}. But "heat" is a *transfer of internal energy*. It does not make sense to ask how much heat an object has. An object can have internal energy, but heat is a transfer of that energy to something else.

The internal energy would depend on how much energy is transferred into the object. Say, we have a large metal block and we set it on a fire. The fire would transfer energy to the block. We could also do work on the block to increase the internal energy of the block, say, by compressing the block. Question 123.10.4

Now we are ready for our path independent quantity for internal energy. We define the change in internal energy as

$$\Delta E_{int} = Q + w_{int}$$

As with the mechanical energy, the change in internal energy does not depend on the details of how the change happens (what we are calling "path"). The change in energy is used because Q and w are transfer variables, so they *do* depend on the details of the transfer. For small changes we can write

$$\Delta E_{int} = \Delta Q + \Delta w_{int}$$

or for very small changes, we could replace the Δ's with small d's

$$dE_{int} = dQ + dw_{int}$$

but be careful! these are not true differentials because dw depends on the path! Integration of dw is trick and you would have to think about it carefully. The relationship

$$\Delta E_{int} = Q + w_{int} \tag{26.4}$$

is known as the *first law of thermodynamics.* This will be the subject of our next lecture.

Let's look at some systems and think about their internal energies.

An Isolated System

An isolated system is a system separated from the rest of the universe. Nothing can act on it. When a system is isolated, we have

$$w_{int} = 0 \tag{26.5}$$

because no work can be done on an isolated system and

$$Q = 0 \tag{26.6}$$

because no heat can flow to or from an isolated system so

$$\Delta E_{int} = w_{int} + Q = 0 \tag{26.7}$$

A Cyclic Process

Suppose we have a process, something that is done to a sample of matter, say a gas. Further suppose that our process repeats over and over again. We call such a process a *cyclic* process. This is like the pistons in your car going through the same motion over and over again to compress the gas in your engine cylinders. Let's practice with the idea of internal energy, work, heat and PV diagrams for such a process. If a process repeats, it starts and ends in the same state so then Question 123.10.5

$$\Delta E_{int} = 0 \tag{26.8}$$

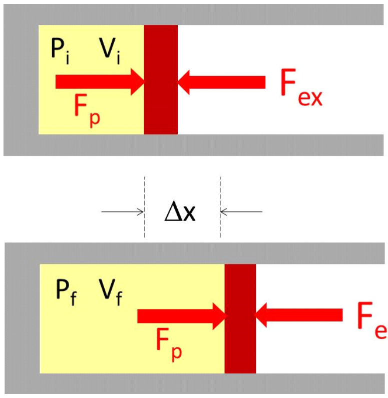
and

$$Q = -w_{int} \tag{26.9}$$

Question 123.10.6

Work and Thermodynamic Processes

Consider the work done by the piston as it compresses the gas. The amount of work must be equal to the amount of energy transfer by heat. We want this to be true! Otherwise you could melt your engine. We want an engine in a car (or train, or airplane) to be a cyclic process. This explains why your car engine gets hot. Work is being done in the cylinders in your engine, but to make the process cyclic some energy must be lost by heat (though there are details of car engines that we won't tackle until later).



Lets take a close look at the gas within the piston. At equilibrium we know

$$P = \frac{F_P}{A}$$

or

$$F_P = PA$$

This force is the force of the gas pushing on the piston. We could label it

$$F_{gp} = F_P$$

Notice that the gas pushes to the right. If we apply an external force to the piston slowly so as to allow the system to remain in thermal equilibrium as we go, then

$$\overrightarrow{F}_{ex} = -F_{ex}\hat{\imath}$$

because we are pushing in the $-x$-direction. Again, we will move the piston slowly so we don't violate our quasi-static rule. As we move the piston the gas inside the piston adjusts in pressure until we are again in equilibrium. So

$$F_{ex} = F_{gp}$$

but the pressure force F_{gp} pushes the opposite direction

$$\begin{aligned}\overrightarrow{F}_{gp} &= +F_{ex}\hat{\imath} \\ &= PA\hat{\imath}\end{aligned}$$

We should also note that since neither the piston nor the gas are accelerating, Newton's third law tells us that the force of the gas on the piston and the force of the piston on the gas are equal in magnitude but opposite in direction

$$F_{gp} = F_{pg}$$
$$\overrightarrow{F}_{gp} = -F_{pg}\hat{\imath}$$

When we move the piston to the left, we have a displacement

$$\overrightarrow{dr} = -dx\hat{\imath}$$

so the work done *on the gas by the piston would be*

$$w_{gp} = \int_{x_i}^{x_f} \overrightarrow{F}_{ex} \cdot d\overrightarrow{r}$$

Remembering how to use a dot product we can write this as

$$w_{gp} = \int_{x_i}^{x_f} F_{ex} dx \cos\theta_{Fx}$$

It is worth remembering that in this formula F_p and dx are magnitudes. Let's write this as

$$w_{gp} = \int_{x_i}^{x_f} |F_{ex}|\,|dx| \cos\theta_{Fx}$$

to remind us. Substituting in $|F_{ex}|$ for $|F_{gp}|$ we have

$$\begin{aligned}w_{gp} &= \int_{x_i}^{x_f} |F_{gp}|\,|dx| \cos\theta_{Fx} \\ &= \int_{x_i}^{x_f} |PA|\,|dx| \cos\theta_{Fx}\end{aligned}$$

and the angle between F_{gp} and dx is 0 (they are both to the left) so

$$w_{gp} = \int_{x_i}^{x_f} |PA|\,|dx|$$

where, to get the total work done on the piston we are integrating over the entire path of the piston travels. It is worth noting that P can only be positive, and the same is true for A. Both P and A are constant, since we are using our quazi-static process and allowing the pressure to equalize as we go. So we could write the work done on the piston by the gas as

$$w_{gp} = PA \int_{x_i}^{x_f} |dx|$$

but now we need to consider $|dx|$. We know that dx is a small displacement, and displacements can be negative. And in our case, as the piston goes to the left, dx certainly is negative. But no matter, we don't just have dx, we have $|dx|$. So the work is positive just as it should be.

But here we can be a little bit clever. If we add in a minus sign so that

$$w_{gp} = -PA\int_{x_i}^{x_f} dx$$

our added minus sign will cancel with the minus sign hiding in the dx, still giving us positive work (think of what we did with springs in PH121 to deal with $S = -k\left(x - x_o\right)$ in work integrals). Now we don't have to worry about integrating with absolute value signs.

Remember that we are using an isobaric process in compressing the gas That is, we keep the pressure the same but allow the temperature to change, so that the volume and temperature change proportionally $\not{V} = (nRT/P)$) So our integral is just

$$\begin{aligned} w_{gp} &= -PA\int_{x_i}^{x_f} dx \\ &= -PA\left(x_f - x_i\right) \end{aligned}$$

We can rewrite this result using what we know about the volume of the gas

$$\not{V} = A\Delta x$$

so the work done by the gas pushing on the piston is

$$\begin{aligned} w_{gp} &= -P\left(Ax_f - Ax_i\right) \\ &= -P\left(\not{V}_f - \not{V}_i\right) \end{aligned}$$

This gives us an idea. We could write the quantity Adx as $d\not{V}$ and change the limits of integration from x_f and x_i to $\not{V}_f$ and $\not{V}_i$. Then our integral would be

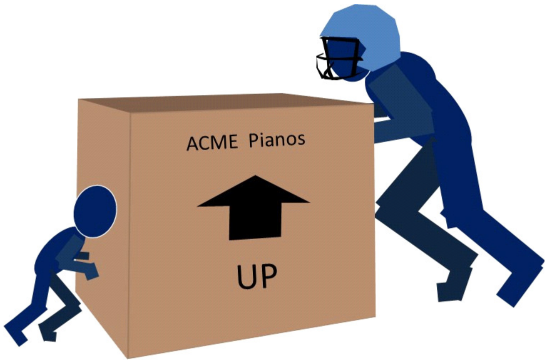
$$\begin{aligned} w_{gp} &= -\int_{x_i}^{x_f} PAdx \\ &= -\int_{\not{V}_i}^{\not{V}_f} Pd\not{V} \end{aligned}$$

This gives the same answer for the work done on the piston for our isobaric case

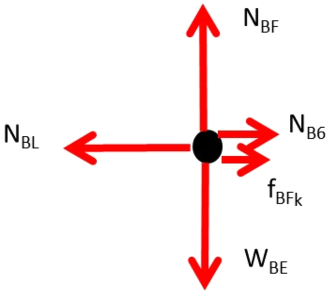
$$w_{gp} = -P\left(\not{V}_f - \not{V}_i\right)$$

But this is the amount of work that the piston does on the gas. Is the gas pushing back on the piston? Of course it is! And that push is also a force, F_{pg}, so it also does work. But let's review what we know about work from PH121.

Suppose we have two people exerting a force on a box. One is a professional American football player, The other is a six-year-old.



They push in opposite directions. How much work is being done? To answer this, let's consider a free-body diagram.



where the subscript B is for the box and the subscript 6 is for the six-year-old and L is for the football player (In this case, a (L)inebacker). We can see that there will be a net force.

$$F_{net_s} = N_{BL} - N_{B6} - f_{BF}$$

But which way will the box go? We could guess that the football player will push the box *and* the six-year-old to the left. So Δs will be negative. Then the work done on the box by the football player will be

$$\begin{aligned} w_{BL} &= \int_{s_i}^{s_f} N_{BL} ds \\ &= N_{BL} \Delta s \end{aligned}$$

but the work done by the child would be

$$\begin{aligned} w_{B6} &= \int_{s_i}^{s_f} -N_{B6} ds \\ &= -N_{BM} \Delta s \end{aligned}$$

The child's work is negative! What can that mean? Well, for starters, the child's force can't be the force that is making the box move. In fact, the child's force is another obstacle that the linebacker's force must overcome to make the box move. This means that the linebacker must do enough work (push hard enough) to make the box go, *and* to overcome the backward push of the child.

But that is just our case with the gas and piston The gas pressure force *is* pushing on the piston. But the gas pressure force on the piston is losing ground (the external force is pushing the piston to the left). So we would expect the work done by the gas on the piston to be negative.

Since we assumed a quasi-static process, the two forces must always be nearly the same,

$$F_{ex} = F_{pg}$$

so the amount of work must be nearly the same, but the sign must be negative for the work on the gas due to the piston's push. We wrote the work done by the piston on the gas as

$$w_{gp} = -\int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V}$$

Then the work done by the gas on the piston must be

$$\begin{aligned} w_{pg} &= -\left(-\int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V}\right) \\ &= \int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V} \end{aligned}$$

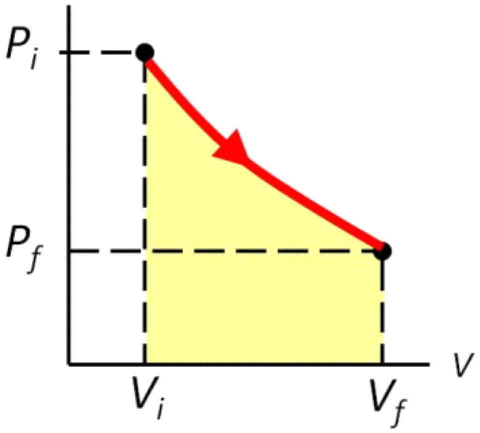
that is, the work done on the gas by the piston is the negative of the work done on the piston by the gas.

We are principally interested in what happens to the gas, (we studied objects like the piston in PH121) so the first equation is what we are looking for. This is the internal work done on the gas!

$$w_{gp} = w_{int} = -\int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V} \tag{26.10}$$

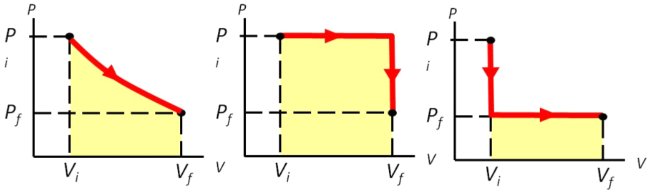
Question 123.10.6

This looks easy enough, but to integrate we must know how P depends on $\cancel{V}$ during the whole process. It was simple for an isobaric process, but it could be much harder for other processes. We can plot P vs. $\cancel{V}$ to see how it varies for a process. We would expect the integral to be the area under a $P\cancel{V}$ curve. This is very like mechanical work, which was the area under the F-x curve.



We can make a general statement: The work done on a gas in a quasi-static process that takes the gas from an initial state to a final state is the negative of the area under the curve on a PV diagram evaluated between the initial and final states.

Unfortunately, the work done depends on the shape of the curve on the $P\cancel{V}$ diagram. Several are shown below. Each has a different amount of work done (notice the different areas under the curves), but the same initial and final points.



Question 123.10.7

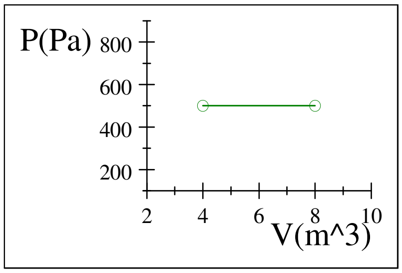
Notice that we have calculated the amount of work done on the gas. You might be wondering if that is what we want to build car or jet engines. Don't we want the gas to expand to push the piston and make the car go? And of course, you are right in your thinking. Eventually we want to know the work done by the gas on the piston, w_{pg}. But for now, we will look at w_{gp}. But from what we have done we know these two works are very related for a quasi-static process.

$$w_{pg} = -w_{gp}$$

Special processes and work

Let’s use our definition of internal work and apply it to ideal gasses. We have three special processes we have studied. Let’s find the work done in each process.

Isobaric process

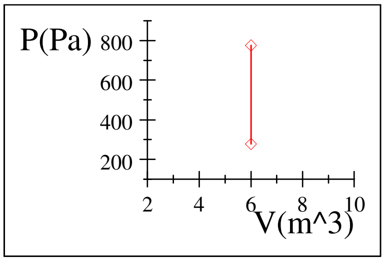


For an isobaric process, we can see that our integral becomes very simple

$$\begin{aligned} w_{int} &= -\int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V} \\ &= -P\int_{\cancel{V}_i}^{\cancel{V}_f} d\cancel{V} \\ &= -P\left(\cancel{V}_f - \cancel{V}_i\right) \end{aligned}$$

where P is constant.

Isochoric (isovolumetric) process

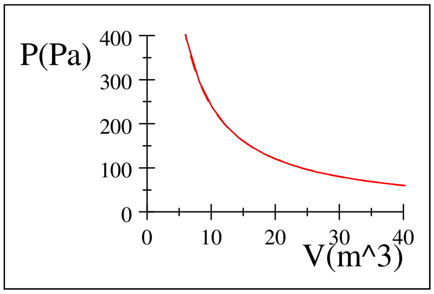


In this case the integral is even easier. The volume does not change. The area under the curve is zero.

$$\begin{aligned} w_{int} &= -\int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V} \\ &= 0 \end{aligned}$$

This means no work is done in an isovolumetric process. This makes some sense. We have neither compressed the gas, nor have we expanded the gas. Think of $\overrightarrow{F} \cdot d\overrightarrow{r}$. Since $d\mathbf{r}$ is zero, the work must be zero.

Isothermal process



For an isothermal process, the temperature does not change, but both the volume and the pressure change. We can rewrite the ideal gas law as

$$P = \frac{nRT}{\cancel{V}} \tag{26.11}$$

Performing our work integral for a gas going from some initial volume $\cancel{V}_i$ to a final volume, $\cancel{V}_f$.

$$\begin{aligned} w_{int} &= -\int_{\cancel{V}_i}^{\cancel{V}_f} P d\cancel{V} \qquad (26.12) \\ &= -\int_{\cancel{V}_i}^{\cancel{V}_f} \frac{nRT}{\cancel{V}} d\cancel{V} \qquad (26.13) \\ &= -nRT \int_{\cancel{V}_i}^{\cancel{V}_f} \frac{1}{\cancel{V}} d\cancel{V} \qquad (26.14) \\ &= -nRT \left[\ln \frac{\cancel{V}_f}{\cancel{V}_i} \right] \qquad (26.15) \end{aligned}$$

then for an isothermal process

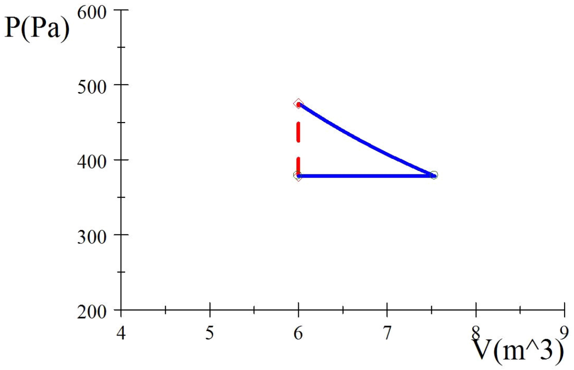
$$w_{int} = nRT\left[\ln\frac{\cancel{V}_i}{\cancel{V}_f}\right] \tag{26.16}$$

So if $\cancel{V}_f > \cancel{V}_i$ the work is negative and if $\cancel{V}_f < \cancel{V}_i$ the work is positive.

Path Dependence

We should do a problem that shows the path dependence of thermodynamics work. Notice that how we compress the gas (quickly vs. slowly, etc.) makes a difference on our PV diagram. For example, quick actions would not let energy leave by heat, so temperature would rise. Slow actions would allow enough time for energy to leave, allowing the temperature to stay the same. So when we say there is a "path dependence" for thermodynamic work, it is equivalent to saying we have different amounts of work for "different paths" on a PV-diagram. The PV-diagram shows the different physical paths as different lines on the graph. Here is an example.

Suppose we change $1\,\mathrm{mol}$ of an ideal gas from one state to another by two different paths. The two paths are shown in the next figure.



We recognize the blue path as a combination of an isobaric and an isothermal process. So for the blue path, the work done will be a combination of

$$\begin{aligned} w_{isobaric} &= -P\left(\cancel{V}_f - \cancel{V}_i\right) \\ w_{isothermal} &= nRT\left[\ln\frac{\cancel{V}_i}{\cancel{V}_f}\right] \end{aligned}$$

Then

$$\begin{aligned} w_{blue} &= -P\left(\cancel{V}_f - \cancel{V}_i\right) + nRT\left[\ln\frac{\cancel{V}_i}{\cancel{V}_f}\right] \\ &= -378.29\,\mathrm{Pa}\left(7.5384\,\mathrm{m}^3 - 6\,\mathrm{m}^3\right) \\ &\quad + (1)\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(343\,\mathrm{K})\left[\ln\frac{7.5384\,\mathrm{m}^3}{6\,\mathrm{m}^3}\right] \\ &= 650.9\,\mathrm{J} \end{aligned}$$

But the red path is an isovolumetric path. We know that for isovolumetric paths the work is zero!

$$\begin{aligned} w_{red} &= -\int_{\cancel{V}_1}^{\cancel{V}_2} P d\cancel{V} \\ &= 0 \end{aligned}$$

and we can see that zero is very different than $651\,\mathrm{J}$! Indeed the work for each PV-diagram path is different. But why?

The red path is what we would get if we put a sealed strong container in a fire. The fire would provide energy to the gas in the container by heat. But the strong walls of the container would keep the gas from expanding. The pressure would rise and the temperature as well. But for the blue path we envision a movable piston in a cylinder of gas. when the fire adds energy by heat, the piston rises, keeping the pressure the same. The temperature does go up. Then we remove the fire, and slowly compress the gas back to its original volume. By doing this slowly the work we do adds a little bit of energy to the gas, but the energy can leave by heat, so the temperature of the gas stays the same.

Note that these are really different processes! So it is no wonder that the internal work done is different. And we see all of this with our PV-diagram!

27 Heat and the First Law of Thermodynamics

Last lecture we learned about work and the first law of thermodynamics. But in the first law there is the energy transfer by heat, Q. We need to understand this better. That is the subject of today's lecture.

Fundamental Concepts

- Doing work on a system can change the system's temperature. This is called the mechanical equivalent of heat.
- There is a sign convention for energy transfer. If energy leaves it is negative, if it comes into the system it is positive.
- In thermodynamics, we try to find ΔE_{int}, Q, and w for a process. We use the ideal gas law and the first law of thermodynamics to do this.
- We have special processes for which it is easy to find ΔE_{int}, Q, and w
- We are still missing something.

Heat and Internal Energy

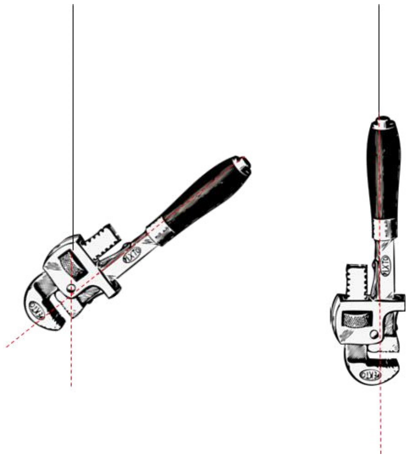
Question 123.11.1

We learned in PH 121 what energy was. We studied work and potential energy, and we studied kinetic energy. But these were always the energy *of* an object. We described these energies in terms of the motion of the center of mass of an object using the particle model. Now we know that there is energy *in* an object. This is different. Particles don't have internal parts, but real things do. So we need to go beyond the particle model here. We will call the energy associated with the inside parts of an object *internal energy.* And this internal energy is associated with thermal energy.

To study this, Let's refine our definition of internal energy by saying that it is all the energy of a system that is associated with its microscopic components – atoms and molecules – when viewed form a reference frame at rest with respect to the center of mass of the system. Any motion of the center of mass is our PH121 mechanical energy. So we want the motion of the parts of the object with respect to the center of mass. Let's review center of mass briefly.

We learned about center of mass in PH 121.

$$r_{cm} = \frac{1}{M}\int \mathbf{r}dm \tag{27.1}$$



For groups of molecules, we remember

$$\overrightarrow{r}_{cm} = \frac{\sum_i m_i \overrightarrow{\mathbf{r}}_i}{M} \tag{27.2}$$

where

$$M = \sum_i m_i \tag{27.3}$$

You might remember this in one dimension for two masses

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{27.4}$$

Now suppose we let all of the masses involved be tied together with spring-like forces. These spring-like forces would allow the masses to oscillate a little. That would change the $\mathbf{r}_i$ involved in our center of mass calculation. But if the oscillations were random, the actual center of mass would not change. This is the kind of motion involved in internal energy.

By making all the little molecules move, we have changed the energy state of the object, but we have not moved it, so the mechanical energy has not changed. The energy we have described is the thermal internal energy (there is still the nuclear internal energy, but that is a subject for PH279). Let's consider forms of internal energy in more detail in this lecture.

Heat

We have mentioned heat, and given it a symbol Q, but in this lecture we want to refine our definition of heat. Heat is defined as the transfer of energy across the boundary of a system due to a temperature difference between the system and its surroundings. When we “heat” a substance, we add energy from it’s surroundings. An object could gain or lose energy *by heat*. We will use the word “heat” for the transfer of energy no matter which direction the energy is flowing. This is a little like using the word “acceleration” no matter whether we are speeding up or slowing down.

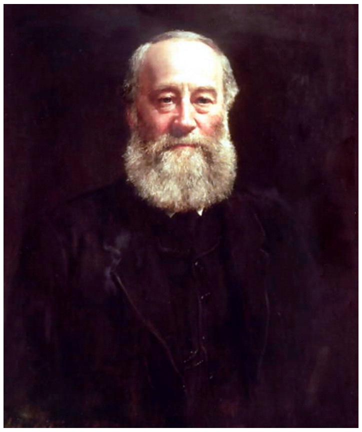
In the dim past, scientists thought heat was a fluid (called caloric). This model of heat is not correct (science changes!). We now define heat as a transfer of energy. But because of this history, we have some left-over names that are not very descriptive of the modern ideas they represent. Examples are *latent heat* and *heat capacity*.

The thing to remember is that a system *has* a temperature but it can *give or receive* energy by heat.

Units

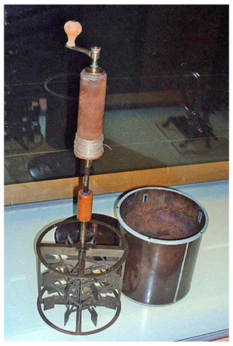
There are several units for heat. The calorie (cal) is the amount of energy transfer necessary to raise the temperature of $1\,\mathrm{g}$ of water from $14.5\,^{\circ}\mathrm{C}$ ($287.65\,\mathrm{K}$) to $15.5\,^{\circ}\mathrm{C}$ ($288.65\,\mathrm{K}$). This is somewhat arbitrary, but it works for many things, especially biological systems. The British thermal unit (Btu) is the unit used by refrigeration and heating contractors. One Btu is the amount of energy transfer necessary to raise the temperature of $1\,\mathrm{lb}$ of water from $63\,^{\circ}\mathrm{F}$ ($290.37\,\mathrm{K}$) to $64\,^{\circ}\mathrm{F}$ ($290.93\,\mathrm{K}$). This is an enormous amount of energy being transferred! If you wish to heat an entire building, this is a fine unit, but for us it is a bit large. In the SI system we already have a unit for energy, the Joule. We will try to stick to this nice, medium sized, unit. But let’s see where it came from.

Mechanical equivalent of heat



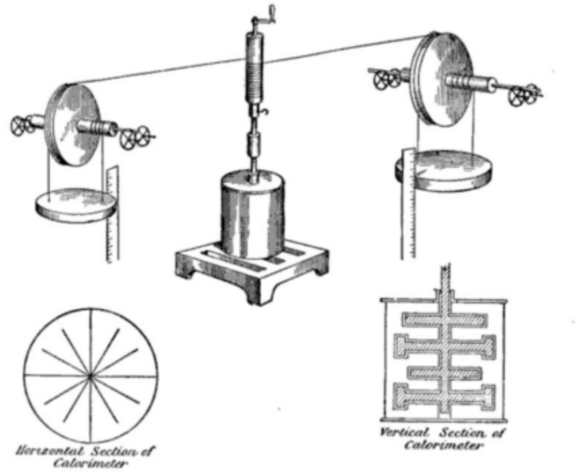
James Prescott Joule. (Image in the Public Domain)

the name "Joule" is a person's name, Joule was an early researcher and is credited with showing that heat transfer is a transfer of energy. Joule was able to build a device to prove that mechanical energy could be converted to heat energy. His device is shown in the picture.



Joule Apparatus (Image in the Public Domain courtesy Gaius Cornelius)

The paddle device was placed in the drum that is sitting behind it. The crank was operated by strings attached to it that were pulled by masses tied to the other end of the string (see the next figure). When the masses were released, the paddle wheel stirred a liquid. The friction from the paddle wheel stirring the liquid raised the temperature of the liquid.



Joule's Water-Churning Apparatus for Determining the Mechanical Equivalent of Heat.

The masses would loose energy equal to

$$\Delta U = 2mgh \tag{27.5}$$

assuming the masses were equal. By carefully measuring the temperature of the water, Joule found that

$$\Delta U = C\Delta T \tag{27.6}$$

with the proportionality constant approximately

$$C = 4.18\frac{\mathrm{J}}{\mathrm{g}}\,{}^{\circ}\mathrm{C} \tag{27.7}$$

which means that $4.18\,\mathrm{J}$ of mechanical energy raises the temperature of $1\,\mathrm{g}$ of water by $1\,{}^{\circ}\mathrm{C}$. (Here C is a generic symbol for a constant). We have more precise measurements now that set this value at $4.186\frac{\mathrm{J}}{\mathrm{g}}\,{}^{\circ}\mathrm{C}$. But you can see that Joule did very good for working with what amounts to a fancy butter churn in the 1800's!

Notice that we are considering raising the temperature of $1\,\mathrm{g}$ of water by $1\,{}^{\circ}\mathrm{C}$. This is the basis of the unit called the calorie. Using the same $4.5\,{}^{\circ}\mathrm{C}$ ($287.\,65\,\mathrm{K}$) to $15.5\,{}^{\circ}\mathrm{C}$ ($288.\,65\,\mathrm{K}$) change in temperature, we realize

$$1\,\mathrm{cal} = 4.186\,\mathrm{J} \tag{27.8}$$

which gives us a convenient way to convert from calories to Jules. But more importantly, it shows that heat is really tied to energy, not to some substance. Work can raise a temperature in a system just like "heating" the system with a fire or stove.

Because of this experiment by Joule, this idea that work can be turned into thermal energy is known as the *mechanical equivalent of heat*.

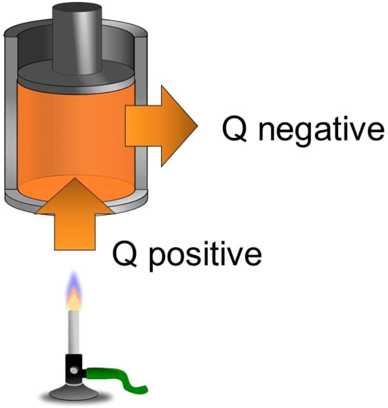
Heat and systems

Heat is a transfer of energy, so it can't be a property of the system, itself. It is an interaction of the system with its environment. This means that heat is not a state variable. It will depend on process path on a PV-diagram, like work does. Only the combination

$$\Delta E_{int} = Q + w$$

is path dependent.

It is customary to refer to energy gained from an environment as positive heat, and energy lost to the environment as negative heat.



This is a sign convention. We could totally define this the other way. But it is useful for everyone to agree to one system of signs, and the system used in physics today is that energy is negative when it leaves a system, and positive when it enters a system. We will need to remember this for our problems.

It is useful to compare the sign convention for heat with that we built for work (on the gas).

	Work	**Heat**
Interaction mechanism	Mechanical	Thermal
Process	Macroscopic forces acting on the system	Microscopic collisions between gas particles in the system
Process Requires	External force and displacement	Temperature Difference
Positive when	A gas is compressed: mechanical energy is transferred in to system	The environment is at a higher temperature than the system
Negative when	A gas expands: mechanical energy is transferred out of the system	The environment is at a lower temperature than the system
Equilibrium condition	No net force or torque on the system	System and environment are at the same temperature

Just like with work, the definition of the word "heat" takes some practice to get used to. Question 123.11.2

It is really important to not use the word "heat" as we use the word with "temperature." Question 123.11.3
Temperature is proportional to the internal energy, Question 123.11.4

$$E_{int} \propto T$$

as we will show soon. But Q is a transfer of energy. A gas can "have" a temperature, but it "experiences" a transfer of energy. That is why it shows up in the first law as a mechanism for the change in internal energy.

$$\Delta E_{int} = Q + w$$

This is not intuitive because of our left-over language from the caloric theory. So we must continually be on our guard, because poor language leads to poor decisions in experiments and engineering designs.

It would help if we all said things like "to do heat" instead of "heat" when we are warming things. That is what we do with work. We "do work" so we should "do heat," and this may be a good way to think about it, even though to say "do heat" like this would make our roommates wonder. Question 123.11.5

We now know that ΔE_{int} is a state variable. It does not depend on path. But it is just one of many state variable for a gas. For ideal gasses, we have

$$P, \cancel{V}, n, T, \Delta E_{int}$$

all as state variables. It takes all of these quantities to describe an ideal gas.

For example, we know that

$$E_{int} \propto T$$

but this does not tell us ΔE_{int}. The quantity ΔE_{int} only tells us about the change in

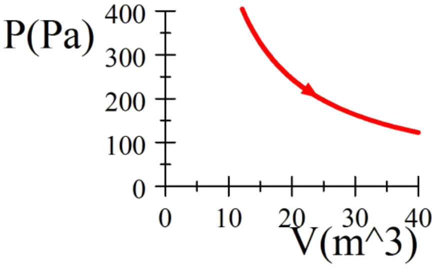
E_{int}. We cannot tell what path was used to accomplish ΔE_{int}. So our first law does not tell us P, $\rlap{-}V$, or n either. We will have to use the first law in cooperation with the ideal gas law.

Special processes

Armed with both the ideal gas law, and the first law of thermodynamics, let's review our special processes–and maybe gain another! Our goal is to see how these processes relate to the first law.

Constant Temperature Process

For an isothermal process



we expect that

$$\Delta E_{int} \propto \Delta T = 0$$

The work we found to be

$$\begin{aligned} W &= -\int P d\rlap{-}V \\ &= nRT\left[\ln\frac{\rlap{-}V_i}{\rlap{-}V_f}\right] \end{aligned}$$

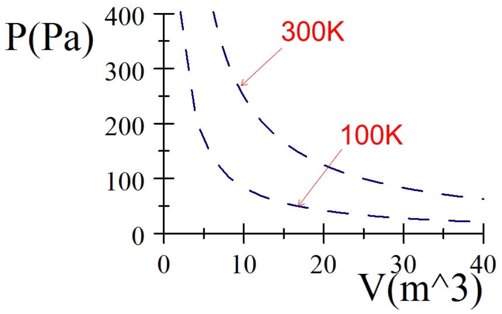
and from the first law

$$\begin{aligned} \Delta E_{int} &= Q + w \\ 0 &= Q + nRT\left[\ln\frac{\rlap{-}V_i}{\rlap{-}V_f}\right] \end{aligned}$$

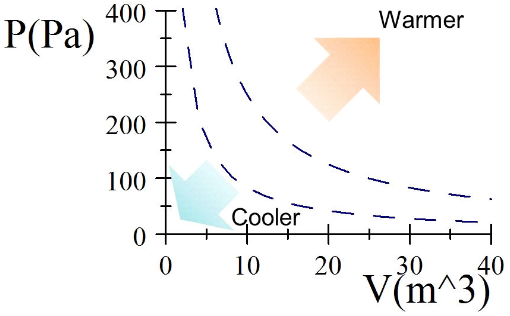
so

$$Q = nRT\left[\ln\frac{\rlap{-}V_f}{\rlap{-}V_i}\right]$$

Looking at an isothermal process can help us understand our other processes. Suppose we have two different isothermal processes, one at $100\,\mathrm{K}$ and one at $300\,\mathrm{K}$. The isotherms look like this on a $P\rlap{-}V$ diagram:



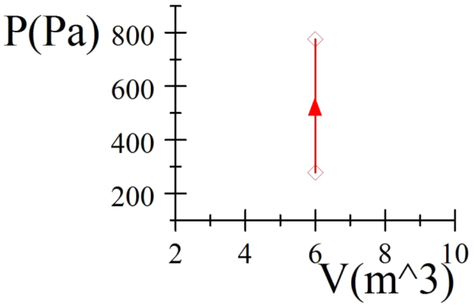
Every point on the 300 K line has the same temperature. Every point on the 100 K line has the same temperature. We can see that points on the $P\mathcal{V}$ diagram that are closer to the origin must have lower temperatures. We can see the points farther from the origin have higher temperatures.



Constant volume process

Question 123.11.6

We know an isochoric process takes us between two states on a $P\mathcal{V}$ diagram as shown in the next figure.



We can see that the temperature will change, and since

$$E_{int} \propto T$$

we expect that

$$\Delta E_{int} \propto \Delta T$$

so we expect

$$\Delta E_{int} \neq 0$$

The work done is

$$W = -\int P d\not{V}$$

but the volume does not change, so

$$W = 0$$

from the first law, we can see that

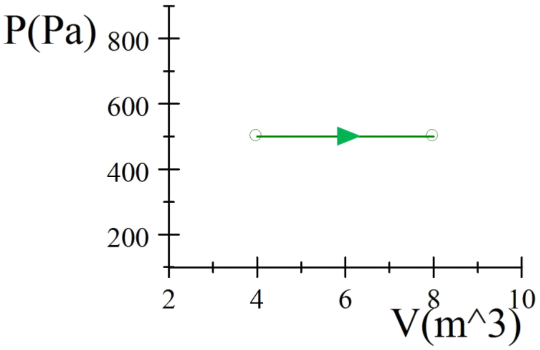
$$\begin{aligned} \Delta E_{int} &= Q + w \\ &= Q + 0 \end{aligned}$$

so

$$Q = \Delta E_{int}$$

Constant Pressure process

Consider the next $P\not{V}$ diagram.



Again we see that the temperature has changed, so

$$\Delta E_{int} \propto \Delta T$$

and we expect that

$$\Delta E_{int} \neq 0$$

The work done is

$$\begin{aligned} W &= -\int P d\rlap{-}V \\ &= P\left(\rlap{-}V_2 - \rlap{-}V_1\right) \end{aligned}$$

From the first law

$$\begin{aligned} \Delta E_{int} &= Q + w \\ &= Q - P\left(\rlap{-}V_2 - \rlap{-}V_1\right) \end{aligned}$$

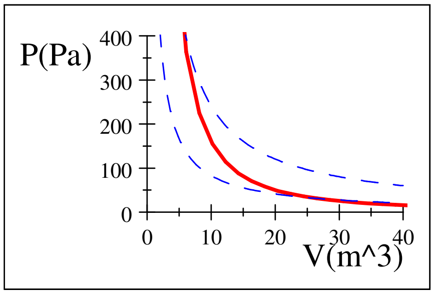
so

$$Q = \Delta E_{int} + P\left(\rlap{-}V_2 - \rlap{-}V_1\right)$$

but we don't have enough information to obtain a numerical value. We must be missing something!

Adiabatic process

We have a process where $w = 0$. It seems reasonable that we would also have a process where $Q = 0$. But none of our special process do that so far. Let's invent a new process such that $Q = 0$ and call it by the name *adiabatic*. The PV diagram looks like this.



The blue dashed lines are isotherms. Notice that the temperature is changing along the adiabatic path. That implies that

$$\Delta E_{int} \propto \Delta T \neq 0$$

but we defined this path such that

$$Q = 0$$

so by the first law

$$\begin{aligned} \Delta E_{int} &= Q + w \\ &= 0 + w \\ \Delta E_{int} &= w \end{aligned}$$

We again don't have enough information to tell what work was done. We suspect there is something missing. We will start to fill in the missing piece in the next lecture.

28 Thermal Properties of Matter

Last lecture we found we can't quite finish the job of finding ΔE_{int}, Q, and w for our special thermodynamic processes. The problem is that although we have a strategy for finding w, we don't have such a strategy for Q. We will start to fill that void in this lecture.

Fundamental Concepts

1. Heat capacity relates energy transfer by heat to temperature change in solids and liquids
2. Specific heat is a heat capacity per unit mass
3. A phase change is changing from solid to liquid, or from liquid to gas for a sample of a material
4. Calorimetry is the use of thermal conservation of energy to study a sample of material

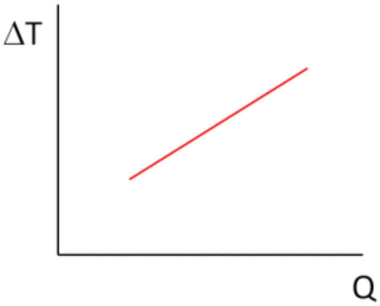
Question 123.12.1

Question 123.12.1.1

Our goal in this lecture is to find the missing piece that allows us to find ΔE_{int}, W, and Q for our systems–at least for our special processes. We have been looking at ideal gases, but let's go back to solids and liquids for a moment and see if we can gain a clue experimentally from these simpler cases.

Suppose you are heating something on your stove, say, a sample of steak.

You would expect that as energy is transferred from the hot stove to the cold beef, the temperature of the steak would increase. It makes sense that the more energy you transfer, the more the temperature will rise. Gasses are more complicated, because their volume will change, but the volume of solids does not change much with temperature (we know that it changes a little) so we don't have as many complications as we would with a gas.



We could guess that the temperature increase might be linear in the amount of energy transfer we give to the solid.

$$\Delta T \propto \Delta E_{int}$$

where our only form of increasing the energy here is Q, since the change in volume of the stake sample is small enough that any work, w, is negligible. Since P and V̶ are not changing much,

$$\Delta T \propto Q$$

Experimentalists did this experiment and found that for many temperature regions this is true. Of course, we could over heat our sample

changing it's form (from stake to charcoal, and smoke), but then the sample is no longer the same substance. In this case ΔT is no longer proportional to Q. But as long as we don't change the composition of the substance (we stay in the linear regime) , we can say that ΔT is proportional to Q.

Physicists don't like proportionality signs, so they invented a constant with the right units that would make this an equality

$$\Delta T = \frac{1}{K} Q$$

But, they found that the internal structure of the solid mattered. There was a different constant, K, for every material.

They called the constant, K, the *heat capacity* of the material. The name is another throwback to the past, but for us it means the constant of proportionality between Q and ΔT that holds all the specific material properties of that substance that change the slope of the ΔT vs. Q graph.

Of course, the more material we have in our sample, the longer it takes to heat it up. If we have a roast, it takes longer to cook than a small steak. So it is customary to define a heat capacity per unit mass, this is called the *specific heat* of the material.

$$\Delta T = \frac{1}{mc} Q$$

and we often rearrange this equation to give

$$Q = mc\Delta T$$

As an example, it takes $4190\,\mathrm{J}$ to raise $1\,\mathrm{kg}$ of water by $1\,\mathrm{K}$. So

$$4190\,\mathrm{J} = (1\,\mathrm{kg})\,(c)\,(1\,\mathrm{K})$$

solving for c yields

$$c_{water} = 4190 \frac{\mathrm{J}}{\mathrm{kg\,K}}$$

Which is familiar to us from Joule's work.

Here are a few more substances and their specific heat values.

Substance	$c\left(\frac{\mathrm{J}}{\mathrm{kg\,K}}\right)$	$c\left(\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)$
Aluminum	900	24.3
Copper	385	24.4
Iron	449	25.1
Gold	129	25.4
Lead	128	26.5
Ice	2090	37.6
Mercury	140	28.1
Water	4190	75.4

Notice that things that heat up or cool down quickly have small specific heats. Things that heat up slowly or cool slowly have large specific heats. Water in any of it's phases has large values of specific heat. This is why living next to a lake or the ocean keeps your temperatures moderate. It takes quite a lot of energy transfer to heat up or cool off the water, so the temperature of the water does not change much. This is why Buffalo temperatures lower when the lake freezes. This is also the source of lake or ocean breezes. In the morning, the land changes temperature more quickly than the water. The warmer air over the land becomes less dense and rises, leaving a lower pressure area. This causes a cool breeze to form from the sea to the land.

Conversely, in a large city, the concrete and blacktop heat up much quicker, so we experience increases in daytime temperature as a city grows.

A specific heat is a per-unit-mass property, but we could just as well define specific heat as a per-unit-mole quantity. Then the specific heat would be the amount of energy it takes to increase the temperature of a mole of material. This incarnation of the idea of specific heat is called a *molar specific heat,* and we will give it the symbol C.

$$Q = nC\Delta T$$

It is worth noting that the molar heat capacities given in the last table are remarkably alike. Except for water, which is a compound, they are all about $25\,\mathrm{J}/\left(\mathrm{mol\,K}\right)$. This gives us a clue towards understanding what is happening microscopically, but we will have to wait to investigate this clue until a further lecture.

WARNING: Specific heat values are really not constant. They do vary with temperature a little. We know from our study of thermal expansion that the volume of a sample of a material will change a small amount as it is heated or cooled. There is a small amount of work done in changing the volume of the material. Not all of the energy will go into changing the temperature. So our heat capacity will change a little as the volume changes. But if the change in temperature is not too big, we can treat heat capacities for solids and liquids as constants. Oh, but for gasses...we know the volume might change quite a lot as the temperature changes for a gas, so we will find that this analysis needs a few changes for gasses. We will study this is a later lecture.

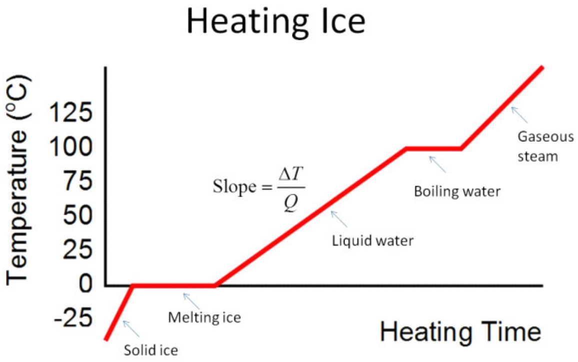
Note that water has a very high heat capacity. This explains our lake effect that we discussed (areas near lakes do not get as cold in the winter). When the water changes temperature, ΔT, the air around the lake receives a large Q.

Phase changes and heat of transformation

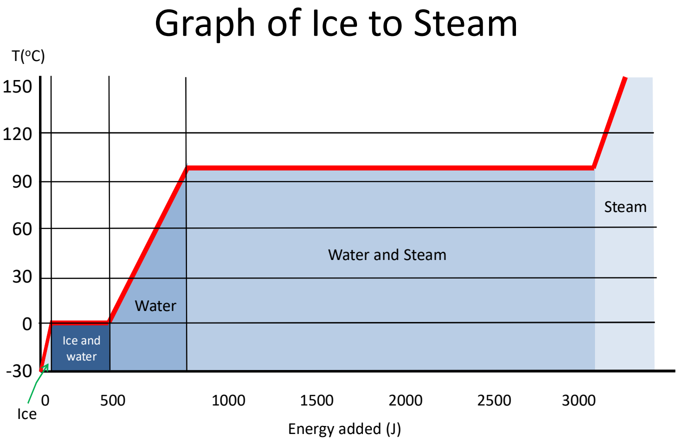
Question 123.12.2

Question 123.12.3

Recall the graph of temperature vs. time for heating a chunk of Rexburg ice.



Here is the graph again, but this time the horizontal axis is labeled “Energy added” because we realize that we are adding internal energy as we wait and watch the ice.



We see regions of the graph where the ice changes temperature, and we can understand that during these times the temperature gain will be proportional to the amount of energy transferred by heat.

$$\text{Slope} = \frac{\Delta T}{Q} = \frac{1}{Mc}$$

But what about the horizontal parts of the graph? We identified these before as phase changes. The slope is zero during the melting and boiling times.

We have an amount of energy transferred by heat. You might guess that how much energy it takes for the phase change to completely change from one state to the other depends on the strength of the bonds of the material. The added energy is going into breaking those bonds. So the amount of energy it takes to, say, melt different substances will be different. From experience we know that it takes more energy to heat large amounts of material than small amounts of material. We can express this mathematically as

$$Q = \pm LM$$

where L is the symbol we give to the constant that expresses the how easily the material melts or freezes. We need at least two of these, one for melting/freezing and one for boiling/condensing. Then

$$\begin{aligned} Q &= \pm L_f M \qquad \text{melt/freeze} \\ Q &= \pm L_v M \qquad \text{boil/condense} \end{aligned}$$

where we call these two constants

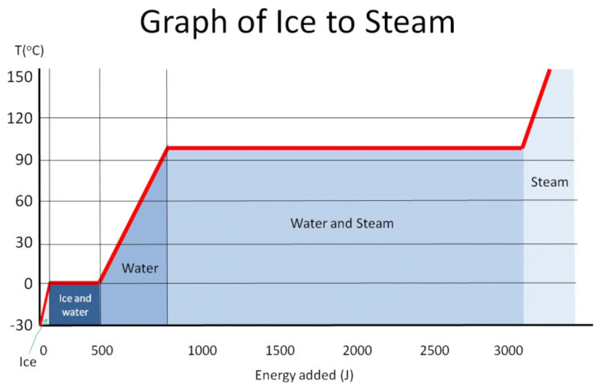
L_f heat of fusion
L_v heat of vaporization

generically we call each of these a *heat of transformation* but generally they are still referred to by their old title of "Latent heat."

Substance	$T_m\ (^\circ\text{C})$	$L_f\left(\frac{\text{J}}{\text{kg}}\right)$	$T_m\ (^\circ\text{C})$	$L_v\left(\frac{\text{J}}{\text{kg}}\right)$
Nitrogen (N_2)	−210	0.26×10^5	−196	1.99×10^5
Ethyl alcohol	−114	1.09×10^5	78	8.79×10^5
Mercury	−39	0.11×10^5	357	2.96×10^5
Water	0	3.33×10^5	100	22.6×10^5
Lead	328	0.25×10^5	1750	8.58×10^5

Notice that our equations each have a "±" sign in them. We must supply the sign by context. Remember that energy that leaves is negative and energy that is gained by the system is positive. Also notice that so long as we warm the ice quasi-statically, that the ice completely melts before the temperature of the water starts to rise.

Let's do a problem. Suppose we want to increase the temperature of $1\ \text{g} = 0.001\ \text{kg}$ of ice at $-30.0\,^\circ\mathrm{C}$ and change it to steam at $120.0\,^\circ\mathrm{C}$. A graph of temperature vs. heat energy is shown. Let's take it one piece at a time.



Part A: Warming up the ice to $0\,^\circ\mathrm{C}$.

This part acts as we learned in the last section. We have

$$\begin{aligned} Q &= m_i c_i \Delta T \\ &= (0.001\ \mathrm{kg})\left(2090\frac{\mathrm{J}}{\mathrm{kg}\,^\circ\mathrm{C}}\right)(0\,^\circ\mathrm{C} + 30\,^\circ\mathrm{C}) \\ &= 62.\,7\,\mathrm{J} \end{aligned}$$

We can get away with using $^\circ\mathrm{C}$ because we only need ΔT and the divisions of the Celsius and Kelvin scales are the same size.

Part B: Melting ice

Now the ice changes to water. We have a heat of transformation. To find out what happens we use the equation

$$\begin{aligned} Q &= m_i L_f \\ &= (0.001\ \mathrm{kg})\left(3.33 \times 10^5 \frac{\mathrm{J}}{\mathrm{kg}}\right) \\ &= 333.0\,\mathrm{J} \end{aligned}$$

The total energy so far is $62.\,7\,\mathrm{J} + 330.0\,\mathrm{J} = 392.\,7\,\mathrm{J}$

Part C: Warming the melted ice (water)

Again we have the normal case where we can use the specific heat of water (now that the ice has melted)

$$\begin{aligned} Q &= m_w c_w \Delta T \\ &= (1\,\mathrm{g}) \left(4190 \frac{\mathrm{J}}{\mathrm{kg}\,^{\circ}\mathrm{C}}\right) (100\,^{\circ}\mathrm{C} - 0\,^{\circ}\mathrm{C}) \\ &= 419.0\,\mathrm{J} \end{aligned}$$

The total energy so far is $62.\,7\,\mathrm{J} + 333.0\,\mathrm{J} + 419.\,0\,\mathrm{J} = 814.\,7\,\mathrm{J}$

Part D: Boiling the melted ice (water)

Now we convert the water to steam.

$$\begin{aligned} Q &= m_w L_v \\ &= (0.001\ \mathrm{kg}) \left(2.26 \times 10^6 \frac{\mathrm{J}}{\mathrm{kg}}\right) \\ &= 2260.0\,\mathrm{J} \end{aligned}$$

The total energy so far is $62.\,7\,\mathrm{J} + 333.0\,\mathrm{J} + 419.\,0\,\mathrm{J} + 2260.0\,\mathrm{J} = 3074.\,7\,\mathrm{J}$

Part E: Warming the steam

Now we have steam, and can use the specific heat of steam (we don't usually do this! we usually use the ideal gas law when it becomes a gas!)

$$\begin{aligned} Q &= m_s c_s \Delta T \\ &= (0.001\ \mathrm{kg}) \left(2010 \frac{\mathrm{J}}{\mathrm{kg}\,^{\circ}\mathrm{C}}\right) (120\,^{\circ}\mathrm{C} - 100\,^{\circ}\mathrm{C}) \\ &= 40.\,2\,\mathrm{J} \end{aligned}$$

The total energy is $62.\,7\,\mathrm{J} + 333.0\,\mathrm{J} + 419\,\mathrm{J} + 2260.0\,\mathrm{J} + 40.\,2\,\mathrm{J} = 3114.\,9\,\mathrm{J}$

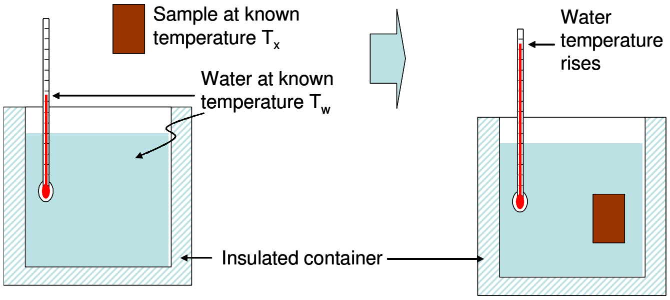
Calorimetry

Question 123.12.4

Question 123.12.5

We can now calculate results for many common experiences. We have all experienced eating something with a large temperature. The solution is to drink something cold. This lowers the temperature. We would like to be able to quantify this. But our mouth is not a great device to use to do quantitative analysis on temperature changes. Let's consider a more suitable apparatus.

Suppose we have an insulated container with a thermometer



Inside of this container let's place a known mass of water and measure the temperature, T_w. Then we can introduce our sample of hot material. Suppose we know that the hot material has a temperature T_x.

Let's assume that the container walls are perfectly insulating, then knowing the sample temperature T_x and the water temperature T_w with $T_w < T_x$ we can find c or K.(or even C). We recognize for this ideal system no energy can leave, because of the insulation. So

$$Q_{\text{cold}} = -Q_{\text{hot}} \tag{28.1}$$

or energy would not be conserved. Then for the water

$$Q_w = m_w c_w \left(T_f - T_w\right)$$

and for the sample

$$Q_x = m_x c_x \left(T_f - T_x\right)$$

so, using energy conservation

$$\begin{aligned} -Q_x &= Q_w \\ -m_x c_x \left(T_f - T_x\right) &= m_w c_w \left(T_f - T_w\right) \end{aligned}$$

and we can solve for c_x

$$c_x = -\frac{m_w c_w \left(T_f - T_w\right)}{m_x \left(T_f - T_x\right)} \tag{28.2}$$

Note the sign convention! If energy leaves the object, the value of Q is negative. Our denominator is, indeed, negative because for the sample T_f is less than T_x.

Let's try a second problem. Suppose we have $0.5\,\text{kg}$ of ice at $0\,^{\circ}\text{C}$ and $1\,\text{kg}$ of water at $50\,^{\circ}\text{C}$. What is the final temperature?

We will need to know $L_f = 3.33 \times 10^5 \frac{\text{J}}{\text{kg}}$ and $c_w = 4190 \frac{\text{J}}{\text{kg K}}$. Again

$$Q_{\text{cold}} = -Q_{\text{hot}}$$

we identify the hot thing as the water and the cold thing as the ice. The water only changes temperature, but the ice experiences a phase change

$$\begin{aligned} Q_{\text{ice}} &= -Q_{\text{water}} \\ L_f M + m_i c_w \left(T_f - T_0\right) &= -m_w c_w \left(T_f - T_w\right) \end{aligned}$$

The final temperature will be the same for both

$$\begin{gathered} L_f M + m_i c_w T_f - m_i c_w T_0 = -m_w c_w T_f + m_w c_w T_w \\ L_f M - m_w c_i T_0 - m_w c_w T_w = -m_w c_w T_f - m_i c_w T_f \\ L_f M - \left(m_i c_w T_0 + m_w c_w T_w\right) = -\left(m_w c_w + m_i c_w\right) T_f \\ T_f = \frac{\left(m_i c_w T_0 + m_w c_w T_w\right) - L_f M}{\left(m_w c_w + m_w c_w\right)} \end{gathered}$$

or

$$\begin{aligned} T_f &= \frac{\left((1\,\text{kg})\left(4190\frac{\text{J}}{\text{kg K}}\right)(273\,\text{K}) + (1\,\text{kg})\left(4190\frac{\text{J}}{\text{kg K}}\right)(273+50)\ \text{K}\right) - \left(3.33\times10^5\frac{\text{J}}{\text{kg}}\right)(1\,\text{kg})}{\left((1\,\text{kg})\left(4190\frac{\text{J}}{\text{kg K}}\right) + (1\,\text{kg})\left(4190\frac{\text{J}}{\text{kg K}}\right)\right)} \\ &= 258.26\,\text{K} \end{aligned}$$

Now let's ask, is this reasonable?

The answer is no! Let's see why. We should have checked in advance to see if all the ice melts. It would take

$$Q = m_i L_f = \left(3.33\times10^5\frac{\text{J}}{\text{kg}}\right)(0.5\,\text{kg}) = 1.665\times10^5\ \text{J}$$

to melt all the ice. We have available from the warm water

$$\begin{aligned} Q &= m_w C_w \left(T_f - T_w\right) \\ &= (1\,\text{kg})\left(4190\frac{\text{J}}{\text{kg K}}\right)(273\,\text{K} - 323\,\text{K}) \\ &= -209\,500\,\text{J} \end{aligned}$$

that is, the warmer water can provide $209.5\,\text{J}$. But since $209.5\,\text{J} < 1.665\times10^5$ J the warm water does not have enough energy available to melt all the ice. Once the water is all at $0\,^\circ\text{C}$, the ice and water are in thermal equilibrium. No energy will be transferred. So our final temperature of the mixture is $273\,\text{K}$ or $0\,^\circ\text{C}$.

Of course if our system of ice and water isn't isolated, eventually all the ice will melt. But that would be due to a transfer of energy by heat from the outside environment.

This problem is a bit of a trick, but such a situation can really happen. So it is really important that we stop to consider whether our numerical answers are reasonable once we have done a calorimetry problem.

We have filled in the missing piece of information for solids and liquids. We can say that so long as we don't change the structure of the material,

$$Q = Mc\Delta T$$

or

$$Q = nC\Delta T$$

so we can find Q, w, and ΔE_{int} for solids and liquids But we said gasses will be more difficult. We will take up the relationship between energy transfer by heat and the change in temperature for gasses in our next lecture.

29 Specific Heat of Gases and Heat Transfer Mechanisms

Last lecture we found a way to find Q for solids and liquids, $Q = Mc\Delta T$ or $Q = nC\Delta T$. We need to do this for gasses so we can complete our calculations of the terms in the first law of thermodynamics. We will take on this task in this lecture.

Fundamental Concepts

1. There are two molar specific heats for gasses, C_V and C_P.
2. The two molar specific heats are related $C_P = C_V + R$
3. The ratio of the molar specific heats is $\gamma = C_P/C_V$
4. The change in internal energy can be expressed as $\Delta E_{int} = nC_V\Delta T$
5. For adiabatic processes we have two new equations $P_i V_i^\gamma = P_f V_f^\gamma$ and $T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$
6. For insolation the power transferred through the insolation material is given by $\mathcal{P} = \frac{Q}{\Delta t} = \frac{A(T_h - T_c)}{\sum_i R_i}$
7. The power transferred by radiation is given by $\mathcal{P} = \frac{Q}{\Delta t} = \sigma A e T^4$

Molar Specific Heat of an Ideal Gas

Question 123.13.1

Remember we said that we would not find gases in specific heat tables? We did find steam. But other gases were missing. The reason is that there is not one unique value for the energy transfer by heat associated with a single temperature change of an ideal gas. We know that Q is path dependent for gasses! But we can still use our formula if we make some limiting assumptions. We will limit ourselves to either isobaric or isovolumetric processes. For these specific cases we can write

$$Q = nC_V \Delta T \qquad \text{isovolumetric} \tag{29.1}$$

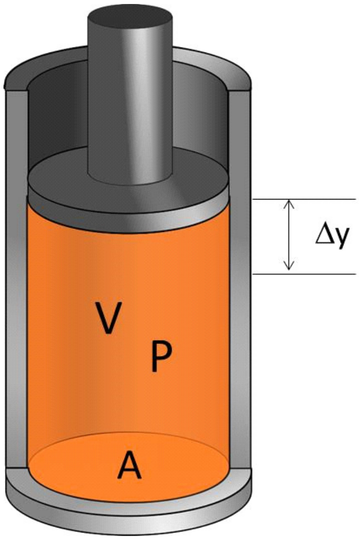
$$Q = nC_P \Delta T \qquad \text{isobaric} \tag{29.2}$$

Note that these are molar specific heats. The quantity n is the number of moles, not the mass. We won't define a regular specific heat for gasses.

This may seem very limiting, but think of our goal. We want to be able to consider P, $\cancel{V}$, n, T, ΔE_{int} and Q and W for each of our specific processes. These equations fill in the gap for two of those processes!

Molar Specific Heat for Constant pressure

Let's take an example. Think again of adding energy by heat to a system at constant pressure. Suppose we know P, n, and $\cancel{V}_i$ and T_i for the initial state, and $\cancel{V}_f$ and T_f for the final state. We want to know ΔE_{int}, Q, and w to finish our description of the process.



We now know that

$$Q = nC_P \Delta T$$

and we also know the work because we have calculated work before for an isobaric process

$$w = -P\left(\cancel{V}_f - \cancel{V}_i\right)$$

so from the first law

$$\Delta E_{int} = nC_P\Delta T - P\left(\cancel{V}_f - \cancel{V}_i\right)$$

and we have achieved our goal! We know Q, w, and ΔE_{int}.

C$_V$ for a monotonic gas

Let's try another example, this time for an isochoric process. So this time we know n, and $\cancel{V}$ and P_i, and T_i for the initial state and P_f and T_f for the final state. We wish to calculate Q, w, and ΔE_{int} for this process. We have calculated the work for an isochoric state before!

$$W = 0$$

so from the first law

$$\Delta E_{int} = Q$$

now we can add

$$\Delta E_{int} = nC_V\Delta T$$

Again we have achieved our goal. We have found Q, w, and ΔE_{int}.

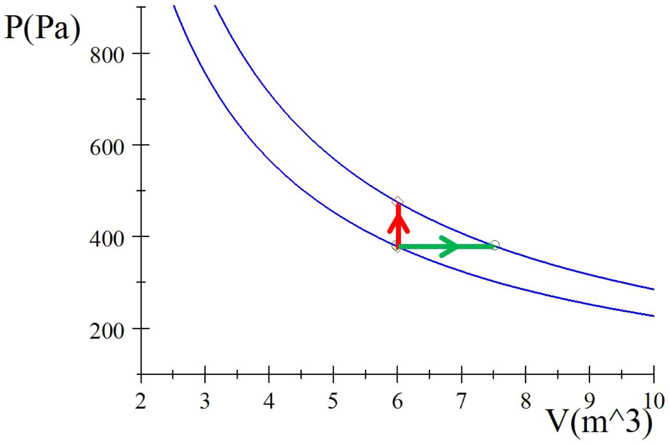
Relationship between C_V and C_P

Question 123.13.2

We noticed before that

$$\Delta E_{int} \propto \Delta T$$

This is really a key relationship in thermodynamics. We have not found the constant of proportionality yet (we will soon!). But this has a profound meaning that we can begin to use here. In the next figure, we have two isotherms (blue lines if you are seeing this in color)



There is also an isovolumetric path marked (red) and an isobaric path (green) Notice that these two paths take the systems between the same temperature change. ΔT. Then we expect they will have the same internal energy change, ΔE_{int}. For the isobaric path, we find that

$$\begin{aligned} Q_P &= \Delta E_{int} + P\Delta \not{V} \\ &= \Delta E_{int} + P\left(\not{V}_f - \not{V}_i\right) \end{aligned}$$

For the isochoric path we have just

$$\begin{aligned} Q_V &= \Delta E_{int} \\ &= \Delta E_{int} \end{aligned}$$

So the energy transfer by heat must supply both the internal energy change and the loss due to work for the isobaric path but only needs to supply the change in internal energy for the isochoric path.. Therefore, for the same n and ΔT

$$Q_{\text{constant } P} > Q_{\text{constant } V}$$

Since

$$\Delta E_{int} = nC_V \Delta T$$

for the isochoric path, and ΔE_{int} is the same for both paths, then we can set our two equations for ΔE_{int} equal to each other and write

$$nC_P\Delta T = nC_V\Delta T + P\left(\not{V}_f - \not{V}_i\right)$$

which gives

$$C_P = C_V + \frac{P\left(\not{V}_f - \not{V}_i\right)}{n\Delta T}$$

and therefore

$$C_P > C_V$$

We can go further, since, for an idea gas,

$$P\rlap{\,-}V = nRT$$

then

$$P\Delta \rlap{\,-}V = nR\Delta T$$

so

$$\begin{aligned} C_P &= C_V + \frac{P\Delta \rlap{\,-}V}{n\Delta T} \\ &= C_V + \frac{nR\Delta T}{n\Delta T} \\ &= C_V + R \end{aligned}$$

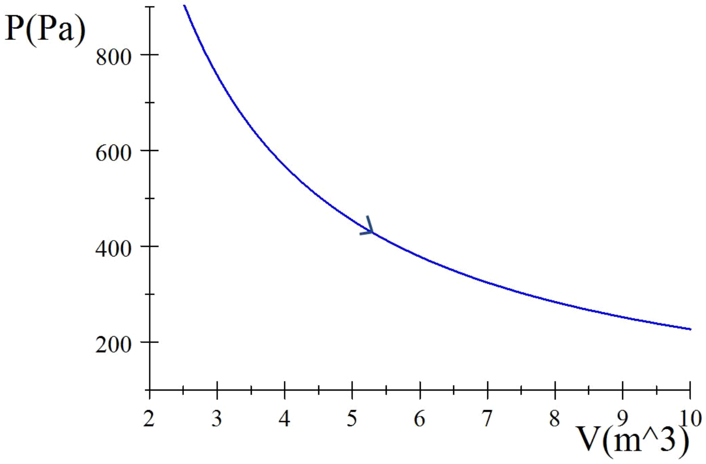
This is a big deal!

$$C_P = C_V + R \tag{29.3}$$

for an ideal gas. So if we know either C_P or C_V for a gas, we can always find the other. This works pretty well in practice for monotonic gasses (and even for diatomic gasses) at standard temperature and pressure. Not only can we find Q_V and Q_P with molar specific heats, but if we know one molar specific heat, we can easily calculate the other!

Isothermal process and Q_V

We now have a powerful new equation to add to our isovolumetric and isobaric processes. But what about an isothermal process?



Suppose we know T and n, and that we know P_i and $\rlap{-}V_i$ and P_f and $\rlap{-}V_f$. Can we find $\Delta E_{int,}$ Q, and w?

We already know how to find w

$$w = -nRT\left[\ln\frac{\rlap{-}V_f}{\rlap{-}V_i}\right]$$

and we know $\Delta E_{int} = 0$, so

$$\begin{aligned} Q &= -w \\ &= nRT\left[\ln\frac{\rlap{-}V_f}{\rlap{-}V_i}\right] \end{aligned}$$

and we did not need our molar specific heat equations! We have $\Delta E_{int,}$ Q, and w for three of our special processes. This completes three of our four special processes!

Adiabatic Processes for and Ideal Gas

With a complete set of equations for three of our four special processes, we should now review our new special process, the adiabatic process. It would be nice to be able to calculate ΔE_{int}, Q, and w. for this process. For and adiabatic process we have

1. No heat transfer $Q = 0$
2. $\Delta E_{int} = w$

It might see like we are home free, $Q = 0$! We don't need our molar specific heat equations for an adiabatic process either.

But we don't yet know how to find w or ΔE_{int} for an adiabatic process. We need to fix this problem. We could do this by finding an expression for the pressure as a function of volume along the adiabatic path, then using

$$w = -\int_{\rlap{-}V_i}^{\rlap{-}V_f} Pd\rlap{-}V$$

Since

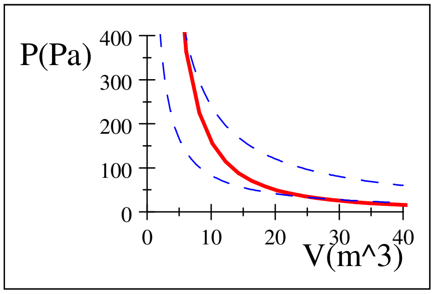
$$\begin{aligned} \Delta E_{int} &= w \\ &= -\int Pd\rlap{-}V \end{aligned}$$

this would give us both w and ΔE_{int} . But it is not obvious how to find P as a function of $\rlap{-}V$ for this path. We need another strategy.

Let's recall that an adiabatic process goes from one temperature to another temperature,

and

$$\Delta E_{int} \propto \Delta T$$



And further remember that ΔE_{int} does not depend on path. So we could find ΔE_{int} for an adiabatic path, and we could find ΔE_{int} for a isochoric path, and as long as the ΔT was the same for both paths, the ΔE_{int} would be the same as well! Also remember that for an isochoric path there is no work done. So

$$Q_{isochoric} = \Delta E_{int}$$

And now we know an equation for $Q_{isochoric}$!

$$Q_{isochoric} = nC_V \Delta T$$

So if we know T_i and T_f, then we could find ΔE_{int} by finding the equivalent $Q_{isochoric}$ for the same two temperatures.

$$\Delta E_{int} = nC_V \Delta T$$

This is tremendous! We have an equation for ΔE_{int}, and $w = \Delta E_{int}$, so with $Q = 0$ we have a complete set.

But it turns out that we can add two more equations to our adiabatic set that will really help. The first is a relationship between pressures and volumes along an adiabatic path. Let's obtain this equation.

We now know for adiabatic processes we can say

$$\Delta E_{int} = nC_V \Delta T$$

A very small change in the internal energy would be

$$dE = nC_V dT$$

Now for our adiabatic we previously noted that there is not energy transfer by heat done, so

$$\begin{aligned} \Delta E_{int} &= 0 + w \\ &= \int P d\!\!\!/V \end{aligned}$$

Then a very small amount of internal energy change, dE_{int} would be

$$dE_{int} = -Pd\mathcal{V}$$

Let's set these two expressions for dE_{int} equal to each other.

$$-Pd\mathcal{V} = nC_V dT$$

or

$$-\frac{Pd\mathcal{V}}{nC_V} = dT$$

Now we know the idea gas law

$$P\mathcal{V} = nRT$$

if we take a total differential for a sample of n moles of a gas

$$Pd\mathcal{V} + \mathcal{V}dP = nRdT$$

we can substitute in our expression we have for an adiabatic process for dT

$$Pd\mathcal{V} + \mathcal{V}dP = -nR\frac{Pd\mathcal{V}}{nC_V}$$

or

$$Pd\mathcal{V} + \mathcal{V}dP = -R\frac{Pd\mathcal{V}}{C_V}$$

We can use our relationship between C_V and C_P that we found in the last section

$$C_V = C_P - R$$

so

$$R = C_P - C_V$$

and let's substitute this in for our R in our total differential equation

$$Pd\mathcal{V} + \mathcal{V}dP = -\frac{C_P - C_V}{C_V}Pd\mathcal{V}$$

or

$$Pd\mathcal{V} + \mathcal{V}dP = \frac{C_V - C_P}{C_V}Pd\mathcal{V}$$

Now let's divide through by $P\mathcal{V}$. It's not apparent that this will help, but it does so

$$\frac{Pd\mathcal{V}}{P\mathcal{V}} + \frac{\mathcal{V}dP}{P\mathcal{V}} = \frac{C_V - C_P}{C_V}\frac{Pd\mathcal{V}}{P\mathcal{V}}$$

and canceling the extra V's and P's gives

$$\frac{d\mathcal{V}}{\mathcal{V}} + \frac{dP}{P} = \frac{C_V - C_P}{C_V}\frac{d\mathcal{V}}{\mathcal{V}}$$

rearranging terms gives

$$\begin{aligned}
\frac{dP}{P} &= \left(\frac{C_V - C_P}{C_V} - 1\right)\frac{d\mathcal{V}}{\mathcal{V}} \\
\frac{dP}{P} &= \left(\frac{C_V - C_P}{C_V} - \frac{C_V}{C_V}\right)\frac{d\mathcal{V}}{\mathcal{V}} \\
\frac{dP}{P} &= \left(-\frac{C_P}{C_V}\right)\frac{d\mathcal{V}}{\mathcal{V}}
\end{aligned}$$

Lets define a new quantity

$$\gamma = \frac{C_P}{C_V} \tag{29.4}$$

so we have

$$\frac{dP}{P} = -\gamma \frac{d\not{V}}{\not{V}}$$

Which is exciting. We are just down to P's and $\not{V}$'s and a gamma. We may be able to get an expression of P as a function of $\not{V}$ yet! We can integrate this last equation

$$\begin{aligned} \int \frac{dP}{P} &= -\gamma \int \frac{d\not{V}}{\not{V}} \\ \ln(P) + \zeta_P &= -\gamma \ln(\not{V}) + \zeta_V \end{aligned}$$

where ζ_P and ζ_V are the constants of integration. We can write this as

$$\begin{aligned} \ln(P) + \gamma \ln(\not{V}) &= -\zeta_P + \zeta_V \\ &= \zeta \end{aligned}$$

where $\zeta = \zeta_V - \zeta_P$ is a constant. Knowing a little bit about logs, we can write this as

$$\ln(P\not{V}^\gamma) = \zeta$$

or we can exponentiate both sides

$$P\not{V}^\gamma = e^\zeta \tag{29.5}$$

where e^ζ is still just a constant. Question 123.13.3

Note that we have done what we set out to do (sort of). We have an expression of how P varies with $\not{V}$.

$$P = \frac{e^\zeta}{\not{V}^\gamma}$$

We could put this into our work equation and integrate to find the work done in an adiabatic process!

$$\begin{aligned} \Delta E_{int} &= w \\ &= -\int \frac{e^\zeta}{\not{V}^\gamma} d\not{V} \end{aligned}$$

But also note that we have a great conservation equation for an adiabatic process.

$$P\not{V}^\gamma = \text{constant}$$

or

$$P_i \not{V}_i^\gamma = P_f \not{V}_f^\gamma \tag{29.6}$$

and we know that conservation equations come in very handy in solving problems! Let's add this last equation to our set of equations for adiabatic processes.

We can get a second equation for adiabatic processes relating the volume and temperature. If we know an initial state

$$P_i \rlap{-}V_i = nRT_i$$

we can use our adiabatic conservation equation from above to write

$$P_f \rlap{-}V_f^{\gamma} = P_i \rlap{-}V_i^{\gamma}$$

we can solve the ideal gas law for P_i and

$$P_i = \frac{nRT_i}{\rlap{-}V_i}$$

and use this in our equation

$$\frac{nRT_f}{\rlap{-}V_f} \rlap{-}V_f^{\gamma} = \frac{nRT_i}{\rlap{-}V_i} \rlap{-}V_i^{\gamma}$$

to yield

$$\frac{T_f}{\rlap{-}V_f} \rlap{-}V_f^{\gamma} = \frac{T_i}{\rlap{-}V_i} \rlap{-}V_i^{\gamma}$$

or

$$T_f \rlap{-}V_f^{\gamma-1} = T_i \rlap{-}V_i^{\gamma-1} \tag{29.7}$$

This is another conservation equation for adiabatic processes! We will add this to our equation set for adiabatic processes

These equations are great shortcuts for problem solving *but remember they are only valid for adiabatic processes!*

Notice that we never did complete the integral for work

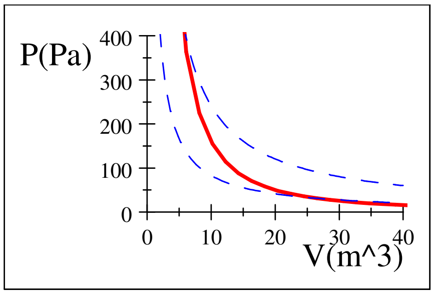
$$\Delta E_{int} = w = -\int \frac{e^{\zeta}}{\rlap{-}V^{\gamma}} d\rlap{-}V$$

We totally could at this point, but we already know the answer to be

$$\Delta E_{int} = nC_V \Delta T$$

where the ΔT is the same as it would be for our actual adiabatic process. Since we know this and it is easy to calculate, we won't often do the integral

Update for Adiabatic processes



Recall that for adiabatic processes we know that

$$Q = 0$$

so that

$$\Delta E_{int} = w$$

and we found that

$$\Delta E_{int} = nC_V \Delta T$$

we can add to these equations

$$P_f \cancel{V}_f^{\gamma} = P_i \cancel{V}_i^{\gamma}$$

and

$$T_f \cancel{V}_f^{\gamma-1} = T_i \cancel{V}_i^{\gamma-1}$$

where

$$\gamma = \frac{C_P}{C_V}$$

This will get us closer to our goal of finding Q, w, and ΔE_{int} for this process.

Heat Transfer Mechanisms

Question 123.13.3

The change in internal energy is the sum of the energy from each type of energy transfer

$$\Delta E_{int} = Q + W$$

Work we know a lot about, but let's become more familiar with Q. We may talk about different ways to transfer energy, then

$$\Delta E_{int} = \sum_i (\text{Transfer mechanism})_i \tag{29.8}$$

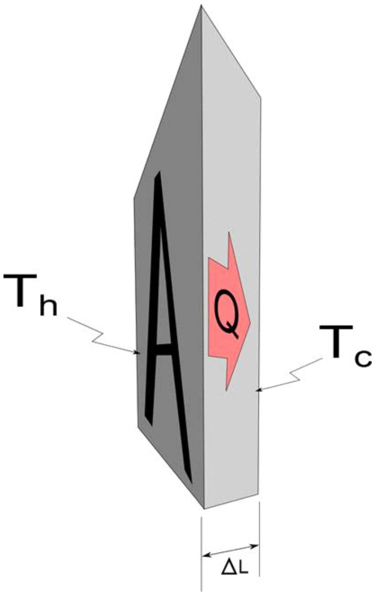
where work is one of those energy transfer mechanisms and there may be many different Q transfer types. In what follows, we will talk about a few.

Thermal Conduction

This is what we have been calling heat up to this point. We define it a little more carefully as the exchange of kinetic energy between microscopic particles (molecules and atoms) through direct contact.

When we add energy by heat to part of a solid, the atoms are locally displaced from their equilibrium locations. These atoms knock into their neighbors, and cause them to be displaced (they increase their kinetic energy). This process continues through the whole solid.

The rate of heat transfer will depend on the properties of the atoms that make up the substance.



We can envision a piece of solid as in the figure. It has thickness Δx and cross-sectional area A. If one face of the slab is at T_c and the other at T_h. We find that energy Q transfers in a time interval Δt from the hotter face to the colder one. The rate

$$\mathcal{P} = \frac{E}{\Delta t} = \frac{Q}{\Delta t}$$

is proportional to A. and ΔT.

$$\mathcal{P} = \frac{Q}{\Delta t} \propto A \frac{\Delta T}{\Delta L}$$

We can take ΔL very small

$$\mathcal{P} = k_{therm} A \left| \frac{dT}{dL} \right| \tag{29.9}$$

where we now have another constant k! This k_{therm} is the *thermal conductivity* of the material and we define $\left|\frac{dT}{dL}\right|$ as the *temperature gradient.*

A gradient is a rate of change, so this gradient is the rate of change of the temperature with respect to position.

For a one dimensional problem, we take a rod. If we assume that k_{therm} is not temperature dependent and the rod is uniform, then

$$\left| \frac{dT}{dL} \right| = \frac{T_h - T_c}{\Delta L}$$

and

$$\begin{aligned}\mathcal{P}_{rod} &= \frac{Q}{\Delta t} = k_{therm} A \frac{T_h - T_c}{\Delta L} \\ &= \frac{A\left(T_h - T_c\right)}{\frac{\Delta L}{k_{therm}}}\end{aligned} \tag{29.10}$$

If we have a complicated multipart rod with n different substances we would have

$$\mathcal{P} = \frac{A\left(T_h - T_c\right)}{\sum_i^n \frac{\Delta L_i}{k_{therm_i}}} \tag{29.11}$$

The amount of energy transferred by thermal conduction is

$$Q = \mathcal{P}\Delta t = \frac{A\left(T_h - T_c\right)\Delta t}{\sum_i^n \frac{\Delta L_i}{k_{therm_i}}}$$

All this wasn't too bad mathematically. But the building industry does some simplifications to our thermal conduction equation. We define a term

$$R = \frac{\Delta L}{k} \tag{29.12}$$

as the "R-value" of a material. Then a wall or multipart solid would have

$$\mathcal{P} = \frac{Q}{\Delta t} = \frac{A\left(T_h - T_c\right)}{\sum_i^n R_i} \tag{29.13}$$

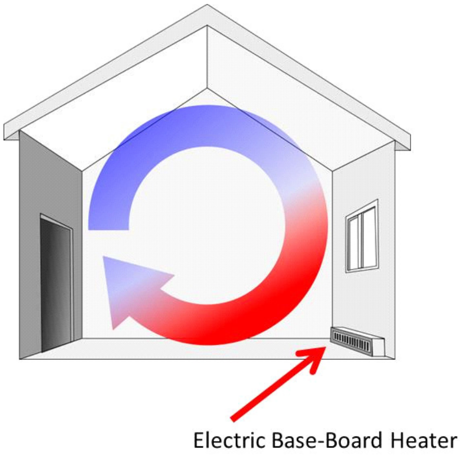
so that

$$Q_{conduction} = \frac{A\left(T_h - T_c\right)\Delta t}{\sum_i^n R_i}$$

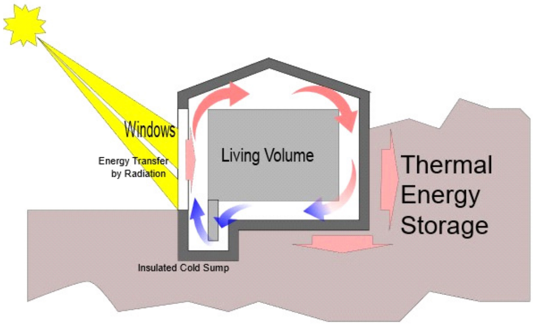
These R-values are used in home insulation and heating design and are usually in English units

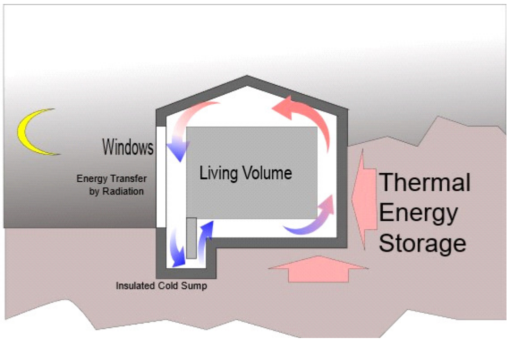
Convection

This is a fundamentally new type of energy transport in our study of thermodynamics, although we are all familiar with it. Convection is the transfer of energy by the movement of the atoms or molecules, themselves. Because the group of moving atoms or molecules has internal energy, moving these molecules also moves the energy. So convection is an energy transfer. This is like circulation of air in your home. Your student apartment might have base-board heading. But your parent's home would likely use forced convection where a fan blows the air.



For several decades houses have been made with the idea that you can heat or cool a house with passive convection. Here is an example.





This house tries to control the thermal conduction mechanisms to keep the interior of the house at nearly the same temperature year round.

Radiation

Light is a form of energy transfer. You will learn more about this in PH220, but for now, what we need to know is that light is a form of wave, and as we know waves carry energy with them. So, light is an energy transfer. We call this energy removal or delivery by light "radiation." This is different than the use of the word "radiation" in nuclear physics.

All objects radiate energy according to Stefan's law

$$\mathcal{P} = \frac{Q}{\Delta t} = \sigma A e T^4 \tag{29.14}$$

The parts of this law are Stephen's constant $\sigma = 5.6696 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m^2\,K}}$, the factor, A, is the surface area, e is the emissivity of the material. This e tells us how like a perfect absorber (called a black body) it is, and T is the temperature. Note there are other forms for these! But this is the form we will use.

The emissivity is a value from 0 to 1. If $e = 0$ then the material is a perfect reflector. This would be a perfect mirror. If the emissivity is 1, then it is a perfect absorber. This is more like blacktop in sunlight. The blacktop warms up, even in the winter, because it absorbs the sunlight. Of course, nothing has an emissivity of exactly 0 or exactly 1. Real objects are somewhere in the middle.

It may seem strange that "black" objects glow, but they do. Think again of that blacktop. The ice on the blacktop melts, because the blacktop not only absorbs the radiation from the sun, it also emits radiation.

An extreme example is the Sun, itself. Light that hits the sun from outside will scatter off the gasses in the sun. It takes a very long time for the light to reemerge, if it ever does. Most likely it will be absorbed by the gasses. So the Sun is considered a "black body!" But it glows, so it does not look black.

We can explain many common experiences using the idea of emissivity. Take this leaf for example. Why is it sinking into the snow?

The leaf has a different emissivity than the snow, so it absorbs more radiation from the sun, the temperature of the leaf rises and it melts the snow around it.

With this understanding, we can see that all objects absorb radiation as well as radiate. If an object is at temperature T and it's surroundings are at temperature T_o. then the net rate of energy gained or lost is

$$\mathcal{P}=\frac{Q}{\Delta t}=\sigma Ae\left(T^{4}-T_{o}^{4}\right) \tag{29.15}$$

and the energy transfer would be

$$Q_{radiation}=\frac{Q}{\Delta t}=\sigma Ae\left(T^{4}-T_{o}^{4}\right)\Delta t$$

Multiple transfer mechanisms

As we said before, we might have many different energy transfer mechanisms. We could have all of these (and more).

$$\Delta E_{int}=Q_{conduction}+Q_{convection}+Q_{radiation}+w$$

Let's think about an example that uses radiation, convection, and insulation to keep something cold. It is called a dewar flask.

Figure 29.4.You might see our liquid nitrogen dewar in demonstrations. The flask is pictured on the left, and a schematic diagram of how a dewar works is given on the right.

This is the same idea as a thermos bottle, only better made (and more expensive). The flask has doubled walls with the space between evacuated to reduce convection and eliminate conduction. The walls are silvered to reduce heat transfer by radiation. The cap has a long piece of insulation attached to it to prevent conduction through the flask opening. New designs use multi-layer films which make their surfaces super Reflective.

30 Microscopic View

So far, as we have considered the motion of atoms with thermal energy we have been able to treat whole moles of a gas at a time. But you might guess that we are sort of taking averages of the motions of the atoms or molecules. Let's make this more explicit in this lecture.

Fundamental Concepts

- In a gas, the molecules have different speeds
- The distribution of speeds is given by the Boltzmann distribution law
- the root mean square speed $\left(v_{rms} = \sqrt{\overline{v^2}}\right)$ is a better measure of the speed of gas molecules than the average velocity
- Pressure for an ideal gas is given by $P = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}m\overline{v^2}\right)$

Molecular Model of an Ideal Gas

We have hinted that the internal energy of a system must be the energy associated with the atoms and molecules that make the system. It's time to make that association in our mental model more concrete, and more mathematical. We will be able to see how the thermodynamic laws we have studied are generated by the basic laws of motion. But we will still limit our study to ideal gases in this class. So let's state some assumptions that follow from the ideal gas approximation.

Ideal Gas Model
The number of molecules in the gas is very large,
The average separation between molecules is large compared the their dimensions
The molecules obey Newton's laws of motion,
On the whole the molecules move randomly
The molecules interact only by short-range forces during elastic collisions
The molecules make elastic collisions with the walls
The gas under consideration is a pure substance; that is; all molecules are identical.

We will often say "molecule" but for an ideal gas atoms and molecules act alike. Question 123.14.1

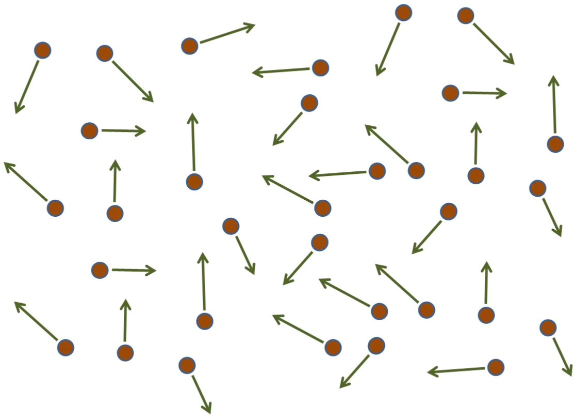
Speed of molecules

The internal energy of a system might consist of nuclear internal energy, or chemical internal energy, but let's reserve those internal energies for PH279. For us, internal energy is mostly due to the motion of the molecules in the system. That motion has kinetic energy associated with it. But not a kinetic energy of the whole system moving together. It is a kinetic energy of the parts of the system, the molecules. Think of driving a car. We could describe the kinetic energy of the whole car. That is the kinetic energy we learned about in PH121. But we could also talk about the kinetic energy of one of the pistons in the engine. These two kinetic energies would be different! Just like this, the kinetic energy of the molecules will be different than the kinetic energy of the system as a whole.

To find this kinetic energy of the molecules, we would need to know how fast the molecules are going in an ideal gas at a particular temperate and pressure in a particular volume.

The average velocity is not helpful, because with our ideal gas model, the molecules move randomly. So we would guess that on the whole their velocities would cancel. If this is not true, we call the situation wind. But if there is no over-all movement, the average velocity, $\bar{\mathrm{v}} = 0$. But we know there must be movement because the molecules have some internal energy. The kinetic energy depends on the speed of the molecules. But not all molecules have the same velocity. So when we say "speed of the molecules" we have to be specific, what speed? The average? The speed the most particles have?

The average velocity won't do.



The molecule motions have random directions. So the average velocity is zero. Some velocities are in negative directions and some are in positive directions, so the velocities cancel out. But if we square the velocities they are all positive. Then we could find the average of the squared velocities and that would be better. But a velocity squared is not a velocity. So to get back to velocity, let's take a square root.

$$v_{rms} = \sqrt{v^2}$$

This quantity is like an average, and gives a representative value that is near the most probable speeds.

Using this, we can get an expression that relates the temperature of our gas to the speed of the molecules. But it we need to know something about how many molecules have what speed. Having the "average" is not enough. To find the *distribution* of speeds, let's remember the idea of number density

$$n_V = \frac{\#\text{ of molecules}}{V\!\!\!\!-}$$

We can express the number of molecules this way assuming we know the volume. But we want the speed of the molecules, and we have learned that concept of energy usually can get us a speed if we don't care to know the direction. Knowing the direction for every air molecule in a room would not be helpful in most cases, so we can be content with just knowing the speed. So lets find a sort of "energy density," that is, the density of molecules that have energy between two amounts of energy, say, E_1 and E_2. We want E_1 and E_2 to be quite close together. So let's let $E_1 = E$ and $E_2 = E + \Delta E$ where ΔE is a small amount of energy. If you take quantum mechanics, or statistical mechanics, you will likely see this quantity quite a lot. Then the number of molecules with a particular energy between E_1 and E_2 could be written as

$$n_V\left(E\right)dE=\frac{\#\text{ of molecules with energy between }E\text{ and }E+\Delta E}{\cancel{V}} \tag{30.1}$$

This is called a *distribution function*. We find distribution functions in statistics. They are associated with probabilities. The standard “bell curve” used sometimes in grading is a distribution function. It tells the total number of students that got a particular number of points in a class.

What we need is the probability that the molecules will have a particular energy (or speed, since this is kinetic energy). A function that gives the amount of molecules that have a particular amount of energy also is called a $distribution$ function. The distribution function that we will use is written symbolically in this cryptic fashion, $n_V\left(E\right)dE$. It is the number of molecules with a particular energy divided by the total number of molecules.

We won’t derive an equation for this quantity, instead we will borrow a result from our junior level thermal physics class.

$$n_V\left(E\right)=n_oe^{-\frac{E}{k_BT}} \tag{30.2}$$

where n_o is the number of molecules per unit volume having energy between $E=0$ and $E=dE$. This distribution function is called the *Boltzmann distribution law.* It tells us that the probability of finding the molecules in a particular energy state varies exponentially as the negative of the energy divided by k_BT.

Distribution of Molecular Speeds

Question 123.14.2

Since there is a distribution of energies, we expect our gas molecules to have a distribution of velocities. That is, the molecules in the gas do not all go the same speed. Again we won’t derive this (at least not in this class!), but the distribution should depend on temperature, T, since we know the internal energy is tied to temperature. The distribution is as follows:

$$N_v=4\pi N\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}v^2e^{-\frac{mv^2}{2k_BT}} \tag{30.3}$$

where m is the mass of the molecule.

If there are dN molecules with speeds between v and $v+dv$ then

$$dN=N_vdv \tag{30.4}$$

so there should be

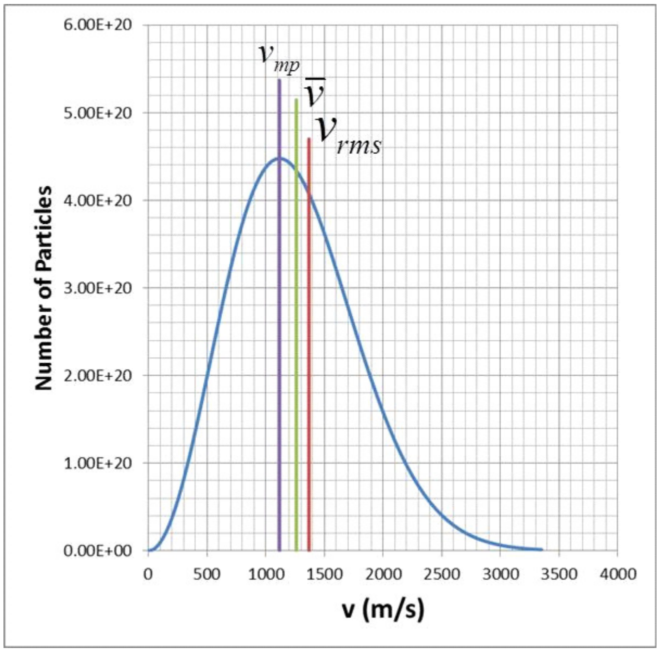
$$N = \int_0^\infty N_v dv \tag{30.5}$$

total molecules.

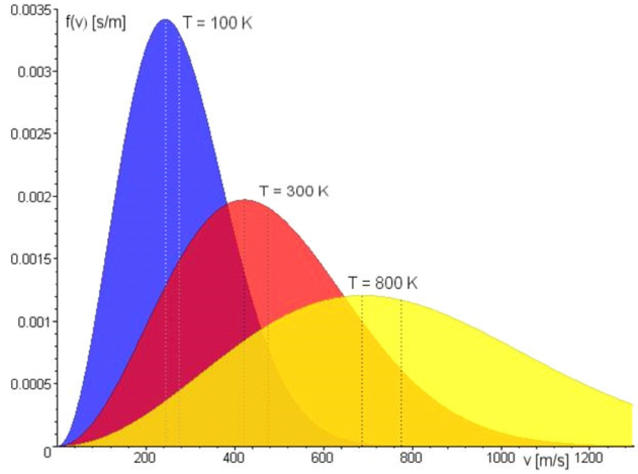
The kinetic energy of the molecules is hiding in the exponent, so we could write this as

$$N_v = 4\pi N \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 e^{-\frac{K}{k_B T}} \tag{30.6}$$

If we plot N_v vs. v we get the figure below. The number of molecules with speeds between v and $v + dv$ is the area under the blue curve. The peak of the curve tells us the most probable speed, that is, the speed the most molecules have, v_{mp}. The curve is not symmetric, so the most probable speed is not the average speed, $\bar{v}$. There is also our new speed estimate marked v_{rms}.



If we plot N_v for different temperatures, we observe that the peak shifts, and the curve broadens



Temperature dependence of the Maxwell-Boltsman distribution (Image in the Public Domain couracy Fred Stober)

A motivated student could now find the most probably speed by finding the maximum of N_v. To do this, we take a derivative

$$\begin{aligned}
\frac{dN_v}{dv} &= \frac{d}{dv}\left(4\pi N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2k_B T}}\right) \\
&= 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \frac{d}{dv}\left(v^2 e^{-\frac{m}{2k_B T}v^2}\right) \\
&= 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \left(-2\frac{m}{2k_B T}v^3 e^{-\frac{m}{2k_B T}v^2} + 2ve^{-\frac{m}{2k_B T}v^2}\right) \\
&= -2ve^{-\frac{mv^2}{2k_B T}}\left(\frac{mv^2}{2k_B T} - 1\right) 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}}
\end{aligned}$$

set this equal to zero

$$\begin{aligned}
0 &= -2ve^{-\frac{mv^2}{2k_B T}}\left(\frac{mv^2}{2k_B T} - 1\right) 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \\
\frac{mv^2}{2k_B T} &= 1 \\
v_{mp} &= \sqrt{\frac{2k_B T}{m}}
\end{aligned}$$

$$v_{mp} = \sqrt{\frac{2k_B T}{m}} \tag{30.7}$$

and this is great! We have related the temperature, T, to the most probable speed of the molecules. But it is more convenient to use $v_{rms,}$ so let's see if we can modify this expression to be in terms of v_{rms}.

The average value of v^n is given by

$$\overline{v^n} = \frac{1}{N}\int_0^\infty v^n N_v dv$$

This motivated student could also use this to find the average speed (not the average velocity, which is zero). He or she will want the average value of v^1

$$\begin{aligned}
\overline{v^1} &= \frac{1}{N}\int_0^\infty v^1 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2k_BT}} dv \\
&= 4\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \int_0^\infty v^3 e^{-\frac{mv^2}{2k_BT}} dv \\
&= 4\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \left(-\frac{1}{2\left(\frac{m}{2k_BT}\right)^2}\left(e^{-\frac{mv^2}{2k_BT}} + \frac{m}{2k_BT}v^2 e^{-\frac{mv^2}{2k_BT}}\right)\right)\Bigg|_0^\infty \\
&= 4\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \left(-\frac{1}{2\left(\frac{m}{2k_BT}\right)^2}\right)(0-(1+0)) \\
&= 2\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \left(\frac{1}{\left(\frac{\pi m}{2\pi k_BT}\right)^2}\right) \\
&= 2\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \left(\left(\frac{m}{2\pi k_BT}\right)^{-\frac{4}{2}}\pi^{-2}\right) \\
&= 2\pi\left(\frac{m}{2\pi k_B T}\right)^{=\frac{1}{2}} \left(\pi^{-2}\right) \\
&= 2\left(\frac{2\pi k_B T}{m}\right)^{\frac{1}{2}} \left(\pi^{-1}\right) \\
&= \sqrt{\frac{8k_BT}{\pi m}}
\end{aligned}$$

so

$$\bar{v} = \sqrt{\frac{8k_BT}{\pi m}}$$

which is also great, but not what we wanted. But it is close. This time let's find the average value of v^2. We hinted that the root mean squared value would be useful. that is, $\overline{v^2} = v_{rms}$. We can find this like we found the average velocity

$$\begin{aligned}\overline{v^2} &= \frac{1}{N}\int_0^\infty v^2 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2k_BT}}dv \\ &= 4\pi\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}\int_0^\infty v^4 e^{-\frac{mv^2}{2k_BT}}dv\end{aligned}$$

But the math is a bit more cumbersome. Let

$$a = \frac{m}{2k_BT}$$

and let then

$$\begin{aligned}\int x^4 e^{-ax^2}dx &= -\frac{1}{8a^{\frac{5}{2}}}\left(6\sqrt{a}xe^{-ax^2} - 3\sqrt{\pi}\operatorname{erf}\left(\sqrt{a}x\right) + 4a^{\frac{3}{2}}x^3e^{-ax^2}\right)\Big|_0^\infty \\ &= -\frac{1}{8a^{\frac{5}{2}}}\left(-3\sqrt{\pi}\right)\end{aligned}$$

The quantity $\operatorname{erf}\left(\sqrt{a}x\right)$ is called the "error function." If you study this function in a good mathematical book on integration you will find that

$$\operatorname{erf}\left(\sqrt{a}\infty\right) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = 1$$

so

$$\overline{v^2} = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}\left(-\frac{1}{8\left(\frac{m}{2k_BT}\right)^{\frac{5}{2}}}\left(-3\sqrt{\pi}\right)\right)$$

or

$$\overline{v^2} = 3\frac{T}{m}k_B$$

now recognize that $v_{rms} = \sqrt{\overline{v^2}}$ so

$$v_{rms} = \sqrt{\frac{3k_BT}{m}} \qquad (30.8)$$

and this is what we wanted. We have an expression that relates the temperature of the gas to the rms speed of the molecules of the gas. Returning to v_{rms} we see that, indeed, v_{rms} is close to the average and most probable speeds. It will show up in the next topic. So we will need to recognize it. Question 123.14.3

That we have some speeds higher than others explains why things do not evaporate or all boil away at once.

	A few rms Speeds for Gasses	
Gas	Molar Mass $(\mathrm{kg/mol})$	v_{rms} at $20\,^\circ\mathrm{C}$ $(\mathrm{m/s})$
H_2	2.02×10^{-3}	1902
He	4.0×10^{-3}	1352
H_2O	18×10^{-3}	637
N_2	28×10^{-3}	511
O_2	28×10^{-3}	478
CO_2	44×10^{-3}	408

Mean Free Path

Question 123.14.4

We started this chapter by saying how molecules interact with the walls of a box. But the molecules really do bounce off each other. The average distance a molecule travels before a molecule-molecule collision is called the *mean free path*.We could give this a symbol, let's re-use the Greek letter λ

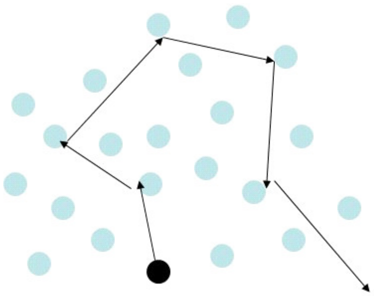


Figure 30.5.

$$\begin{aligned}\lambda &= \frac{\text{Length traveled}}{\text{Number of collisions}} \\ &= \frac{L}{N_{\text{col}}}\end{aligned}$$

This will give the average length before a collision happens.

The average length traveled is easy

$$L = \bar{v}\Delta t$$

But to predict the number of collisions is harder. To do this problem, let's play a geometrical trick. First let's assume the molecules are spheres with diameter d. We see that a collision does not happen unless the distance between the molecules is less than

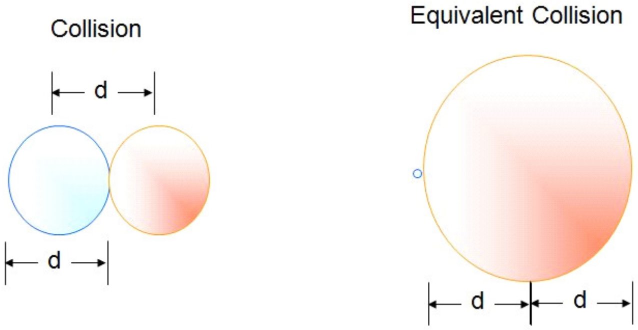
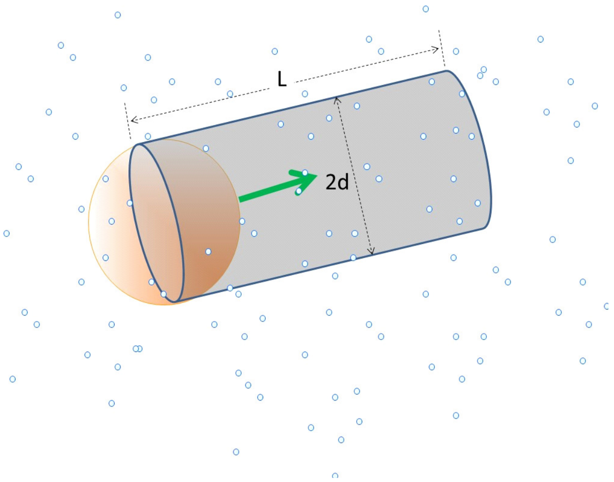


Figure 30.6.

or equal to $2d$. We can model this interaction as a large particle of size $2d$ and many point particles. We let the large particle move in a straight line to create a cylindrical path.



All point particles in the cylinder will collide with our large particle. The volume of such a big cylindrical path is

$$V\!\!\!/ = (\pi d^2)L$$

but the length L of the cylinder will be

$$L = \bar{v}\Delta t$$

so the volume will be

$$V\!\!\!/ = \pi d^2 \bar{v}\Delta t \tag{30.9}$$

then, if n_V is the number of molecules per unit volume then

$$N_{\text{col}} = \pi d^2 \bar{v} \Delta t n_V \tag{30.10}$$

is the number of point molecules in the cylinder.

The mean free path is the average distance traveled in a time interval Δt divided by the number of collisions that occur in Δt

$$\begin{aligned} \lambda &= \frac{\bar{v}\Delta t}{\pi d^2 \bar{v} \Delta t n_V} \\ &= \frac{1}{\pi d^2 n_V} \end{aligned} \tag{30.11}$$

The frequency of collisions is

$$f = \pi d^2 \bar{v} n_V \tag{30.12}$$

and the inverse of the frequency (like a period) is the mean free time.

It turns out that our simple model is off by a factor of $\sqrt{2}$ because we assumed the molecules are stationary.

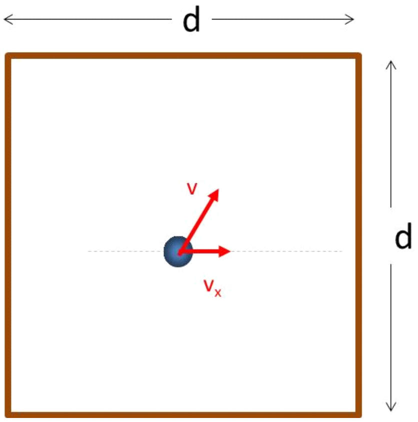
$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n_V} \tag{30.13}$$

$$f = \sqrt{2}\pi d^2 \bar{v} n_V \tag{30.14}$$

$$= \frac{\bar{v}}{l} \tag{30.15}$$

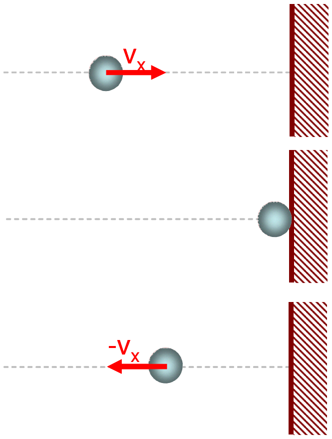
Microscopic Definition of Pressure

Armed with a better understanding of the motions of gas particles, we want to express macroscopic quantities in terms of microscopic effects. The easiest to start with is pressure



Picture a particle in a cubical box of side length d. Question 123.14.5

As the particle approaches the wall of the box we may say that it has velocity v_x. The particle impacts the wall and bounces off again. Now it travels with $-v_x$.



We can think of why this would be. We know momentum is conserved. Before the collision we may take the velocity of the wall to be zero, and we may imagine the wall to have a very large mass (which it does, compared to the particle!). If we call the particle mass m and the wall mass M then

$$\begin{aligned} p_i &= M v_{wall_i} + m v_{x_i} \\ &= m v_{x_i} \end{aligned}$$

but the wall will not move with any speed after the collision so $v_{wall_f} = 0$ also. Then

$$\begin{aligned} p_f &= M v_{wall_f} + m v_{xf} \\ &= -m v_{x_f} \end{aligned}$$

then the magnitude of the velocities must be equal and the change in momentum must be

$$\begin{aligned} \Delta p &= m v_{x_i} - \left(-m v_{x_f}\right) \\ &= 2 m v_{x_i} \end{aligned}$$

Question 123.14.6

Recall that the impulse from the collision of the particle on the wall would be

$$\begin{aligned} \bar{F}_{i,\text{on molecule}} \Delta t_{\text{collision}} &= \Delta p_{x_i} \\ &= -2 m v_{x_i} \end{aligned}$$

This is the impulse momentum theorem from PH121. The time of the collision is very short. This force is the force of the wall on the particle.

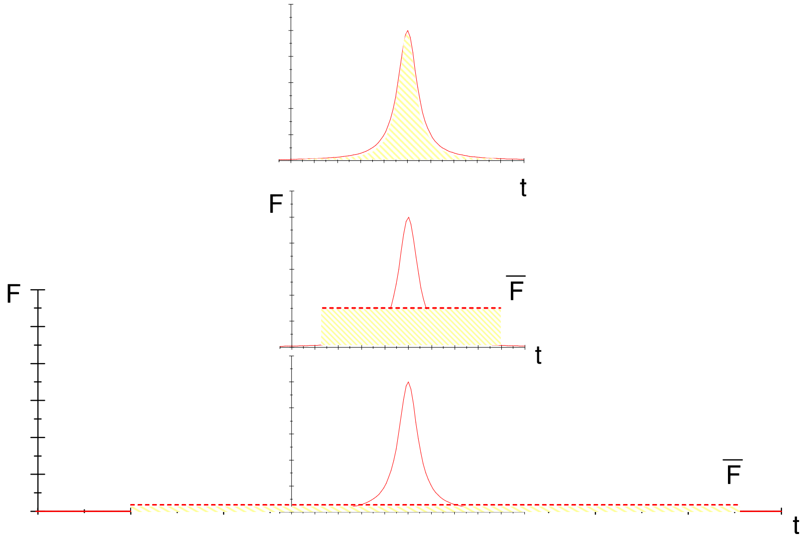
Now consider that if a particle bounces off one wall in the x-direction, it must travel to the opposite wall and back before it can bounce off the same wall again. That is, the time between bounces is roughly

$$\Delta t=\frac{2d}{v_{x_i}} \tag{30.16}$$

Averages are funny things, it is perfectly legal to redefine the time of our average force to be this travel time Δt. What does this mean? In the first graph of the next figure we have the full definition of impulse

$$\Delta p=\int_{t_i}^{t_f}\mathbf{F}dt \tag{30.17}$$

which says the impulse is the area under the F vs. t graph.



The average force $\bar{F}$ is pictured in the second graph. It has the same area as the first graph, but is flat, representing the average force (red dotted line). The final graph has a $\bar{F}$ averaged over a much larger Δt. The area is the same, but the magnitude of the average force is much smaller.

This may seem useless, but at any rate, we can do this. So let's take our average of F over the time

$$\Delta t=\frac{2d}{v_{x_i}}$$

which is the time it takes our particle to travel away from our wall, and bounce back to the wall. We know that sometime within this Δt the collision with our wall actually occurs. The change in momentum is still

$$\bar{F}\Delta t = -2mv_{x_i}$$

so we can write

$$\begin{aligned}\bar{F}\frac{2d}{v_{x_i}} &= -2mv_{x_i} \\ \bar{F} &= -\frac{mv_{x_i}^2}{d}\end{aligned}$$

So far our force has been the force of the wall on the particle, but we know by Newton's third law that the particle force must be equal and opposite.

$$\bar{F}_{i,\text{on molecule}} = -\bar{F}_i$$

then

$$\bar{F}_i = \frac{mv_{x_i}^2}{d}$$

for one particle (molecule or atom).

For many particles

$$\begin{aligned}\bar{F} &= \sum_{i=1}^{N}\frac{mv_{x_i}^2}{d} \\ &= \frac{m}{d}\sum_{i=1}^{N}v_{x_i}^2\end{aligned}$$

We have found the average force on the container wall! Question 123.14.7

Now remember how to take an average

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N}x_i$$

so if we were to calculate the average squared velocity

$$\overline{v_{x_i}^2} = \frac{1}{N}\sum_{i=1}^{N}v_{x_i}^2$$

We almost have this in our force equation. Let's put in the missing parts

$$\begin{aligned}\bar{F} &= \frac{m}{d}\frac{N}{N}\sum_{i=1}^{N}v_{x_i}^2 \\ &= \frac{m}{d}N\overline{v_{x_i}^2}\end{aligned}$$

Going back to a single molecule, we remember that

$$v_i^2 = v_{x_i}^2 + v_{y_i}^2 + v_{z_i}^2$$

then if we take the averages of each piece like we did with $v_{x_i}^2$

$$\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$$

There is no real reason to prefer the x-direction, we could have chosen the y-direction and had the same value. Then

$$v_{x_i}^2 = v_{y_i}^2 = v_{z_i}^2$$

and Question 123.14.8

$$\overline{v^2} = 3v_{x_i}^2$$

So our force on the wall is

$$\begin{aligned}
\bar{F} &= \frac{m}{d} N \overline{v_{x_i}^2} \\
&= \frac{m}{d} N \frac{1}{3} \overline{v^2} \\
&= \frac{mN}{3d} \overline{v^2} \\
&= \frac{N}{3d} \left(m\overline{v^2} \right)
\end{aligned}$$

Finally we can find the pressure. We know Question 123.14.9

$$\begin{aligned}
P &= \frac{F}{A} \\
&= \frac{F}{d^2} \\
&= \frac{1}{d^2} \frac{N}{3d} \left(m\overline{v^2} \right) \\
&= \frac{1}{d^3} \frac{N}{3} \left(m\overline{v^2} \right) \\
&= \frac{1}{3} \frac{N}{\cancel{V}} \left(m\overline{v^2} \right) \\
&= \frac{2}{3} \frac{N}{\cancel{V}} \left(\frac{1}{2} m\overline{v^2} \right)
\end{aligned}$$

$$P = \frac{2}{3} \frac{N}{\cancel{V}} \left(\frac{1}{2} m\overline{v^2} \right) \tag{30.18}$$

and we see that the pressure of the gas is proportional to the number of molecules per unit volume and to the average kinetic energy of the molecules! In fact this shows some of the things we have learned. For example, P is inversely proportional to $\cancel{V}$!

We already knew most of this. But now we have a mathematical form so we can do calculations! We can do this for our other thermodynamic quantities. We will start by finally defining temperature in our next lecture.

31 What is Temperature? and Values for C_V and C_P

Finally we can define temperature in terms of the basic physics of the atoms and molecules that make up the substance.

Fundamental Concepts

- Temperature is related directly to the average kinetic energy of the molecules of a gas
- We call a way the molecule can move a *degree of freedom.*
- For an ideal gas $C_V = \frac{3}{2}R$ and $C_P = \frac{5}{2}R$
- For an ideal gas $E_{int} = \frac{3}{2}k_BT$

Temperature

Question 123.15.1

Question 123.15.2

Equation on Board

Last lecture we found that

$$P = \frac{2}{3}\frac{N}{V\!\!\!\!-}\left(\frac{1}{2}m\overline{v^2}\right)$$

We can rearrange this to look a little like the ideal gas law.

$$PV\!\!\!\!- = \frac{2}{3}N\left(\frac{1}{2}m\overline{v^2}\right)$$

But temperature does not appear in this equation. Let's recall that we can write the ideal gas law as

$$PV\!\!\!\!- = Nk_BT$$

If we set these two equations equal to each other

$$Nk_BT = \frac{2}{3}N\left(\frac{1}{2}m\overline{v^2}\right)$$

then

$$T = \frac{2}{3}\frac{1}{k_B}\left(\frac{1}{2}m\overline{v^2}\right) \tag{31.1}$$

This is fantastic! We have finally defined temperature. Temperature is related directly to the average kinetic energy of the molecules! The average kinetic energy is

$$\bar{K}_{mol} = \frac{1}{2}m\overline{v^2} = \frac{3}{2}k_BT$$

Note that in each direction we should have

$$\bar{K}_x = \frac{1}{2}m\overline{v_x^2} = \frac{1}{2}k_BT$$

so

$$\bar{K}_y = \frac{1}{2}k_BT$$

and

$$\bar{K}_z = \frac{1}{2}k_BT$$

So in each direction the molecules have

$$\bar{K}_i = \frac{1}{2}k_BT \tag{31.2}$$

We call a way the molecule can move a *degree of freedom.* Our molecules can move in the x, y, and z, direction, so our molecules have three degrees of freedom. Each degree of freedom contributes $\frac{1}{2}k_BT$ worth of energy.

Of course, we only allow translational energy in our formulation for ideal gasses, but for more complex molecules we could have rotational energy (another way to move) and vibrational energy (yet another way to move) etc. And each way the molecule can move is another degree of freedom would contribute $\frac{1}{2}k_BT$. But remember, these more complex molecules would not be ideal gasses.

For the collection of N molecules, we have all together

$$\begin{aligned} K &= N\left(\frac{3}{2}k_BT\right) && (31.3) \\ &= \frac{3}{2}Nk_BT && (31.4) \\ &= \frac{3}{2}nRT \end{aligned}$$

and for our ideal gas we see that the internal energy depends only on the temperature.

This is only true under our assumption that the molecules or atoms are structureless. So this works well for actual monotonic gases. Question 123.15.3

Let's do a problem. We found earlier by borrowing results from Statistical Mechanics that

$$v_{rms} = \sqrt{\frac{3k_BT}{m}}$$

Let's show this from knowing that each degree of freedom contributes $\frac{1}{2}k_BT$. We just found that

$$\bar{K} = \frac{K}{N} = \frac{3}{2}k_BT$$

and we know that

$$\bar{K} = \frac{1}{2}m\overline{v^2}$$

Setting these equal gives

$$\begin{aligned} \frac{3}{2}k_BT &= \frac{1}{2}m\overline{v^2} \\ 3Nk_BT &= m\overline{v^2} \\ \frac{3Nk_BT}{m} &= \overline{v^2} \end{aligned}$$

so

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_BT}{m}}$$

which is just what we found before, but this was a much easier and more understandable way to solve the problem. This shows the power of the simple idea of the ideal gas approximation.

The rms speed tells us that if the molecule or atom is tiny (has little mass) it must move very fast. If it is larger, it will move more slowly. Question 123.15.4

C_V and C_P values

In the problems we have done so far, we had to look up the C_V and C_P. But they must come from somewhere. We should discover why they are what they are. To do this, remember for an ideal gas, we only have translational kinetic energy, so it must be true that

$$E_{int} = K_{tran} = \frac{3}{2}nRT$$

Now let's use our special processes. If we take the system through a process at constant volume,

$$Q = \Delta E_{int}$$

and

$$Q = nC_V\Delta T$$

so

$$\Delta E_{int} = nC_V\Delta T$$

We can write this as

$$\begin{aligned} E_{int_f} - E_{int_i} &= nC_V\left(T_f - T_i\right) \\ &= nC_V T_f - nC_V T_i \end{aligned}$$

this strongly suggests that if C_V is constant

$$E_{int} = nC_V T$$

We can solve for C_V

$$C_V = \frac{1}{n}\frac{\Delta E_{int}}{\Delta T}$$

and for very small changes in temperature

$$C_V = \frac{1}{n}\frac{dE_{int}}{dT}$$

But for an ideal gas the internal energy is kinetic energy so we know that

$$E_{int} = \frac{3}{2}nRT$$

So we can easily find $\frac{dE_{int}}{dT}$

$$\frac{dE_{int}}{dT} = \frac{3}{2}nR$$

Then our value for C_V is

$$C_V = \frac{1}{n}\frac{dE_{int}}{dT}$$

$$\begin{aligned} C_V &= \frac{1}{n}\frac{3}{2}nR \\ &= \frac{3}{2}R \end{aligned}$$

Knowing $R = 8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}$, the numerical value is

$$\begin{aligned} C_V &= \frac{3}{2}8.314\frac{\mathrm{J}}{\mathrm{mol\,K}} \\ &= 12.471\frac{\mathrm{J}}{\mathrm{mol\,K}} \end{aligned}$$

for all monotonic gasses. It turns out that this is a very good approximation.

Finding C_P is easy because we know that

$$C_P = C_V + R$$

so

$$C_P = \frac{5}{2}R$$

For adiabatic processes we need the ratio of C_P to C_V or γ. We now know that

$$C_V = \frac{3}{2}R$$

$$C_P = \frac{5}{2}R$$

and we may take the ratio of these values

$$\begin{aligned}\gamma &= \frac{C_P}{C_V} \\ &= \frac{\frac{5}{2}R}{\frac{3}{2}R} \\ &= \frac{5}{3}\end{aligned}$$

This works well for monotonic gasses, but fails badly for more complex gasses. So we see our ideal gas formulation starts to break down with more complex gasses at this point. To go farther, we would need to include the rotational and vibrational energy of the molecules. Still, we have come a long ways with our simple ideal gas assumptions!

Note that for solids and liquids $\Delta \cancel{V}$ is very small so very little work is done. This means that $C_P \approx C_V$ which is why we we could get away with only one value for the molar heat capacity in the tables for solids and liquids.

The Equipartiction of Energy

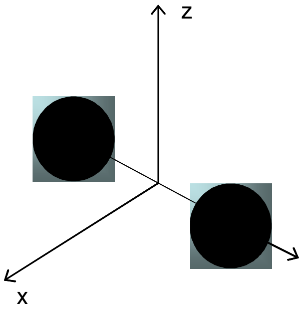
We can extend our ideal gas model a little by using what we know about degrees of freedom. Remember that we found that for each degree of freedom the internal energy was

Movie of Equapartition

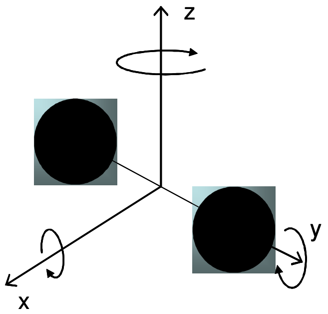
$$E_i = \frac{1}{2}k_B T \tag{31.5}$$

we found for an ideal monotonic gas that the internal energy was

$$E_{int} = 3E_i = \frac{3}{2}k_B T \tag{31.6}$$



where each E_i came from a translational degree of freedom. But a diatomic molecule has several more degrees of freedom. It can rotate about any of the axes.



Here rotation about the y axis does not contribute significantly because the moment of inertia of a sphere (we will take the atom to be roughly spherical) about it's axis is

$$\mathbb{I} = \frac{2}{5}mr^2 \tag{31.7}$$

where m is the mass of the atom. Most of the mass is centered in the nucleus (proton mass $= 1.67 \times 10^{-27}$ kg, electron mass $= 9.11 \times 10^{-31}$ kg), which has a radius of about $r = 1.7 \times 10^{-5}$ Å is the radius of the atom. The moment of inertia for rotation about the center of the two mass system is

$$\mathbb{I} = \sum_i m_i R_i^2 \tag{31.8}$$

where R is the distance from the center of mass. For diatomic hydrogen, $R = \frac{1}{2}1.06$ Å so we can see that the rotation about the y-axis is not very important, so we are left with three translational and two rotational degrees of freedom. This gives

$$E_{int} = \left(\frac{3}{2}k_BT\right)_{trans} + \left(\frac{2}{2}k_BT\right)_{rot} = \frac{5}{2}k_BT \tag{31.9}$$

Writing this in molar terms

$$E_{int} = \frac{5}{2}nRT \tag{31.10}$$

then

$$C_V = \frac{1}{n}\frac{dE_{int}}{dT} = \frac{5}{2}R \tag{31.11}$$

and

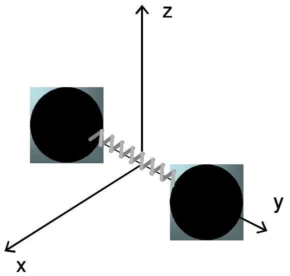
$$C_P = C_V + R \tag{31.12}$$

gives

$$C_P = \frac{7}{2}R \tag{31.13}$$

Measurements of C_V and C_P at STP agree with these values.

BUT WAIT, we did not include vibration! The atoms are bond together with an electrical attraction that acts quite like a spring force. So vibration along the axis is possible and we need to add in one more degree of freedom. We also have potential energy involved for a spring force, so we expect an additional degree of freedom for vibration.



When we add all these up, we get

$$E_{int} = \left(\frac{3}{2}k_BT\right)_{trans} + \left(\frac{2}{2}k_BT\right)_{rot} + \left(\frac{2}{2}k_BT\right)_{vib} = \frac{7}{2}k_BT \qquad (31.14)$$

which gives

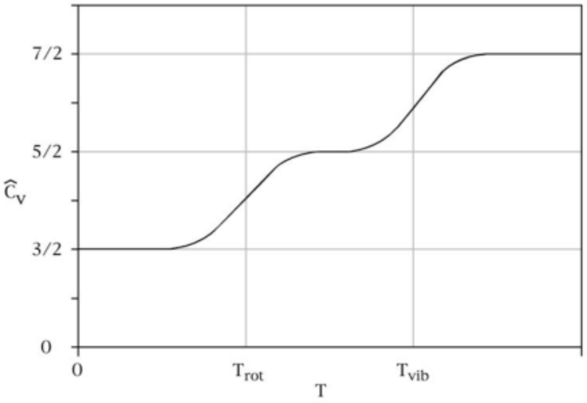
$$C_V = \frac{7}{2}R \qquad (31.15)$$

and

$$C_P = \frac{9}{2}R \qquad (31.16)$$

We should pause to ask, what values do we use? for diatomic gasses, is $C_V = \frac{7}{2}R$ all the time?

It turns out that when energy is added to a collection of molecules, it does not pick randomly from the degrees of freedom.

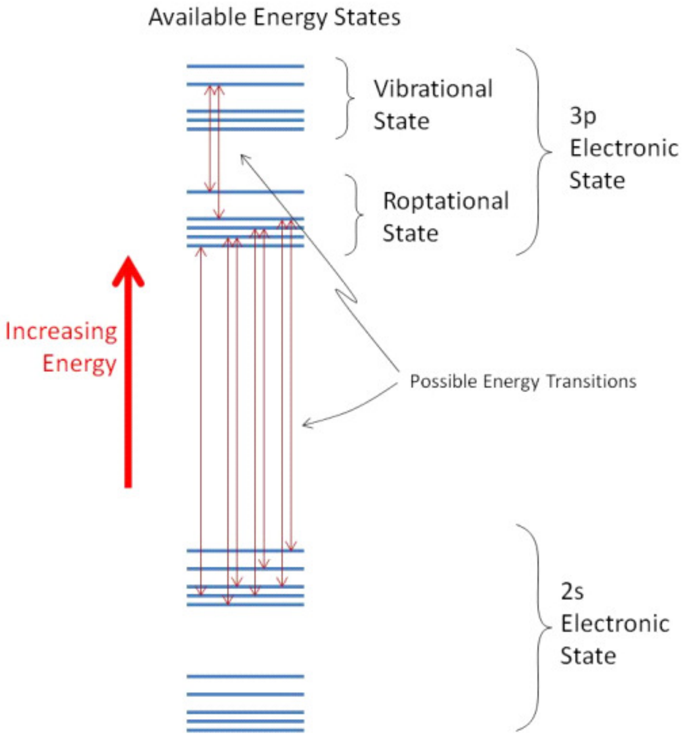


At low temperature, the translational degrees of freedom are preferred. Then, as the temperature rises, the rotational degrees of freedom are filled. Finally the vibrational degrees of freedom are used. The figure shows this relationship for diatomic Hydrogen. Note that there are plateaus at each of our values of C_V that we found ($\frac{3}{2}R$, $\frac{5}{2}R$, $\frac{7}{2}R$).

We have talked about the simplest of molecules. If we had a more complex molecule, there would be more complex rotational and vibrational degrees of freedom. That is why for larger molecules the values of C_V in table in the book don't follow a simple fraction of R.

Quantization

We have come to the end of what classical theory can explain. We know that different degrees of freedom are filled in groups, but we don't know why. We will only hint at the reason, but it has to do with wave motion that we will soon study!



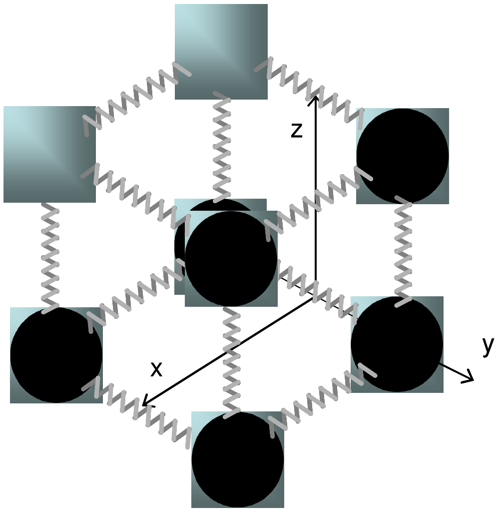
Quantum theory tells us that atoms can be described as waves under boundary conditions. Like a string with fixed ends, these atomic waves have quantized frequencies. The energy of the molecule is proportional to the frequency in quantum theory, so quantum mechanics tells us that the energy of the molecule will be quantized. The figure shows an energy level diagram for a molecule. Higher energy is in the y direction. The vibrational and rotational states are marked. (the maroon arrows show transitions from higher to lower states. These transitions create radiation).

The lowest allowed state is called the *ground state*. It is analogous to the fundamental frequency (something we will study soon!). At low temperatures the molecule only has enough energy to populate the lower states, but when the temperature rises, there is enough energy transfer due to collisions to push molecules into the rotational states. With higher temperatures, you will see the collisions transfer molecules to the higher vibrational states.

Molar specific heat of solids

Let's do a problem. Let's find C_V for a solid, and compare it to the specific heat of elemental solids.

We can view the atoms of solids as having a structure of springs and atoms.



We can see that we should have three translational degrees of freedom for each atom. We also have three degrees of freedom associated with the potential energy from the spring-like bonding forces. Recall that

$$E_x = \frac{1}{2}mv_x^2 - \frac{1}{2}kx^2 \tag{31.17}$$

and we have and E_y and E_z as well.

From equipartition of energy we have

$$E_i = 6\left(\frac{1}{2}k_BT\right) \tag{31.18}$$

per atom! So we have

$$\begin{aligned} E_{int} &= 3Nk_BT && (31.19)\\ &= 3nRT && (31.20) \end{aligned}$$

and then for solids

$$C_V = \frac{1}{n}\frac{dE_{int}}{dT} = 3R \tag{31.21}$$

numerically this is

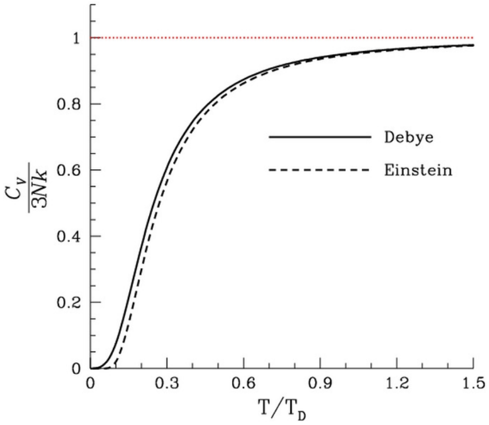
$$\begin{aligned} C_V &= 3\times 8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\\ &= 24.942\frac{\mathrm{J}}{\mathrm{mol\,K}} \end{aligned}$$

This is the *DuLong-Petit law*. Here is our table of specific heat values again.

Substance	$c\left(\frac{\mathrm{J}}{\mathrm{kg\,K}}\right)$	$C_V\left(\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)$
Aluminum	900	24.3
Copper	385	24.4
Iron	449	25.1
Gold	129	25.4
Lead	128	26.5
Ice	2090	37.6
Mercury	140	28.1
Water	4190	75.4

we see that our simple analysis did not do too bad! For elemental solids we get about the right number.

This law also breaks down for very cold temperatures where we need even more quantum mechanics to explain what happens. Einstein and Debye both contributed to explaining this departure, but their work is beyond this freshman course. This is a topic for a solid state physics course which is an elective that you might consider if you are interested is seeing a more complete solution.



Debye and Einstein models. (Image in the Public Domain courtesy Frédéric Perez)

This almost completes our study of ideal gasses. Next lecture we will use our microscopic view to explain energy transfer, and we will tackle the second law of thermodynamics. Following that we will use what we know to build idealized engines and refrigerators.

32 Microscopic Energy Transfer Considerations

Fundamental Concepts

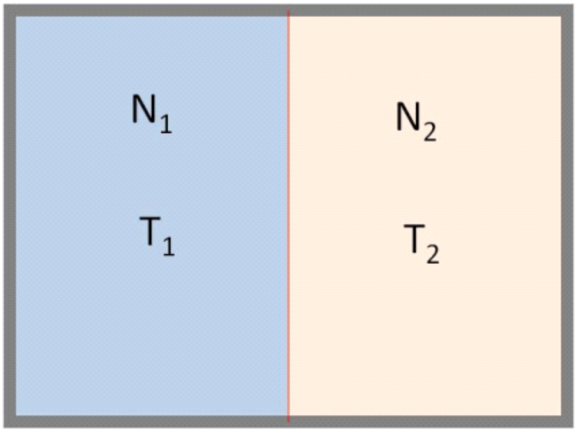
1. Microscopic idea of Energy Transfer
2. Microscopic idea of disorder
3. Entropy

Microscopic View of Energy Transfer

Question 123.16.1

Suppose we have a chamber divided into two volumes. The volumes are separated by a membrane. If they have different temperatures, we expect that energy will transfer, but how?

Think of a gym. In our gyms, we have large nets that divide courts to keep balls from passing from one court to another. Now think of millions of basket balls in each side of the gym. The basket balls all are moving. Some of the basket balls will strike the net. If a ball on the other side strikes the net at the same place at the same time, the collision will transfer energy. This would be a conservation of momentum and energy problem The slower ball would gain energy, the faster ball would loose energy.



This is analogous to our gasses in chambers. The molecules, like the basketballs, transfer energy by collision.

The internal energy for each side of the chamber depends on both N and T

$$\begin{aligned} E_{1i} &= \frac{3}{2}N_1 k_B T_{1i} \\ E_{2i} &= \frac{3}{2}N_2 k_B T_{2i} \end{aligned}$$

The total energy is

$$E_{tot} = E_{1i} + E_{2i}$$

if our container does not change volume (no work done) and is insolated ($Q = 0$) then E_{tot} will not change. But eventually the two sides come to the same temperature

$$T_f = T_{1f} = T_{2f}$$

this happens when the average energy of the molecules (or basketballs) are the same. On average, neither side gains energy due to the collisions on the membrane (net). We can rewrite our energy equation for the final condition as

$$\bar{E}_{tot} = \bar{E}_{1i} = \bar{E}_{2f}$$

because the average energies will all be the same. Using

$$\begin{aligned} E_{1f} &= N_1 \frac{3}{2} k_B T_{1f} = N_1 \bar{E}_1 \\ E_{2f} &= N_2 \frac{3}{2} k_B T_{2f} = N_2 \bar{E}_2 \end{aligned}$$

we can rewrite this as an energy per molecule

$$\frac{E_{tot}}{N_1 + N_2} = \frac{E_{1f}}{N_1} = \frac{E_{2f}}{N_2}$$

which gives a set of expressions for the final energy of each side

$$\begin{aligned} E_{1f} &= \frac{N_1 E_{tot}}{N_1 + N_2} \\ E_{2f} &= \frac{N_2 E_{tot}}{N_1 + N_2} \end{aligned}$$

and we can verify that

$$\begin{aligned} E_{tot} &= E_{1f} + E_{2f} \\ &= \frac{N_1 E_{tot}}{N_1 + N_2} + \frac{N_2 E_{tot}}{N_1 + N_2} \\ &= E_{tot} \end{aligned}$$

Let's take what we have learned and mathematically expressed as macroscopic formula and show it at a microscopic level as a problem example.

Let's show that if $Q_{tot} = 0$, and no work is done, then

$$Q_1 = -Q_2$$

This is a calorimetry problem.

Start with the first law,

$$\begin{aligned} Q_1 &= \Delta E_1 = E_{1f} - E_{1i} \\ Q_2 &= \Delta E_2 = E_{2f} - E_{2i} \end{aligned}$$

since no work is done. Let's take Q_1 first

$$Q_1 = E_{1f} - E_{1i}$$

since

$$E_{1f} = \frac{N_1 E_{tot}}{N_1 + N_2}$$

and

$$E_{tot} = E_{1i} + E_{2i}$$

then

$$Q_1 = \frac{N_1 \left(E_{1i} + E_{2i}\right)}{N_1 + N_2} - E_{1i}$$

Then, applying some algebra,

$$\begin{aligned} Q_1 &= \frac{N_1 \left(E_{1i} + E_{2i}\right)}{N_1 + N_2} - E_{1i} \frac{N_1 + N_2}{N_1 + N_2} \\ &= \frac{N_1 E_{1i} + N_1 E_{2i} - E_{1i} N_1 - E_{1i} N_2}{N_1 + N_2} \\ &= \frac{N_1 E_{2i} - E_{1i} N_2}{N_1 + N_2} \end{aligned}$$

Expanding this out gives

$$\begin{aligned}
Q_1 &= \frac{N_1 E_{2i}}{N_1+N_2} - \frac{E_{1i}N_2}{N_1+N_2} \\
&= \frac{N_1 E_{2i}}{N_1+N_2} - \frac{E_{1i}N_2}{N_1+N_2} + \frac{N_2 E_{2i}}{N_1+N_2} - \frac{N_2 E_{2i}}{N_1+N_2} \\
&= \frac{N_1 E_{2i}}{N_1+N_2} + \frac{N_2 E_{2i}}{N_1+N_2} - \frac{E_{1i}N_2}{N_1+N_2} - \frac{N_2 E_{2i}}{N_1+N_2} \\
&= \frac{N_1 E_{2i} + N_2 E_{2i}}{N_1+N_2} - \frac{E_{1i}N_2}{N_1+N_2} - \frac{N_2 E_{2i}}{N_1+N_2}
\end{aligned}$$

The first term is just E_{2i}, and using $E_{tot} = E_{1i} + E_{2i}$ again we see

$$\begin{aligned}
Q_1 &= E_{2i} - \left(\frac{(E_{1i}N_2 + N_2 E_{2i})}{N_1+N_2}\right) \\
&= E_{2i} - \left(\frac{N_2 (E_{1i} + E_{2i})}{N_1+N_2}\right) \\
&= E_{2i} - \left(\frac{N_2 (E_{tot})}{N_1+N_2}\right) \\
&= E_{2i} - E_{2f} \\
&= -Q_2
\end{aligned}$$

which is what we expect. We have shown our calorimetry equation using the ideas of microscopic energy transport for ideal gasses.

The arrow of time.

Why does time go only one way? The answer to this question may surprise you. *We don't know.* There is nothing in our laws of motion that tells us that time should only go one way. Take motion under constant acceleration.

$$y_f = y_i + v_o \Delta t + \frac{1}{2} a \Delta t^2$$

and let's start with $t_i = 0$ so

$$y_f = y_i + v_o t_f + \frac{1}{2} a t_f^2$$

this equation describes the motion of an object under constant acceleration, like that of gravity near the Earth's surface. But nothing in the equation tells us that time must take on ever grater more positive values.

The truth is that there is only one physical law that tells us that time goes only one way, and that law is an empirical law that says essentially "time only goes one way." We experience time going one way, so we developed a law that says so. This is not a terribly convincing argument. But this is the state of physics.

Some processes (really all processes) can’t actually run forward or backward. Idealized processes can be run in reverse, but not real process. We call processes that can only work one direction *irreversible processes*. If we shot a movie of an irreversible process and ran the movie backwards, it would be funny, because we would know that the reverse process can’t happen.

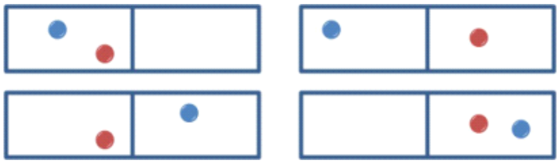
Let’s take an example. Suppose our process is dropping a marker on the floor. Because the marker bounces on the floor are not perfectly elastic, the potential energy of the marker is converted into heat energy. If we added the lost energy back to the marker on the floor, it won’t bounce up to a height for us to catch. This process of dropping the marker is irreversible.

To make this idea of irreversible processes and the arrow of time sound a little better, let’s study thermodynamic situations and see that at least the arrow of time make sense.

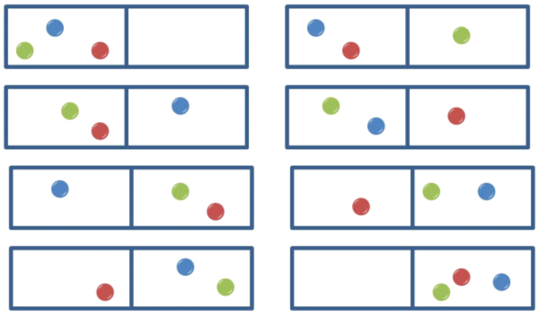
Disorder

Suppose we have a system that has particles and places for those particles to be placed. We can start with a small space (see figure below). If I have one particle moving randomly, and divide my space into two compartments, then I have a probability of $\frac{1}{2}$ that the particle will be in the left compartment.

Now if I have two particles, the probability that two particles will be in the left compartment is $\left(\frac{1}{2}\right)^2$ or $\frac{1}{4}$.



I can keep adding particles. For three particles the probability of being in the left side is $\left(\frac{1}{2}\right)^3$ or $\frac{1}{8}$.



For 50 particles it is

$$\left(\frac{1}{2}\right)^{50}=\frac{1}{1125\,899\,906\,842\,624}$$

We can extend this thinking to divisions of particles. Suppose I have 100 particles. We know that they will have a distribution of speeds (Maxwell-Boltzman distribution). What is the probability that the 50 fastest particles will be in the left side and the 50 slowest will be on the right side? This is the product of the last result with itself

$$\begin{aligned}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50} &= \\ &= \frac{1}{1267\,650\,600\,228\,229\,401\,496\,703\,205\,376} \\ &= 7.888\,6\times 10^{-31}\end{aligned}$$

BYU 50 particle demonstrationn

This is not very probable. And we are only working with 100 particles and two sides of a box!

The point to this exercise is to realize there are very may *states* or ways to place particles in a divided box. Although each configuration is equally probable, suppose we want a particular outcome, say, the fastest 50 molecules on one side and the slowest 50 molecules on the other. There are many many more ways to *not* have this configuration that there are to have this configuration.

Each configuration of molecules and locations is called a *microstate*. There are a huge number of microstates in our example.

The collection of microstates is called a *macrostate*. There is more than one configuration of particles in our fast-slow division that places all the fast on one side and all the slow on another side. But the collection of all microstates that make up this division would be a macrostate. For example, we could have an even distribution of particles (within their fast/slow groups) on each side, or we could have all the fastest particles in each group on the top, and the slowest on the bottom within each major fast or slow side of the room. These two microstates sill preserve the division between the 50 fastest on the left and the slowest 50 on the right. They just rearrange the molecules within their groups.

Extension to large spaces and many particles

Now let's take a room full of molecules. We count the number of microstates by considering the number of places we can put a molecule. Each molecule has a volume $\cancel{V}_m$. So there are

$$w = \frac{\cancel{V}}{\cancel{V}_m} \tag{32.1}$$

places to put the molecule that are distinct. Now suppose we have N molecules. If we ignore the probability that we might have two molecules in one space (which is different than the case above!) we have

$$(w)^N = \left(\frac{\cancel{V}}{\cancel{V}_m}\right)^N \tag{32.2}$$

ways to place these molecules among our w states. I'll call this

$$W_i = \left(\frac{\cancel{V}_i}{\cancel{V}_m}\right)^N \tag{32.3}$$

where $\cancel{V}_i$ is a particular volume (we will let it change to $\cancel{V}_f$ in just a minute).

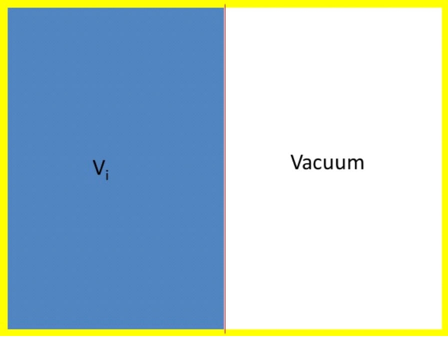
A Measure of Disorder

It would be convenient to find a way to measure the disorder. Then we could see how a thermodynamic process affects the amount of disorder. The traditional way to do this is to define a quantity

$$S \equiv k_B \ln(W) \tag{32.4}$$

This has the property that S increases as the number of states, W, increases.

Let's take an example, our old friend, the adiabatic free expansion.



For this case we start with all the atoms on one side of a membrane. No energy transfers out by heat. No work is done, we just puncture the membrane in the center and let the gas flow from one side to the other. How does the disorder change in this process? Consider the number of states.

$$W_i = \left(\frac{\cancel{V}_i}{\cancel{V}_m}\right)^N$$
$$W_f = \left(\frac{\cancel{V}_f}{\cancel{V}_m}\right)^N$$

We can write our disorder measure as

$$S_i \equiv k_B \ln\left(W_i\right)$$

and

$$S_f \equiv k_B \ln\left(W_f\right)$$

then the change is order is

$$\begin{aligned}\Delta S &= S_f - S_i \\ &= k_B \ln\left(W_f\right) - k_B \ln\left(W_i\right) \\ &= k_B \ln\left(\frac{W_f}{W_i}\right)\end{aligned}$$

If we look at the ratio of the number of states

$$\begin{aligned}\frac{W_f}{W_i} &= \frac{\left(\frac{\cancel{V}_f}{\cancel{V}_m}\right)^N}{\left(\frac{\cancel{V}_i}{\cancel{V}_m}\right)^N} \\ &= \left(\frac{\cancel{V}_f}{\cancel{V}_i}\right)^N\end{aligned}$$

so

$$\begin{aligned} \Delta S &= k_B \ln\left(\left(\frac{\cancel{V}_f}{\cancel{V}_i}\right)^N\right) \\ &= N k_B \ln\left(\left(\frac{\cancel{V}_f}{\cancel{V}_i}\right)\right) \qquad (32.5) \end{aligned}$$

We should ask if the order increased or decreased. Since $\cancel{V}_f$ is larger for our case than $\cancel{V}_i$, then ΔS will be a positive number. This means the disorder has increased in this case.

It is a good question to ask, will ΔS ever be negative? In other words, will the gas ever line up on only one side every again? If the answer is no, then this is an irreversible process. We can guess that this process is irreversible. We would have to do work to pump the gas back into one side.

Definition of entropy

Question 123.16.1

We have learned to use one state variable, ΔE_{int}. The trick is to realize that ΔE_{int} only depends on ΔT and use any easy path to calculate it's value. That is why we can use

$$\Delta E_{int} = n C_V \Delta T$$

no matter what process we have. State variables, like ΔE_{int} give us powerful ways to calculate what will happen in a system.

Question 123.16.2

There is another state variable in Thermodynamics, *change in entropy*, ΔS. We will again find that we can calculate ΔS by any path because it depends only on the starting and ending points. And from our previous discussion we recognize that this new state variable, it is just a change in our disorder parameter!

$$\Delta S = k_B \ln\left(\frac{W_f}{W_i}\right)$$

Note that this is a difference between two disorder parameters. We will call our disorder parameter the *entropy* of the system.

$$S \equiv k_B \ln(W) \qquad (32.6)$$

Entropy was originally defined at the macroscopic level. Statistical Mechanics (the microscopic study of thermodynamics) gives us our modern definition we have followed. We can state this definition as

Definition 32.1 *Entropy is a measure of the amount of disorder in a system and its environment.*

Entropy and the Second Law

In our example, because, $Ꝟ_f > Ꝟ_i$ we found that ΔS will be positive. This implies that the disorder has increased. We can see that this is true, because there are now molecules distributed among many more states (position, occupation combinations) than before.

We will find this to be true in general, for irreversible processes (all natural process that are recognized). The total entropy of an isolated system that undergoes a change cannot decrease.

What about non-isolated systems? We must then consider the entropy of the system *and* its surroundings. We again find that entropy of a system *and* its surroundings cannot decrease. But the entropy of a system can decrease if the entropy of the surroundings increases more so the total effect is an increase in entropy.

These statements constitute the most commonly seen form of the second law of thermodynamics.

Macroscopic view

We don't really want to count the microstates of a process and look for changes in their number to do calculations. It would get tedious very fast! And so far our definition of ΔS was done only for a free expansion. We expect that in some way disorder would increase if we transferred in energy by heat, or did some kind of work on the system. We need a macroscopic version of our definition of entropy.

We will give a definition and then show that it works. The change in entropy is given by

$$\Delta S = \frac{\Delta Q_r}{T} \tag{32.7}$$

where Q_r is the energy transferred by heat through a reversible path and T is the absolute temperature (K) of the system. The assumption of a single temperature is because we have assumed ΔQ_r is small. But why do we use a reversible path?

Suppose we have an irreversible process. It might be hard to compute Q. But remember that, like ΔE_{int}, ΔS does not depend on path,! it depends only on the initial and final states. So we may calculate it using any convenient path, and a reversible path is more convenient.

Note that macroscopically we have not defined entropy but the change in entropy, ΔS.

This is the meaningful quantity (like ΔE_{int}) for thermodynamic calculations.

Let's do an example: What is the change in entropy when a block of ice melts? We will use

$$\Delta S = \int \frac{dQ_r}{T}$$

Note that while the ice is melting, we will have ice water at a constant $T_{melt} = 273\,\mathrm{K}$. Then our integral is

$$\Delta S = \frac{1}{T_{melt}} \int dQ_r$$

and the integral is not too hard

$$\Delta S = \frac{Q_r}{T_{melt}}$$

and for melting we know what Q_r could be. We can freeze water as well as melt ice. So we could model this as a reversible process. The amount of energy transfer needed to melt the ice is just

$$Q = mL$$

where m is the mass of the ice and L is the latent heat of fusion for water. So

$$\Delta S = \frac{mL}{T_{melt}}$$

This wasn't too hard!

Let's do another example to tie our macroscopic ΔS to our microscopic ΔS. Let's find the change in entropy for a gas brought through an isothermal expansion. Again we have

$$\Delta S = \int_{\mathcal{V}_i}^{\mathcal{V}_f} \frac{dQ_r}{T}$$

and since the process is isothermal, T is constant

$$\Delta S = \frac{1}{T} \int_{\mathcal{V}_i}^{\mathcal{V}_f} dQ_r$$

and the integral would seem easy. It is just Q, but it is a particular Q for an isothermal process. For an isothermal process we know

$$\Delta E_{int} = 0$$

and

$$\Delta E_{int} = Q + w$$

so

$$Q = -w$$

and further we know that for an isothermal process

$$\begin{aligned} w &= -\int P d\mathcal{V} \\ &= -nRT\ln\left(\frac{\mathcal{V}_f}{\mathcal{V}_i}\right) \end{aligned}$$

so the Q_r for our process could be just

$$Q = nRT\ln\left(\frac{\mathcal{V}_f}{\mathcal{V}_i}\right)$$

Then

$$\begin{aligned} \Delta S &= \frac{1}{T}nRT\ln\left(\frac{\mathcal{V}_f}{\mathcal{V}_i}\right) \\ &= nR\ln\left(\frac{\mathcal{V}_f}{\mathcal{V}_i}\right) \end{aligned}$$

and remember that $nR = Nk_B$ so

$$\begin{aligned} \Delta S &= \\ &= Nk_B\ln\left(\frac{\mathcal{V}_f}{\mathcal{V}_i}\right) \end{aligned}$$

which is just what we found from our microscopic view of entropy change! Now we realize that when we let the gas flow from one side of our divide box to the other, we did this isothermally. And that is true, the gas did not change the average kinetic energy per molecule as it flowed from one side of the box to the other. Since our system was isolated, $Q = 0$ and $W = 0$ for the whole system, so $\Delta E_{int} = 0$ and the system did not change temperature. But surely something changed! And that something was ΔS!

Heat death of the Universe

We should ask the question, is the universe an isolated system? We don't know the answer to this for sure[31], but we believe this is true that the universe is isolated.[32] That means that the change in entropy of the universe will always be positive. The universe is "running down." The ultimate end of this increase in energy would mean that all mater in the universe will be spread out and cold. This is called the *heat death of the universe*.

Note that this is a little different than the eschatological view that we have in the Church of Jesus Christ of Latter Day Saints.

Throughout my career, many colleagues and acquaintances have said it is foolish to

---

[31] After all, the "Flash" and his colleagues seem to be able to come and go from different "universes all the time."

[32] Dispite what the television shows seem to say.

believe in a church and a resurrection because the second law of thermodynamics tells us that the universe will die in a cold blur. How do we respond?

I can only give my opinion. But I think it is worth discussing, so here is a disclaimer for what follows.

WARNING: This is not an official statement of doctrine This is an answer to a question that many scientists must face from colleagues who will question their faith using the principals of thermodynamics. Official statements by the General Authorities supersede any opinions expressed here

The First Law and LDS Thought[33]

I suppose we should ask, do we believe in the first law of thermodynamics. If this law does not work for us, we have no reason to expect the second law to work. The first law is basically conservation of energy. Here are some quotes to consider.

> And there stood one among them that was like unto God, and he said unto those who were with him: We will go down, for there is space there, and we will take of these materials, and we will make an earth whereon these may dwell; (Abraham 3:24)
> You ask the learned doctors why the say the world was made out of nothing; and they will answer, ‘Doesn’t the Bible say He created the world?’ And they infer, from the word create, that it must have been made out of nothing. Now, the word create came from the word baurau which does not mean create out of nothing; it means to organize; the same as a man would organize materials and build a ship. (Joseph Smith, King Follett Discourse)

The idea of the first law of thermodynamics seems in harmony with LDS thought.

So let’s take on the Second Law.

The Second Law and LDS Thought

Again here are some quotes

---

[33] Comparing physical theory to LDS thought has it’s dangers, I am aware. But as scientists, we can’t escape thinking about this. Just as a caution, John A. Widsoe wrote a book called *Joseph Smith as Scientist* in which Widsoe tried very hard to show that LDS though is in harmony with Universal Ether Theory. But we now know Universal Ether Theory is not correct, so being in harmony with it is meaningless. So we should be cautious!

> Wherefore, the first judgment which came upon man must needs have remained to an endless duration. And if so, this flesh must have laid down to rot and to crumble to its mother earth, to rise no more. (2 Nephi 9:7)
> And our spirits must have become like unto [Satan], and we become devils, angels to a devil, to be shut out from the presence of our God, and to remain with the father of lies, in misery, like unto himself. (2 Nephi 9:9)

We seem to accept the concept of the second law as well! –This will be a surprise to our non-member friends.

But then how can we expect eternal life and exaltation? Does something counter the second law (a Third or Fourth Law of Thermodynamics)?

A Third Law of Thermodynamics?

Again let's go to the scriptures. Jacob describes another force not recognized by current scientific theory that counteracts this tendency to decay

> Wherefore, it must needs be an infinite atonement—save it should be an infinite atonement this corruption could not put on incorruption. Wherefore, the first judgment which came upon man must needs have remained to an endless duration. And if so, this flesh must have laid down to rot and to crumble to its mother earth, to rise no more. (2 Nephi 9:7-8)
> O how great the goodness of our God, who prepareth a way for our escape from the grasp of this awful monster; yea, that monster, death and hell, which I call the death of the body, and also the death of the spirit. (2 Nephi 9:10)

Atonement means to bring together. This is an apt description of a process or law that would counter or balance the second law[34]. Since the atonement can only be accessed through Christ, it is not likely to be discovered in a laboratory, but is a very real, physical, process. (see for example Luke 24:39)

This is not doctrine, but is a way to think about the second law until we get direct revelation on the subject. We really should not be worried about a conflict between thermodynamics and our doctrine. I don't believe one exists as you can see.

---

[34] Nibley, The meaning of the temple, reprinted in the Collected Works of Hugh Nibley: volume 12, Deseret Book, Salt Lake City, UT, 1992

33 Producing Useful Work

So far we have talked about doing work on gas in cylinders. That may not seem all that useful. But if we think that car engines have gas in cylinders with pistons, then it may seem like this could be important.

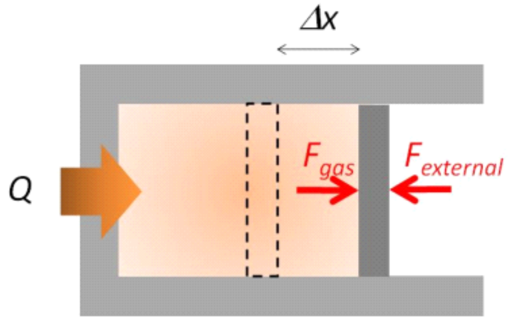
Fundamental Concepts

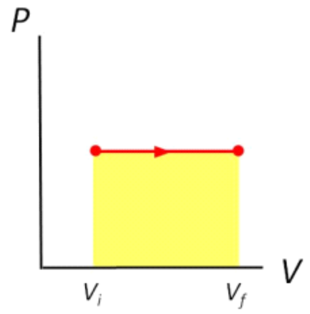
- $W_{\text{useful}} = -W_{\text{on the gas}}$

Useful work.

Question 123.17.1

Suppose that we want to produce useful work from a thermodynamic device. For building such a device we have a nomenclature issue.





Consider our piston. If we add energy by heat, and the piston is free to move, then we do an amount of work on the gas

$$w = -P\left(\text{Ꝟ}_f - \text{Ꝟ}_i\right)$$

*due to*This is a negative amount of work, since Ꝟ_f is lager than Ꝟ_i. But think of a steam engine. We want the force by the gas on the piston to win, because this pushing of the piston is what we can use to push the wheel of a steam engine, or a turbine, etc. That is negative work on the gas. And it is inconvenient to have the useful work be negative. Let's define the useful work done by a thermodynamic device as the negative of work done on the gas.

$$w_{\text{useful}} = -w$$

But, w_{useful} is kind of a awkward name. Usually a machine that does useful work is called an engine, so let's call our useful work W_{eng} for "work done by an engine."

$$w_{eng} = -w$$

Reservoirs

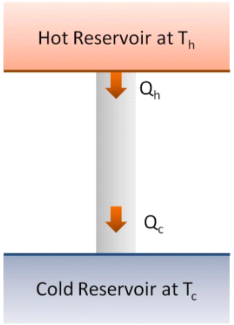
Question 123.17.2

Think of a water reservoir. In the western US we know what these are. It is a large man-made body of water that is designed to provide irrigation water for farms and homes.

Think of what happens when you water your lawn with water from the reservoir. Some water does leave from the reservoir, but compared to the whole reservoir of water, one lawn watering makes very little difference. Often there is more water coming into the reservoir, so really there is negligible water loss from your individual lawn.

A heat reservoir is like this. A hot reservoir is a source of thermal energy that can be tapped without causing any measurable change in temperature. We can also have a cold reservoir that has a large lack of thermal energy. A small amount of thermal energy transferred to this cold reservoir won't measurably change its temperature

The next figure is called an *Energy Transfer Diagram*.

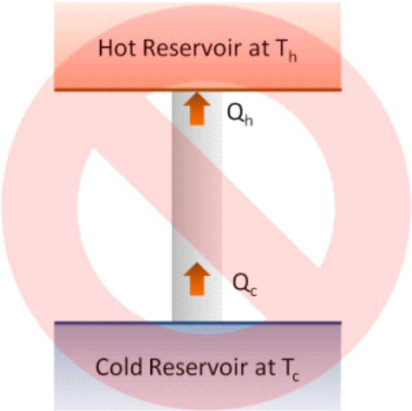


It shows energy moving by heat from a hot reservoir to a cold reservoir.

Question 123.17.3

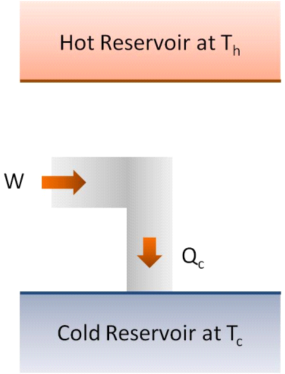
Using our understanding of the second law of thermodynamics, we can see that an amount of energy, Q_h, can flow from the hot reservoir and an amount of energy, Q_c, can flow by heat to the cold reservoir. We expect that if there is no other mechanism for dissipating the energy, then $Q_c = Q_h$. We will define both Q_h and Q_c as positive quantities. If they are negative, we will explicitly write a negative sign.

Look at this energy transfer diagram.



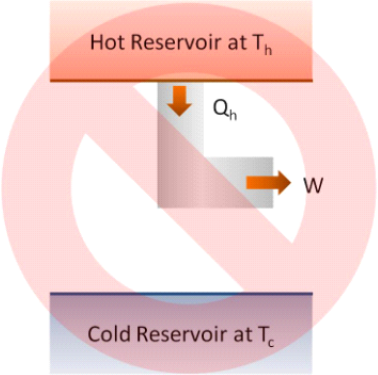
This diagram says that energy flows from the cold reservoir to the hot reservoir. From the second law of thermodynamics we would say that this won't happen. This would increase the order of the cold reservoir spontaneously. This does not happen. Question 123.17.4

Here is another energy transfer diagram



This diagram says that we do work that is directly converted into heat. This is a common experience. Rub your hands together. Your hands warm up. The work has created more internal energy, But that energy quickly dissipates into the room The room temperature is not affected by this increase in internal energy of the air. The air in the room is a cold reservoir.

Here is another diagram.



This diagram says that energy from the hot reservoir is converted directly into useful work. This is not something we expect to happen. And our second law of thermodynamics again would prohibit this situation.

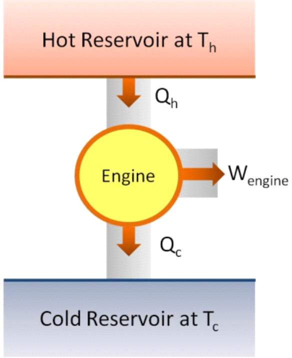
This would be a perfect engine, converting all the energy transferred from the hot reservoir into useful work. In fact, this idea could give us a perfectly good restatement of the second law of thermodynamics

---

Perfect engines are not possible

Heat Engines

Using our energy transfer diagram, let's introduce a more practical engine. Here is the diagram. We have added the engine explicitly on the diagram.



The engine takes energy from the heat reservoir and turns it into useful work. But there is some energy that is waisted and this energy ends up in the cold reservoir. The amount of energy that becomes work must be

$$W_{\text{engine}} = Q_h - Q_c$$

We call $Q_h - Q_c$ the net heat

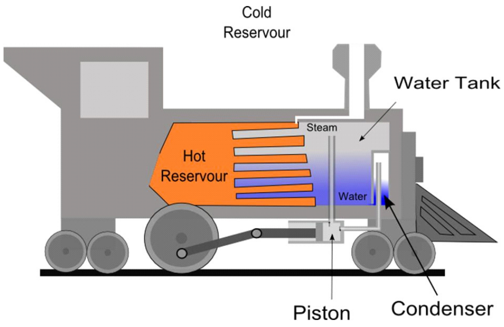
$$Q_{net} = Q_h - Q_c$$

Notice that Q_h is positive *for the engine* and Q_c is negative *for the engine.*

A useful heat engine is a device that takes in energy by heat and operates in a cycle, If it did not periodically return to its original state, then it could not keep going. The basics of the cycle are as follows

1. The working substance absorbs energy by heat from the high temperature reservoir
2. Work is done by the engine
3. Energy is expelled by heat to a low temperature reservoir

Example: Steam engine



In a steam engine

1. Water in the boiler absorbs energy from burning fuel (hot reservoir) and evaporates to steam
2. The steam does work by expanding against a piston
3. The steam cools and condenses releasing heat to the outside air (cold. reservoir) , and the liquid water returns to the boiler.

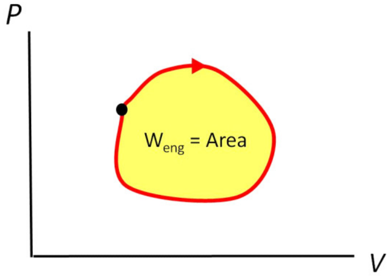
Since the engine goes through a cyclical process,

$$\Delta E_{int} = 0$$

Its initial and final internal energies are the same, so

$$Q_{net} = W_{eng} \tag{33.1}$$

The work done by the engine equals the *net* energy absorbed by the engine. The work is equal to the area enclosed by the curve of the PV diagram.



Thermal efficiency is defined as the ratio of the net work done by the engine during one cycle to the energy input at the higher temperature

$$\eta = \frac{W_{eng}}{|Q_h|} \tag{33.2}$$

$$= \frac{Q_{net}}{Q_h} \tag{33.3}$$

$$= \frac{|Q_h| - |Q_c|}{Q_h} \tag{33.4}$$

$$= 1 - \frac{|Q_c|}{|Q_h|} \tag{33.5}$$

We can think of this as

$$\eta = \frac{\text{what you gain in work}}{\text{what you gave in energy by heat}} \tag{33.6}$$

For a car engine

$$\eta = 20\% \tag{33.7}$$

is a very good number (most student and faculty cars are far less efficient than this). Question 123.17.5

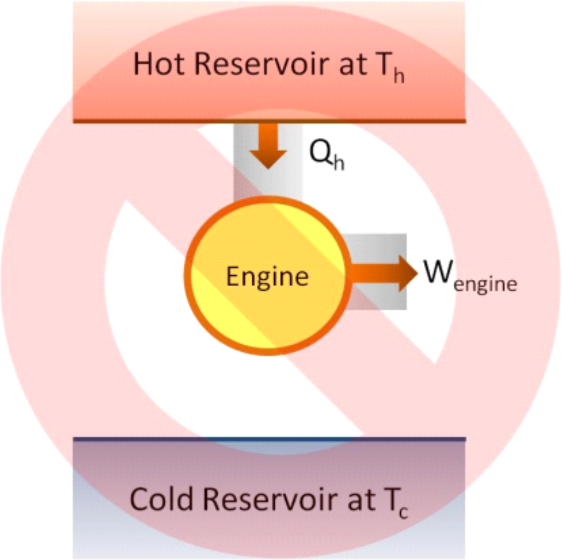
Question 123.17.6

Suppose that we have a 100% efficient engine. We know that this is not possible, but consider what this would mean.

$$\eta = 1 - \frac{|Q_c|}{|Q_h|} \tag{33.8}$$

$$= 1 \tag{33.9}$$

this can only happen when $|Q_c| = 0$ which means that no energy is transferred out of the engine by heat – which is not really possible. The fact that real engines have efficiencies much lower than 1 leads to another statement of the second law of thermodynamics.



It is impossible to construct a heat engine that, operating in a cycle, inputs energy by heat from a hot reservoir and converts the energy entirely into useful work.

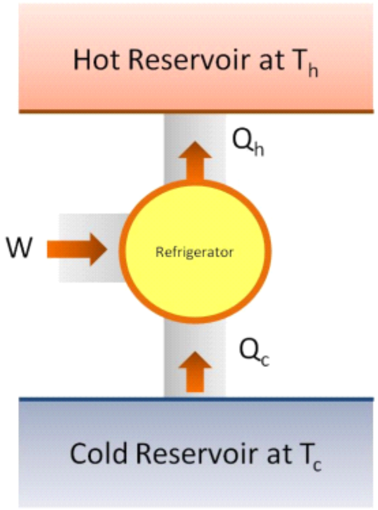
This is called the Kelvin-Plank form of the second law, and it means that

$$W_{eng} < |Q_h| \tag{33.10}$$

so the device represented by our drawing above is impossible to achieve.

Heat Pumps and Refrigerators

Let's look at a special form of a heat engine, a head pump.. Question 123.17.7



A heat pump is an engine run in reverse. We wish to pump heat from a cold reservoir to the heat reservoir. To do this we must do work *on* the heat pump. This is not a natural process, so by adding in work we are adding energy to make it happen.

A window air conditioner unit is a good example of a heat pump. It removes energy by heat from the inside of your apartment and transfers that energy to the outside of your apparent by heat.process. To make this happen, you need mechanical device that does work

A refrigerator is another example of a heat pump. It transfers energy by heat from the cold interior to the warmer environment of your apartment kitchen. It would be great if we could transfer heat from our cold reservoir (room of a house or freezer compartment of a fridge) to a hot reservoir without doing work, but this is not possible. This fact was discovered by Clausius and lead to the Clausius statement of the second law–*It is impossible to construct a cyclical machine whose sole effect is to transfer energy continuously by heat from one object to another object at a higher temperature without the input of energy by work.*

Or more simply, as we have already said: Energy does not transfer spontaneously by heat from a cold object to a hot object.

Coefficient of Performance

How well a heat pump works can be described by the *coefficient of performance* (COP).

$$COP_h = \frac{\text{energy transferred at high temperature}}{\text{work done on heat pump}} \tag{33.11}$$

$$= \frac{|Q_h|}{W} \tag{33.12}$$

COP is similar to efficiency. Think of our air conditioner. Q_h is typically higher than W. That is, we usually transfer more energy by heat to the outside world that we provide in work. The outside part of the airconditioner does indeed get hotter than the outside air temperature. That is why energy will transfer by heat. Values of COP are generally greater than 1, though it is possible for them to be less than 1. Of course we would like the COP to be as high as possible

For refrigeration, the important thing is how much energy we transfer out of the cool area, Q_c, so for cooling the COP is

$$COP_c = \frac{|Q_c|}{W} \tag{33.13}$$

A good refrigerator should have a high COP, Typical values are 5 or 6.

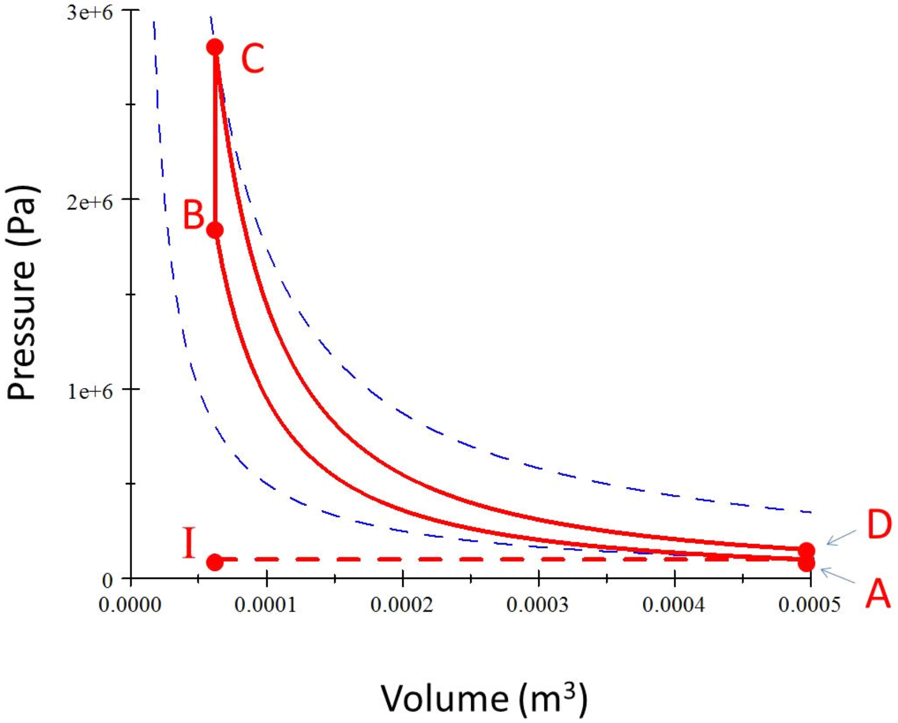
34 Ideal Gas Heat Engines

So far we have talked about idealized heat engines and refrigerators. We should try a somewhat realistic engine. Our goal in such a problem is to find for each process the values of ΔE_{int}, Q, W, and W_{eng}, Q_h, Q_c. We also want to find the ideal gas state variables P, $\not{V}$, n, and T at the transition points between processes.

Fundamental Concepts

- To describe a heat engine cycle you need to find P, $\not{V}$, n, T, ΔE_{int}, Q, W, for each part of the cycle and W_{eng}, Q_h, Q_c for the entire cycle.
- The Otto Cycle

An example: The Otto Cycle



The $P\cancel{V}$ diagram above shows the Otto cycle. This cycle is approximately what happens in a gasoline engine, like those in cars or, more like what we find in lawn mowers.

For our engine, suppose the compression ratio is

$$\frac{\cancel{V}_A}{\cancel{V}_B} = 8.00$$

Let's take

$$\begin{aligned}
\cancel{V}_A &= 500\,\mathrm{cm}^3 = 0.000\,5\,\mathrm{m}^3 \\
P_A &= 100\,\mathrm{kPa} \\
T_A &= 293\,\mathrm{K}
\end{aligned}$$

then we can find $\cancel{V}_B$

$$\cancel{V}_B = \frac{\cancel{V}_A}{8} = 62.5\,\mathrm{cm}^3 = 6.25 \times 10^{-5}\,\mathrm{m}^3$$

and we will need to measure the temperature at C

$$T_C = 1023\,\mathrm{K}$$

and the intake temperature

$$T_I = 273\,\mathrm{K}$$

(which probably tells us that this must be a road trip in Rexburg, not Phoenix). Further suppose our engine uses a diatomic ideal gas with

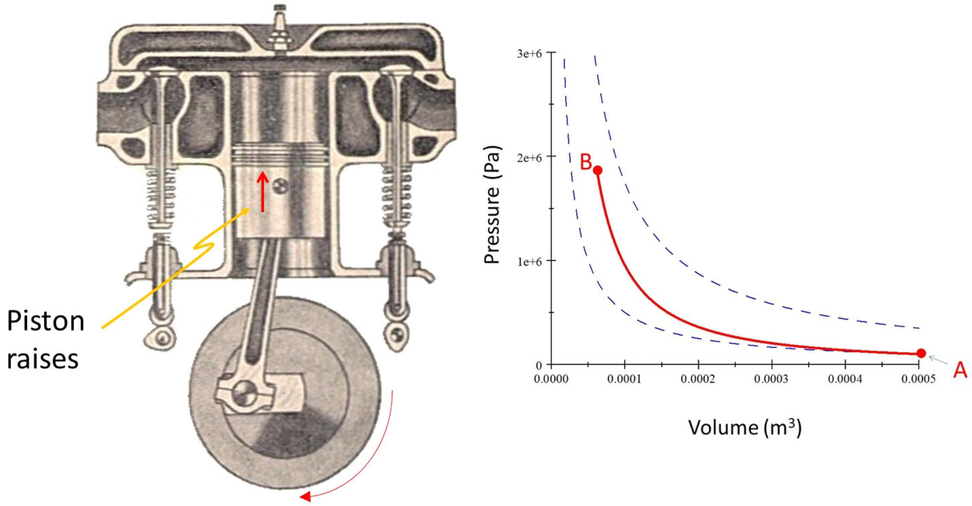
$$\begin{aligned} C_V &= \frac{5}{2}R \\ R &= 8.314\frac{\mathrm{J}}{\mathrm{mol\,K}} \end{aligned}$$

so

$$\gamma = 1.4$$

Let's follow the process through to see what the state variables are at each point on the PV diagram, and what Q_h and Q_c are for each process, and what the work, W, is and what the change in internal energy will be, ΔE_{int}. Finally, we will find the efficiency of the engine.

Process AB



Process $A \to B$ is an adiabatic compression. The piston moves upward, compressing the fuel-air mixture. This happens rapidly, so there is no time for energy to leave by heat, $Q = 0$. In this process $\Delta E_{int} = W$.

we can use

$$P_i \cancel{V}_i^{\gamma} = P_f \cancel{V}_f^{\gamma}$$

to find the final pressure for this process at B

$$P_B = P_A \frac{\mathcal{V}_A^\gamma}{\mathcal{V}_B^\gamma}$$

We can find the work from

$$\begin{aligned}
W_{AB} &= -\int_{\mathcal{V}_A}^{\mathcal{V}_B} P d\mathcal{V} \\
&= -\int_{\mathcal{V}_A}^{\mathcal{V}_B} P_A \frac{\mathcal{V}_A^\gamma}{\mathcal{V}^\gamma} d\mathcal{V} \\
&= -P_A \mathcal{V}_A^\gamma \int_{\mathcal{V}_A}^{\mathcal{V}_B} \frac{1}{\mathcal{V}^\gamma} d\mathcal{V} \\
&= -P_A^\gamma \mathcal{V}_A^\gamma \left(\frac{\mathcal{V}^{1-\gamma}}{1-\gamma} \right|_{\mathcal{V}_A}^{\mathcal{V}_B} \\
&= -\frac{P_A \mathcal{V}_A^\gamma}{1-\gamma} \left(\mathcal{V}_B^{1-\gamma} - \mathcal{V}_A^{1-\gamma} \right)
\end{aligned}$$

We can rearrange this into a more convenient formula

$$\begin{aligned}
W_{AB} &= \frac{P_A \mathcal{V}_A^\gamma}{\gamma - 1} \left(\frac{1}{\mathcal{V}_B^{\gamma-1}} - \frac{1}{\mathcal{V}_A^{\gamma-1}} \right) \\
&= \frac{1}{\gamma - 1} \left(\frac{P_A \mathcal{V}_A^\gamma}{\mathcal{V}_B^{\gamma-1}} - \frac{P_A \mathcal{V}_A^\gamma}{\mathcal{V}_A^{\gamma-1}} \right) \\
&= \frac{1}{\gamma - 1} \left(\frac{P_A \mathcal{V}_A^\gamma}{\mathcal{V}_B^\gamma \mathcal{V}_B^{-1}} - \frac{P_A \mathcal{V}_A^\gamma}{\mathcal{V}_A^{\gamma-1}} \right) \\
&= \frac{1}{\gamma - 1} \left(P_B \mathcal{V}_B - P_A \mathcal{V}_A \right)
\end{aligned}$$

which we can generalize into another basic equation for adiabatic processes.

$$W_{AB} = \left(\frac{1}{\gamma - 1} \right) \left(P_f \mathcal{V}_f - P_i \mathcal{V}_i \right)$$

then for process $A \rightarrow B$ we have

$$\begin{aligned}
W_{AB} &= \left(\frac{1}{\gamma-1}\right)\left(P_B \not{V}_B - P_A \not{V}_A\right) \\
&= \left(\frac{1}{\gamma-1}\right)\left(P_A \frac{\not{V}_A^{\gamma}}{\not{V}_B^{\gamma}} \not{V}_B - P_A \not{V}_A\right) \\
&= \left(\frac{1}{\gamma-1}\right)\left(P_A \frac{\not{V}_A^{\gamma}}{\left(\frac{\not{V}_A}{8}\right)^{\gamma}} \not{V}_B - P_A \not{V}_A\right) \\
&= \left(\frac{1}{\gamma-1}\right)\left(8^{\gamma} P_A \left(\frac{\not{V}_A}{8}\right) - P_A \not{V}_A\right) \\
&= \left(\frac{\not{V}_A}{\gamma-1}\right)\left(8^{\gamma-1} P_A - P_A\right) \\
&= \left(\frac{\not{V}_A P_A}{\gamma-1}\right)\left(8^{\gamma-1} - 1\right)
\end{aligned}$$

so the work is

$$\begin{aligned}
W_{AB} &= \left(\frac{\not{V}_A P_A}{\gamma-1}\right)\left(8^{\gamma-1} - 1\right) \\
&= \left(\frac{\left(0.000\,5\,\mathrm{m}^3\right)\left(100\,\mathrm{kPa}\right)}{1.4-1}\right)\left(8^{1.4-1} - 1\right) \\
&= 162.\,17\,\mathrm{J}
\end{aligned}$$

But this is the work done on the gas, the useful work is the negative of this

$$W_{eng_{AB}} = -162.\,17\,\mathrm{J}$$

and the change in internal energy is

$$\Delta E_{AB} = 162.\,17\,\mathrm{J}$$

and we can find

$$E_{int} = \frac{5}{2} nRT$$

but we need n. We can get it from what we know at A

$$\begin{aligned}
n &= \frac{P_A \not{V}_A}{RT_A} \\
&= \frac{\left(100\,\mathrm{kPa}\right)\left(0.000\,5\,\mathrm{m}^3\right)}{\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)\left(293\,\mathrm{K}\right)} \\
&= 2.\,052\,5 \times 10^{-2}\,\mathrm{mol}
\end{aligned}$$

then

$$\begin{aligned}
E_A &= \frac{5}{2} nRT_A \\
&= \frac{5}{2}\left(2.\,052\,5 \times 10^{-2}\,\mathrm{mol}\right)\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)\left(293\,\mathrm{K}\right) \\
&= 125.\,00\,\mathrm{J}
\end{aligned}$$

We used

$$P_B = P_A \frac{\mathcal{V}_A^\gamma}{\mathcal{V}_B^\gamma}$$

but we should find a numeric answer for P_B

$$\begin{aligned} P_B &= P_A \frac{\mathcal{V}_A^\gamma}{\left(\frac{\mathcal{V}_A}{8}\right)^\gamma} \\ &= P_A \frac{\mathcal{V}_A^\gamma}{\frac{\mathcal{V}_A^\gamma}{8^\gamma}} \\ &= 8^\gamma P_A \\ &= 8^{1.4}\,(100\,\mathrm{kPa}) \\ &= 1837.9\,\mathrm{kPa} \end{aligned}$$

and finally, from the ideal gas law

$$\begin{aligned} T_B &= \frac{P_B \mathcal{V}_B}{nR} \\ &= \frac{(1837.9\,\mathrm{kPa})\left(6.25 \times 10^{-5}\,\mathrm{m}^3\right)}{(2.0525 \times 10^{-2}\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)} \\ &= 673.15\,\mathrm{K} \end{aligned}$$

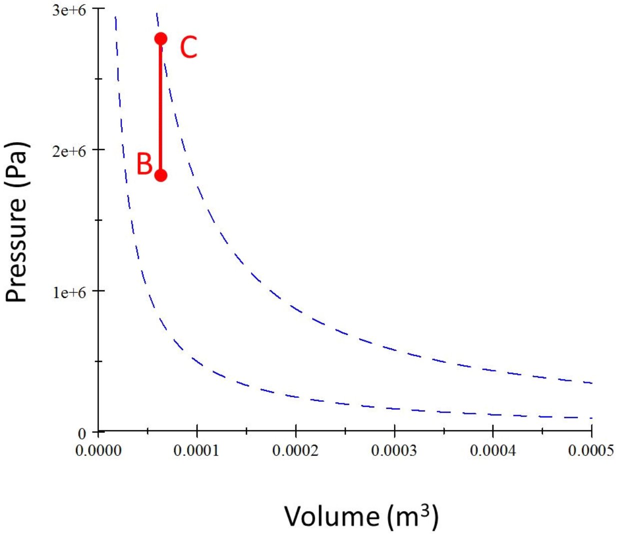
at last we can find E_B

$$\begin{aligned} E_B &= \frac{5}{2} n R T_B \\ &= \frac{5}{2}\left(2.0525 \times 10^{-2}\,\mathrm{mol}\right)\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(673.15\,\mathrm{K}) \\ &= 287.17\,\mathrm{J} \end{aligned}$$

what we have learned is summarized in this table

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{kPa})$	$\mathcal{V}\,\left(\mathrm{m}^3\right)$	$\mathbf{n}\,(\mathrm{mol})$	$\mathbf{E}_{int}\,(\mathrm{J})$	**–**
A	293	100	0.0005	2.0525×10^{-2}	125.00	**–**
B	673.15	1837.9	6.25×10^{-5}	2.0525×10^{-2}	287.17	**–**
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	$\mathbf{Q}_c$
AB	0	162.17 J	162.17 J	−162.17 J	**0**	**0**

That was just one process!



Let's take on the next process $B \rightarrow C$. This is where the spark plug ignites the air-fuel mixture created by the fuel injectors. A great deal of energy is released in a hurry. The temperature jumps up and the pressure does too, but the volume is the same because this process happens in a flash–literally!

So this must be an isochoric process. We expect $W_{BC} = 0$, so $\Delta E_{BC} = Q_{BC}$. The pressure increases, so we expect Q_{BC} to be positive. This must be part of Q_h! We know the temperature at C and we know the volume so we can find the final pressure

$$
\begin{aligned}
P_C &= \frac{nRT_C}{\not{V}_C} \\
&= \frac{\left(2.0525 \times 10^{-2}\right)\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(1023)}{6.25 \times 10^{-5}\,\mathrm{m}^3} \\
&= 2.7931 \times 10^{6}\,\mathrm{Pa} \\
&= 2793.1\,\mathrm{kPa}
\end{aligned}
$$

We can find the internal energy at C now

$$
\begin{aligned}
E_C &= \frac{5}{2}nRT_C \\
&= \frac{5}{2}\left(2.0525 \times 10^{-2}\,\mathrm{mol}\right)\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(1023\,\mathrm{K}) \\
&= 436.42\,\mathrm{J}
\end{aligned}
$$

We can use

$$\begin{aligned} Q &= nC_V\Delta T \\ &= n\frac{5}{2}R\left(T_C - T_B\right) \end{aligned}$$

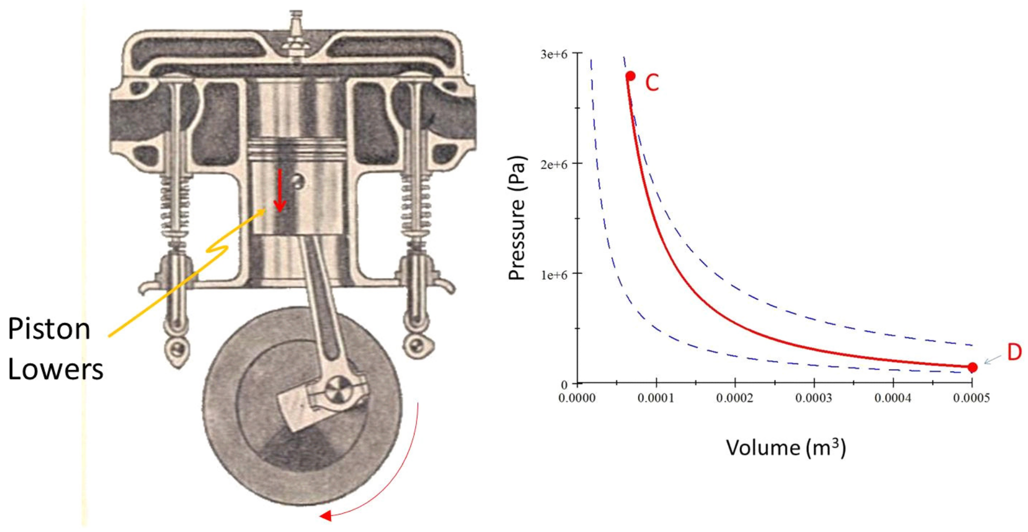
For the process BC to find the energy transfer by heat

$$\begin{aligned} Q_{BC} &= \left(2.052\,5\times10^{-2}\,\mathrm{mol}\right)\frac{5}{2}\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)\left(1023\,\mathrm{K} - 673.15\,\mathrm{K}\right) \\ &= 149.25\,\mathrm{J} \end{aligned}$$

We already mentioned that $Q_{BC} = \Delta E_{int_{BC}}$ and $W_{BC} = 0$, so we can complete our table of state variables and process variables for this process.

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{kPa})$	$\cancel{V}\,(\mathrm{m}^3)$	$\mathbf{n}\,(\mathrm{mol})$	$\mathbf{E}_{int}\,(\mathrm{J})$	–
B	673.15	1837.9	6.25×10^{-5}	$2.052\,5\times10^{-2}$	287.17	–
C	1023	2793.1	6.25×10^{-5}	$2.052\,5\times10^{-2}$	436.42	–
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	Q_c
BC	149.25	0	149.25	$\mathbf{0}$	149.25	0

Process CD (Power Stroke)



The next process is $C \rightarrow D$. This is another adiabatic process, this time an expansion. This is sometimes called the “power stroke” because the hot, high pressure gas pushes the piston down. This push is what makes the car go. We know that $Q_{CD} = 0$ so $\Delta E_{CD} = W_{CD}$

To find the ideal gas state variables at D let's start with

$$P_i \rlap{-}V_i^{\gamma} = P_f \rlap{-}V_f^{\gamma}$$

again, then

$$P_D = P_C \frac{\rlap{-}V_C^{\gamma}}{\rlap{-}V_D^{\gamma}}$$

but from our PV diagram we see $\rlap{-}V_C = \rlap{-}V_B$ and $\rlap{-}V_D = \rlap{-}V_A$. This makes sense because the cylinder and piston system maximum and minimum volumes don't change. So we can write

$$P_D = P_C \frac{\rlap{-}V_B^{\gamma}}{\rlap{-}V_A^{\gamma}} = 8^{-\gamma} P_C$$

and knowing this we can find the temperature at D using the ideal gas law

$$T_D = \frac{P_D \rlap{-}V_D}{nR}$$

let's numerically calculate P_D and T_D at this point.

$$\begin{aligned} P_D &= 8^{-1.4} (2793.1) \\ &= 151.97\,\mathrm{kPa} \end{aligned}$$

and

$$\begin{aligned} T_D &= \frac{(151.97\,\mathrm{kPa}) \left(0.0005\,\mathrm{m}^3\right)}{(2.0525 \times 10^{-2}\,\mathrm{mol}) \left(8.314 \frac{\mathrm{J}}{\mathrm{mol\,K}}\right)} \\ &= 445.28\,\mathrm{K} \end{aligned}$$

and while we are at it we can complete our state D by calculating E_D

$$\begin{aligned} E_D &= \frac{5}{2} nRT_D \\ &= \frac{5}{2} \left(2.0525 \times 10^{-2}\,\mathrm{mol}\right) \left(8.314 \frac{\mathrm{J}}{\mathrm{mol\,K}}\right) (445.28\,\mathrm{K}) \\ &= 189.96\,\mathrm{J} \end{aligned}$$

Now for the work done, we again use our new adiabatic equation

$$\begin{aligned} W_{CD} &= \left(\frac{1}{\gamma - 1}\right) (P_D \rlap{-}V_D - P_C \rlap{-}V_C) \\ &= \left(\frac{1}{\gamma - 1}\right) \left(8^{-\gamma} P_C \rlap{-}V_A - P_C \rlap{-}V_B\right) \\ &= \left(\frac{P_C}{\gamma - 1}\right) \left(8^{-\gamma} \rlap{-}V_A - \rlap{-}V_B\right) \end{aligned}$$

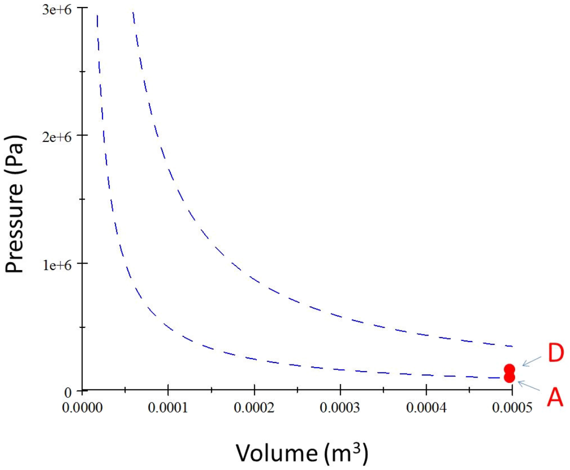
all of which we know, so

$$\begin{aligned} W_{CD} &= \left(\frac{2793.1\,\mathrm{kPa}}{1.4 - 1}\right) \left(8^{-1.4} \left(0.0005\,\mathrm{m}^3\right) - 6.25 \times 10^{-5}\,\mathrm{m}^3\right) \\ &= -246.46\,\mathrm{J} \end{aligned}$$

which completes our set for process CD

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{kPa})$	$\cancel{V}\,(\mathrm{m}^3)$	$\mathbf{n}\,(\mathrm{mol})$	$\mathbf{E}_{int}\,(\mathrm{J})$	–
C	1023	2793.1	6.25×10^{-5}	2.0525×10^{-2}	436.42	–
D	445.28	151.97	0.0005	2.0525×10^{-2}	$189.96\,\mathrm{J}$	–
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	$\mathbf{Q}_c$
CD	0	-246.46	-246.46	246.46	**0**	**0**

We are nearing the end of the cycle, In process $D\rightarrow A$ a valve opens letting the pressure drop quickly.



Since this happens quickly, the piston does not have time to change position, so the volume does not change. Process DA is (almost) isochoric. We already know P_A, $\cancel{V}_A$, n, and T_A. But we need to find the process work, heat and change in internal energy. Since the volume does not change, $W_{DA}=0$ and $\Delta E_{DA}=Q_{DA}$ which we can find using

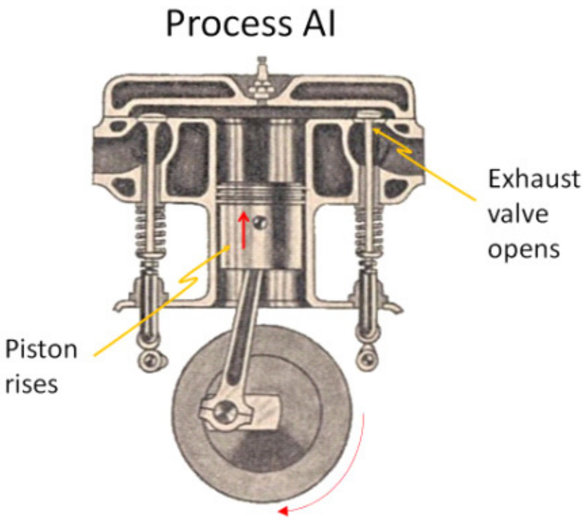
$$\begin{aligned} Q_{DA} &= nC_V\Delta T \\ &= n\frac{5}{2}R\left(T_A-T_D\right) \end{aligned}$$

which gives

$$\begin{aligned} Q_{DA} &= \left(2.0525\times10^{-2}\,\mathrm{mol}\right)\frac{5}{2}\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)\left(293\,\mathrm{K}-445.28\,\mathrm{K}\right) \\ &= -64.964\,\mathrm{J} \end{aligned}$$

which tells us that we have lost energy by heat. So Q_{DA} must contribute to Q_c.

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{kPa})$	$\cancel{V}\,(\mathrm{m}^3)$	$\mathbf{n}\,(\mathrm{mol})$	$\mathbf{E}_{int}\,(\mathrm{J})$	–
D	445.28	151.97	0.0005	2.0525×10^{-2}	$189.96\,\mathrm{J}$	–
A	293	100	0.0005	2.0525×10^{-2}	125.00	–
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	$\mathbf{Q}_c$
DA	-64.964	0	-64.964	0	0	64.964



But we are not done. The opening of the valve let the spent fuel leave as exhaust. But we can help remove the exhaust by pushing the piston up. This is done at constant pressure, because the valve is open to the exhaust system air pressure. We will lose gas by venting it. We can find the new number of moles using the ideal gas law

$$P_I \rlap{-}V_I = nRT$$

since $P_I = P_A$ and

$$\frac{\rlap{-}V_A}{\rlap{-}V_I} = 8.00$$

$\rlap{-}V_I = 8/\rlap{-}V_A$ so we can guess that we should have

$$\frac{P_A \rlap{-}V_A}{P_I \rlap{-}V_I} = \frac{n_A R T_A}{n_I R T_I}$$

$$8 = \frac{n_A T_A}{n_I T_I}$$

so

$$n_I = \frac{n_A T_A}{8 T_I}$$

or

$$\begin{aligned} n_I &= \frac{\left(2.052\,5 \times 10^{-2}\,\mathrm{mol}\right)(293\,\mathrm{K})}{8\,(273\,\mathrm{K})} \\ &= 2.753\,6 \times 10^{-3}\,\mathrm{mol} \end{aligned}$$

and then E_I will be

$$\begin{aligned} E_D &= \frac{5}{2} n_I R T_I \\ &= \frac{5}{2}\left(2.7536 \times 10^{-3}\,\mathrm{mol}\right)\left(8.314 \frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(273\,\mathrm{K}) \\ &= 15.625\,\mathrm{J} \end{aligned}$$

and the work done for an isobaric process is

$$W_{AI} = P_A\left(\not{V}_I - \not{V}_A\right)$$

but this assumed that n did not change. Really we are just moving the gas, so our work is

$$\begin{aligned} W &= F\Delta x \\ &\approx 0 \end{aligned}$$

because it takes very little force to push out the gas, so we can ignore W_{AI}.

Finding Q_{AI} we would like to use

$$Q = nC_P\Delta T$$

but we quickly run into trouble again because n changed! So what has gone wrong?

Well we are using convection to remove heat instead of conduction. We can get around this by finding ΔE_{int} directly.

$$\begin{aligned} \Delta E_{AI} &= E_I - E_A \\ &= 15.625\,\mathrm{J} - 125.00\,\mathrm{J} \\ &= -109.38\,\mathrm{J} \end{aligned}$$

From this and the first law

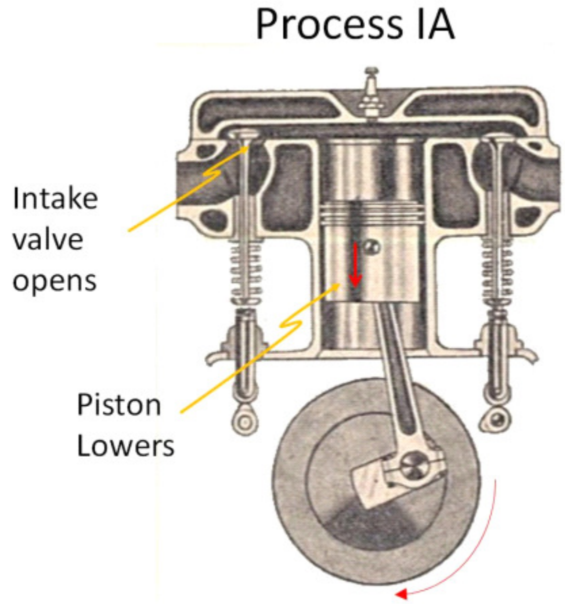
$$\Delta E_{int} = Q + W$$

we can find Q

$$\begin{aligned} Q &\approx \Delta E_{int} \\ &\approx -109.38\,\mathrm{J} \end{aligned}$$

Energy is leaving with the gas, so we seem to pick up a contribution to Q_c. But really this is not part of our thermodynamic process, because we mechanically removed the energy by removing the gas. Let's denote this by putting our energy in parenthesis to remind us that this is a more mechanical than thermal process.

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{kPa})$	$\not{V}\,(\mathrm{m}^3)$	$\mathbf{n}\,(\mathrm{mol})$	$\mathbf{E}_{int}\,(\mathrm{J})$	–
A	293	100	0.000 5	2.0525×10^{-2}	125.00	–
I	273	100	6.25×10^{-5}	2.7536×10^{-3}	15.625	–
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	$\mathbf{Q}_c$
AI	(-109.38)	~ 0	(-109.38)	~ 0	0	(109.38)

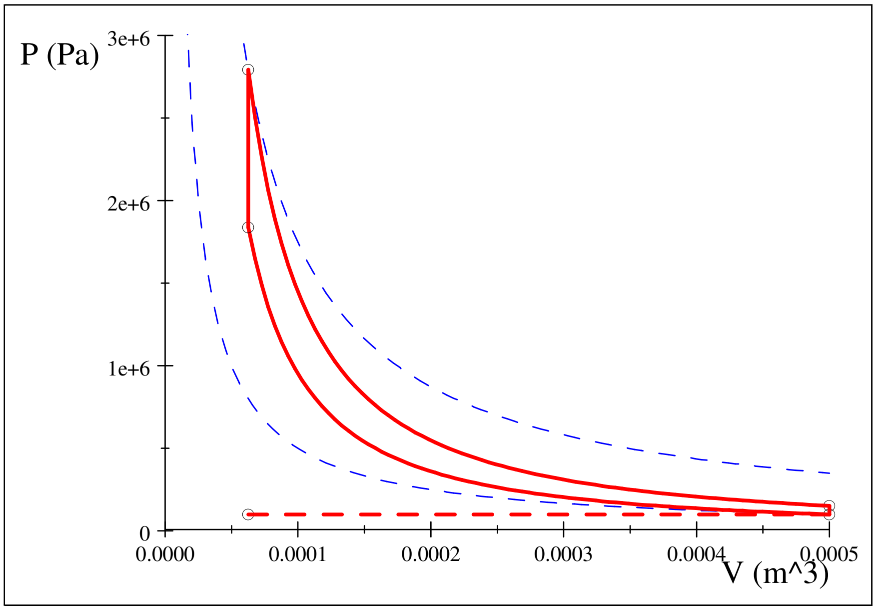


Now the exhaust valve closes, and the intake valve opens. The new fuel mixture is brought in. We essentially reverse process AI in process IA.

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{kPa})$	$\not{V}\,(\mathrm{m}^3)$	$\mathbf{n}\,(\mathrm{mol})$	$\mathbf{E}_{int}\,(\mathrm{J})$	–
I	293	100	6.25×10^{-5}	2.5657×10^{-3}	15.625	–
A	293	100	0.000 5	2.0525×10^{-2}	125.00	–
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	$\mathbf{Q}_c$
IA	(109.38)	~ 0	(109.38)	~ 0	(109.38)	0

But there is a difference. The new gas has useful fuel. so we are ready to complete the cycle again by compressing the gas, igniting it, and receiving the power stroke and exhaust. Note that Processes AI and IA have a net zero effect thermodynamically, but are very important to making your car or lawn mower go!

We can plot what we have found. Note that since n changed for the $AI - IA$ processes, it is not really possible to include it on the PV diagram. That is why it has a dashed line. We know the volume and pressure along the path, but the predicted temperature from the graph may not be right because n changed too.



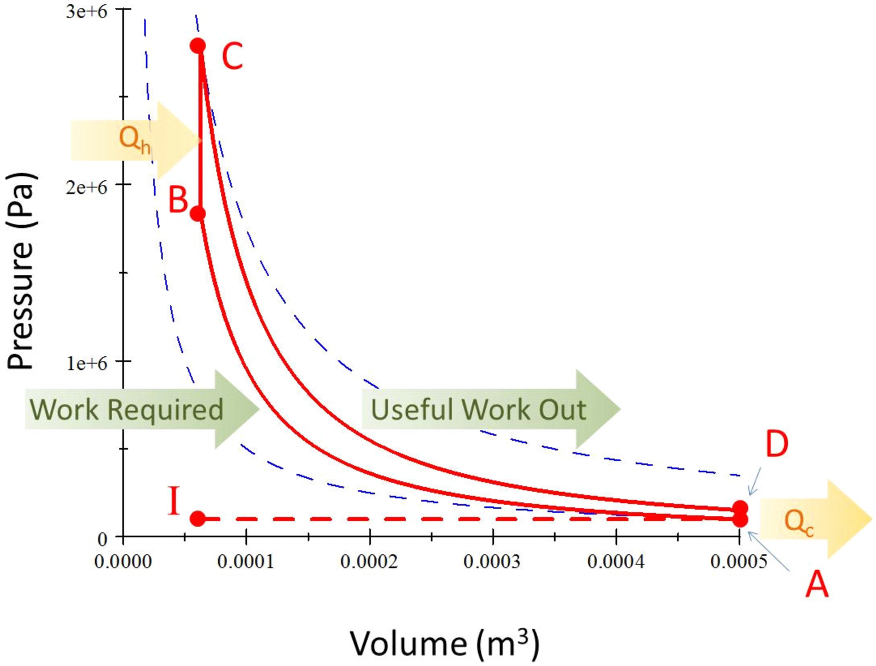
Let's summarize what we found in large tables. First the ideal gas state variables and E_{int}

State	**T** (K)	**P** (kPa)	Ꝟ (m^3)	**n** (mol)	$\mathbf{E}_{int}$ (J)
A	293	100	0.0005	2.0525×10^{-2}	125.00
B	673.15	1837.9	6.25×10^{-5}	2.0525×10^{-2}	287.17
C	1023	2793.1	6.25×10^{-5}	2.0525×10^{-2}	436.42
D	445.28	151.97	0.0005	2.0525×10^{-2}	189.96 J
I	293	100	6.25×10^{-5}	2.5657×10^{-3}	15.625

and now the values of Q, W, and E_{int} for the processes. We have included W_{eng}, Q_h and Q_c as well

Process	**Q** (J)	$\mathbf{W}_{int}$ (J)	$\mathbf{\Delta E}_{int}$ (J)	$\mathbf{W}_{eng}$ (J)	$\mathbf{Q}_h$	$\mathbf{Q}_c$
AB	0	162.17	162.17	−162.17	**0**	**0**
BC	149.25	0	149.25	**0**	149.25	0
CD	0	−246.46	−246.46	246.46	**0**	**0**
DA	−64.964	0	−64.964	0	0	64.964
AI	(−109.38)	(∼ 0)	(−109.38)	(∼ 0)	(0)	(109.38)
IA	(109.38)	(∼ 0)	(109.38)	(∼ 0)	(109.38)	(0)
Whole Cycle	84.286	−84.29	0	84.29	149.25	64.964

Note that the last row of the table is a total for the entire cycle. We see that we have useful work of 84.3 J and that $Q_h = 258.63$ J and $Q_c = 174.34$. We can identify Q_h and Q_c on our graph. The useful work is the difference between the work done in the two adiabatic processes.



We can see that it takes some work to make process AB happen. That is why we need a starter motor to get our engine started. But once the engine get's going the momentum from the piston and crank shaft keep the engine running. But still process AB requires energy input by work, so it is like a resistance for the cycle.

What is the efficiency of this cycle? We don't have a really good way with our simple view of thermodynamics to accurately account for the change in chemical potential energy involved with state I. The energy transfer in processes AI and IA are, in a sense, mechanical energy processes. So for this class it might be more fair to say that processes AI and IA have no net effect (except for refueling) and exclude their effect from Q_c and Q_h then ignoring the I processes

$$\begin{aligned}\eta &= \frac{W_{eng}}{Q_h}\\ &= \frac{84.29}{149.25}\\ &= 0.56476\end{aligned}$$

For a gasoline engine, it is often quoted that the efficiency is

$$\eta_{otto} = 1 - \frac{1}{\left(\frac{\cancel{V}_A}{\cancel{V}_B}\right)^{\gamma-1}}$$

Let's test this.

$$\begin{aligned}\eta_{otto} &= 1 - \frac{1}{\left(\frac{0.000\,5}{6.\,25\times10^{-5}}\right)^{1.4-1}} \\ &= 0.564\,72\end{aligned}$$

But you might guess that this is very optimistic. Efficiencies for actual engines come in at values around 20%.

35 The Limit of Efficiency for Heat Engines

We now know that we can design acutual engines that power cars and useful machines by combining several of our special thermodynamic procces into larger, more complex processes that form a cycle. You might guess that there are more than just isochoric, isobaric, isothermal, and adiabatic processes that exist. A good question is what processes should we combine to form our engine? Past researchers have asked this question over and over and many new engines have been developed. Of course we want our engines to be as efficient as possible, and the second law of thermodynamics says the engines can't be 100%y efficient. But is there a "most efficient" engine, one that no other engine can beat in efficiency? If there is, we could use this to judge new engine designs. Any engine with a better efficiency would be a mistake that won't work, and that approach this maximum efficiency would be a possible improvement in technology.

There is such a maximally efficient engine. We will study it in this lecture.

Fundamental Concepts

- The Carnot cycle represents the maximum theoretical limit of efficiency for a heat engine.
- The efficiency of the Carnot cycle depends only on the two extreme temperatures $\eta_C = 1 - \frac{T_c}{T_h}$.
- An engine design that is more efficient that a Carnot engine must have a fundamental flaw in it's calculations.

Question 123.19.1

Carnot was a French Engineer. He thought of an engine that would operate on an ideal reversible cycle. He hit upon the most efficient cycle possible. Sadly, it is not possible to create his engine in practice. Carnot's engine assumed no friction and ideal thermodynamic processes. So why is it useful to know about Carnot's engine design? The Carnot cycle provides us with a useful upper limit for all practical engines. To tell if a design is practical, we can compare it to the Carnot cycle. If our design looks like it will outperform the Carnot engine, we know we have done something wrong. Or conversely, if our system design requires an engine to be better than a Carnot engine, we need to go back to the drawing board.

Knowing just this much we can state Carnot's theorem.

No real heat engine operating between two energy reservoirs can be more efficient than a Carnot engine operating between the same two reservoirs

---

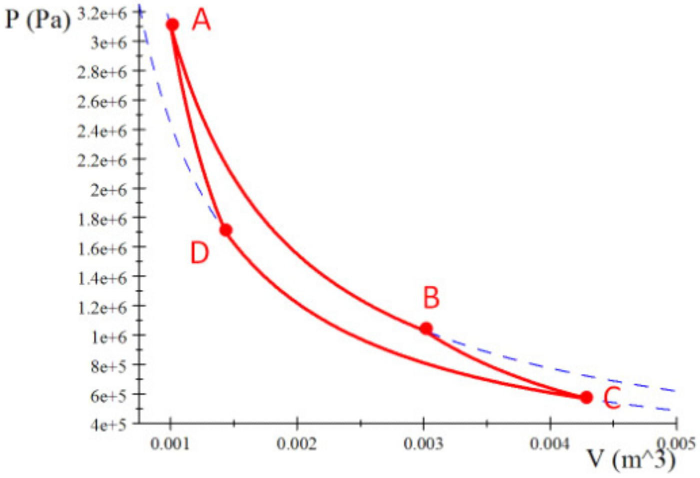
All real engines are less efficient than a Carnot engine because they do not operate through a reversible cycle. The efficiency of a real engine is reduced by friction, energy losses through conduction, etc.

The Carnot Cycle

Carnot found a set of special processes that when combined made a theoretical heat engine. And he proved mathmatically that this particular combination is theoretically the most efficient possible. Let's take a look at this engine design.

The Carnot cycle is shown in the figure. It consists of the following parts:

1. Process $A \rightarrow B$ is an isothermal expansion
2. Process $B \rightarrow C$ is an adiabatic expansion
3. Process $C \rightarrow D$ is an isothermal contraction
4. Process $D \rightarrow A$ is an adiabatic compression



Let's take an example. Let's choose a Carnot engine that operates between the same two extreme temperatures as our Otto cycle example.

$$\begin{aligned} T_c &= 273\,\mathrm{K} \\ T_h &= 1023\,\mathrm{K} \end{aligned}$$

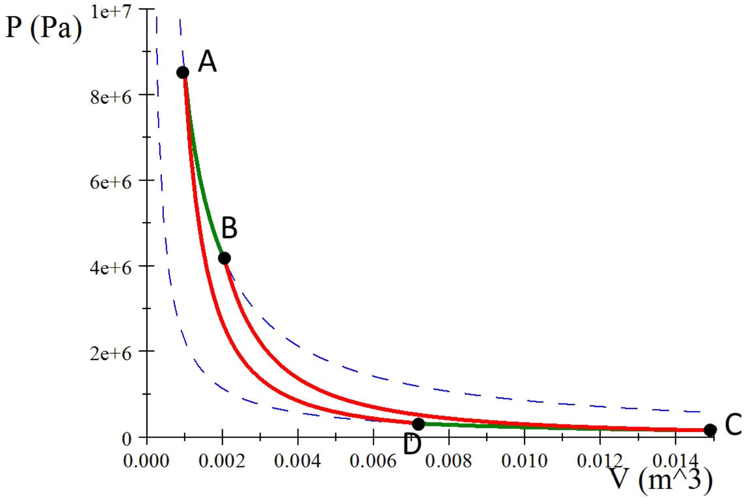
Let's suppose the carnot engine is made from a piston with a monotonic idea gas. Let's choose the minimum and maximum volumes to be

$$\begin{aligned} \not{V}_A &= 0.001\,\mathrm{m}^3 \\ \not{V}_C &= 0.015\,\mathrm{m}^3 \end{aligned}$$

and let's have

$$n = 1\,\mathrm{mol}$$

of gas. For these numbers, our graph looks like this:

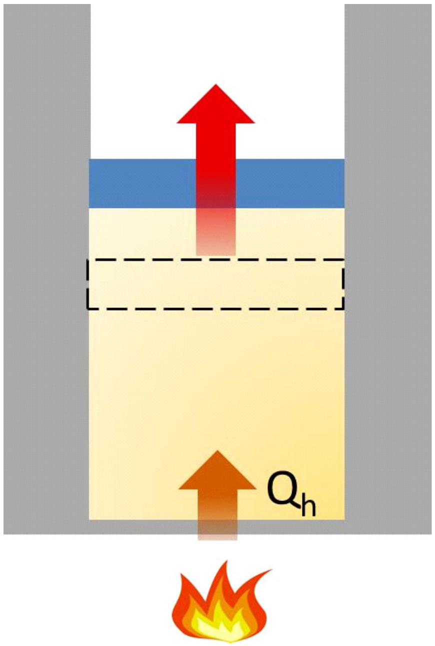


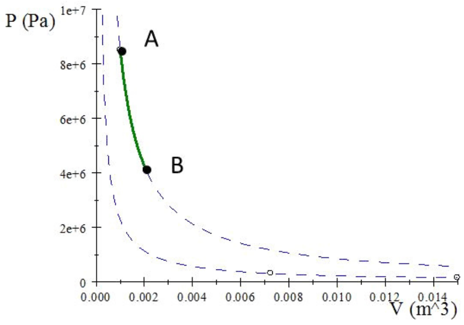
Our job is to find P, $\not{V}$, T, W, Q, and ΔE_{int} for each part of the cycle and Q_h, Q_c, and W_{eng} for the whole cycle.

Let's take each process separately like we did for the Otto cycle.

Process $A \rightarrow B$ isothermal expansion

The gas is placed in contact with the high temperature reservoir, T_h. The gas absorbs heat $|Q_h|$. The gas does work W_{AB} in raising the piston





Remember for isothermal processes we can write

$$\frac{P_i \rlap{-}V_i}{P_f \rlap{-}V_f} = \frac{nRT_i}{nRT_i}$$

because the temperature does not change, so

$$P_A \rlap{-}V_A = P_B \rlap{-}V_B$$

and we know for an isothermal process that

$$W_{AB} = nRT_A \ln\left(\frac{\rlap{-}V_A}{\rlap{-}V_B}\right)$$

and that

$$\Delta E_{int} = 0$$

so

$$|Q_{AB}| = |-W_{AB}|$$

and we can identify this as $|Q_h|$ and $T_A = T_h$

We can complete state A using the ideal gas law

$$\begin{aligned} P_A &= \frac{nRT_h}{\rlap{-}V_A} = \frac{(1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(1023\,\mathrm{K})}{0.001\,\mathrm{m}^3} \\ &= 8.505\,2 \times 10^6\,\mathrm{Pa} \end{aligned}$$

To complete state B we need the volume and the pressure. At point B both the adiabat and the isotherm have the same pressure and volume. So we know

$$P_B = \frac{nRT_h}{\cancel{V}_B}$$

and

$$P_B = \frac{K_{BC}}{\cancel{V}_B^{\gamma}}$$

where the constant $K_{BC} = P_B \cancel{V}_B^{\gamma}$ from our adiabatic equations. But we don't know K_{BC}, P_B, or $\cancel{V}_B$

We do know that at C

$$\begin{aligned} P_C &= \frac{nRT_c}{\cancel{V}_C} \\ P_C &= \frac{K_{BC}}{\cancel{V}_c^{\gamma}} \end{aligned}$$

and at point C and we do know $\cancel{V}_{C,}$ so we can find K_{BC}

$$\frac{nRT_c}{\cancel{V}_C} = \frac{K_{BC}}{\cancel{V}_C^{\gamma}}$$

$$\begin{aligned} K_{BC} &= \frac{nRT_c}{\cancel{V}_C} \cancel{V}_C^{\gamma} \\ &= nRT_c \cancel{V}_C^{\gamma-1} \end{aligned}$$

so

$$\begin{aligned} K_{BC} &= nRT_c \cancel{V}_C^{\gamma-1} = (1\,\mathrm{mol}) \left(8.314 \frac{\mathrm{J}}{\mathrm{mol\,K}}\right) (273\,\mathrm{K}) \left(0.015\,\mathrm{m}^3\right)^{\frac{2}{3}} \\ &= 138.\,05\,\mathrm{J\,m}^2 \end{aligned}$$

then moving back to point B we can set the two equations for pressure equal

$$\begin{aligned} P_B &= \frac{nRT_h}{\cancel{V}_B} \\ P_B &= \frac{K_{BC}}{\cancel{V}_B^{\gamma}} \end{aligned}$$

to get

$$\frac{nRT_h}{\cancel{V}_B} = \frac{K_{BC}}{\cancel{V}_B^{\gamma}}$$

then

$$\frac{\cancel{V}_B^{\gamma}}{\cancel{V}_B} = \frac{K_{BC}}{nRT_h}$$

$$\cancel{V}_B^{\gamma-1} = \frac{K_{BC}}{nRT_h}$$

which gives us an awkward

$$\cancel{V}_B = \sqrt[\gamma-1]{\frac{K_{BC}}{nRT_h}}$$

but it works.

$$\begin{aligned} \cancel{V}_B &= \sqrt[\frac{2}{3}]{\frac{138.\,05\,\mathrm{J\,m}^2}{(1\,\mathrm{mol}) \left(8.314 \frac{\mathrm{J}}{\mathrm{mol\,K}}\right) (1023\,\mathrm{K})}} \\ &= 2.\,067\,9 \times 10^{-3}\,\mathrm{m}^3 \end{aligned}$$

and we are back to the ideal gas law to find the pressure

$$\begin{aligned} P_B &= \frac{nRT_B}{\rlap{\,-}V_B} \\ &= \frac{(1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(1023\,\mathrm{K})}{2.0679\times 10^{-3}\,\mathrm{m}^3} \\ &= 4.1130\times 10^{6}\,\mathrm{Pa} \end{aligned}$$

Armed with all of this, we can compute the work and heat.

$$W_{AB} = nRT_h \ln\left(\frac{\rlap{\,-}V_A}{\rlap{\,-}V_B}\right)$$

$$\begin{aligned} W_{AB} &= (1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(1023\,\mathrm{K})\ln\left(\frac{0.001\,\mathrm{m}^3}{2.0679\times 10^{-3}\,\mathrm{m}^3}\right) \\ &= -6179.3\,\mathrm{J} \end{aligned}$$

We can find

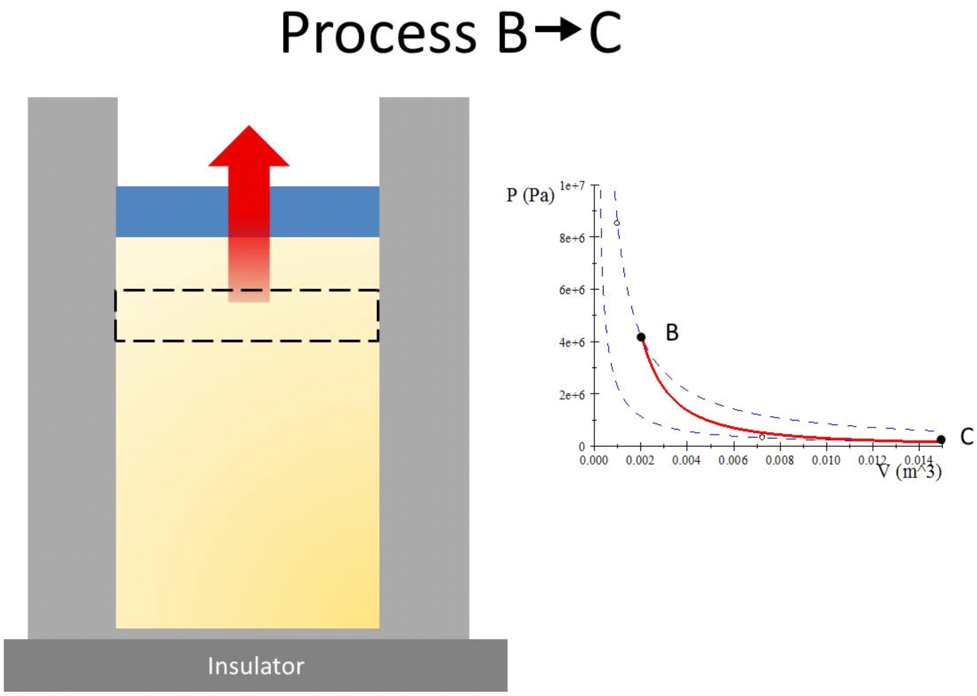
State	**T** (K)	**P** (Pa)	$\rlap{\,-}V$ (m^3)	**n** (mol)	**–**	**–**
A	1023	8.5052×10^{6}	0.001	1	**–**	**–**
B	1023	4.1130×10^{6}	2.0679×10^{-3}	1	**–**	**–**
Process	**Q** (J)	$\mathbf{W}_{int}$ (J)	$\mathbf{\Delta E}_{int}$ (J)	$\mathbf{W}_{eng}$ (J)	$\mathbf{Q}_h$	$\mathbf{Q}_c$
AB	6179.3	−6179.3	0	6179.3	6179.3	0

Process $B \rightarrow C$ adiabatic expansion

The base of the cylinder is replaced by a thermally nonconducting wall. No energy enters or leaves the system by heat.

$$Q = 0$$

The temperature falls from T_h to T_c. The gas does work W_{BC}.



Remember that for an adiabatic process

$$P_A V_A^{\gamma} = P_B V_B^{\gamma}$$

and we usually find a more convenient path for calculating ΔE_{int}.

$$\Delta E_{int} = nC_V \Delta T$$

We can find the pressure at C using the ideal gas law

$$\begin{aligned} P_C &= \frac{nRT_c}{V_C} = \frac{(1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(273\,\mathrm{K})}{0.015\,\mathrm{m}^3} \\ &= 1.5131 \times 10^5\ \mathrm{Pa} \end{aligned}$$

and

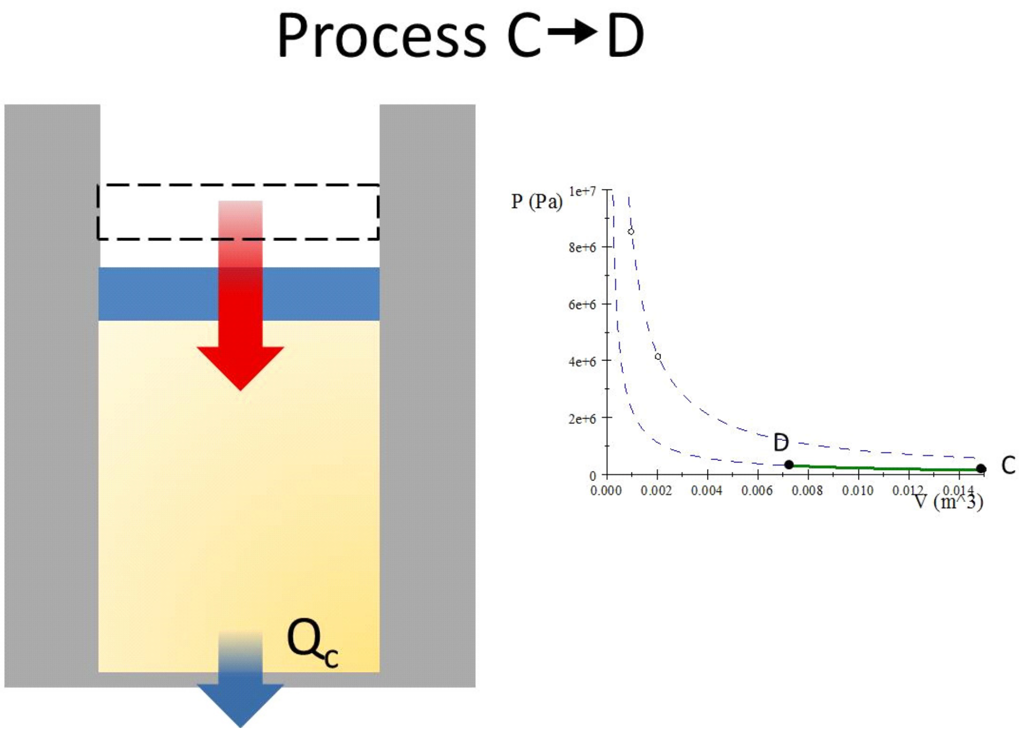
$$\begin{aligned} \Delta E_{int} &= (1\,\mathrm{mol})\,\frac{3}{2}\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(273\,\mathrm{K} - 1023\,\mathrm{K}) \\ &= -9353.3\,\mathrm{J} \end{aligned}$$

we can summarize all this.

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{Pa})$	$\rlap{-}V\,(\mathrm{m}^3)$	$\mathbf{n}\,(\mathrm{mol})$	–	–
B	1023	4.1130×10^{6}	2.0679×10^{-3}	1	–	–
C	273	1.5131×10^{5}	$0.015\,\mathrm{m}^3$	1	–	–
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	$\mathbf{Q}_c$
BC	0	-9353.3	-9353.3	9353.3	0	0

Process $C \to D$ isothermal compression

The gas is placed in contact with the cold temperature reservoir. The gas expels energy $|Q_c|$ and work W_{CD} is done on the gas.



The process is isothermal, so $T_C = T_D$ and $nRT_C = nRT_D$ so we have

$$P_C \rlap{-}V_C = P_D \rlap{-}V_D \tag{35.1}$$

and

$$W_{CD} = nRT_c \ln\left(\frac{\rlap{-}V_C}{\rlap{-}V_D}\right) \tag{35.2}$$

again

$$\Delta E_{int} = 0 \tag{35.3}$$

so

$$|Q_{CD}| = |-W_{CD}| \tag{35.4}$$

and we can identify this as $|Q_c|$

We again need to find Ꝟ_D and we can use the same method as before, finding Ꝟ_B. At position D

$$P_D = \frac{nRT_c}{\text{Ꝟ}_D}$$

and

$$P_D = \frac{K_{DA}}{\text{Ꝟ}_D^{\gamma}}$$

but we don't know K_{DA}, P_D, or Ꝟ_D

But we also know that at A

$$\begin{aligned} P_A &= \frac{nRT_h}{\text{Ꝟ}_A} \\ P_A &= \frac{K_{DA}}{\text{Ꝟ}_A^{\gamma}} \end{aligned}$$

and at point A and we do know $\text{Ꝟ}_{A,}$ so we can find K_{DA}

$$\frac{nRT_h}{\text{Ꝟ}_A} = \frac{K_{DA}}{\text{Ꝟ}_A^{\gamma}}$$

so

$$K_{DA} = nRT_h \text{Ꝟ}_A^{\gamma-1}$$

or

$$\begin{aligned} K_{DA} &= (1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(1023\,\mathrm{K})\left(0.001\,\mathrm{m}^3\right)^{\frac{2}{3}} \\ &= 85.052\,\mathrm{J\,m}^2 \end{aligned}$$

then moving back to point D we have

$$\frac{nRT_c}{\text{Ꝟ}_D} = \frac{K_{DA}}{\text{Ꝟ}_D^{\gamma}}$$

and

$$\text{Ꝟ}_D = \sqrt[\gamma-1]{\frac{K_{DA}}{nRT_c}}$$

numerically we get

$$\begin{aligned} \text{Ꝟ}_D &= \sqrt[\frac{2}{3}]{\frac{85.052\,\mathrm{J\,m}^2}{(1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(273\,\mathrm{K})}} \\ &= 7.2537\times10^{-3}\,\mathrm{m}^3 \end{aligned}$$

And we are back to the ideal gas law to find the pressure

$$\begin{aligned} P_D &= \frac{(1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(273\,\mathrm{K})}{7.2537\times10^{-3}\,\mathrm{m}^3} \\ &= 3.1291\times10^5\,\mathrm{Pa} \end{aligned}$$

And so we can find the work as

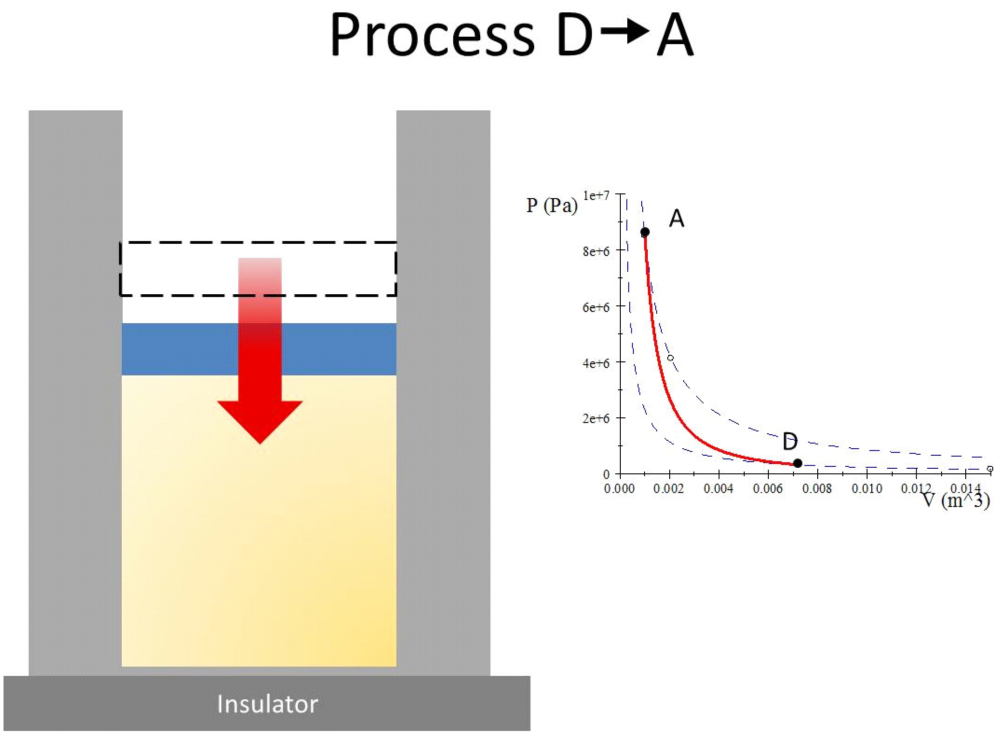
$$\begin{aligned} W_{CD} &= (1\,\mathrm{mol})\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(273\,\mathrm{K})\ln\left(\frac{0.015\,\mathrm{m}^3}{7.2537\times10^{-3}\,\mathrm{m}^3}\right) \\ &= 1649.0\,\mathrm{J} \end{aligned}$$

and our summary is as follows

State	$\mathbf{T}\,(\mathrm{K})$	$\mathbf{P}\,(\mathrm{Pa})$	$\not{V}\,(\mathrm{m}^3)$	$\mathbf{n}\,(\mathrm{mol})$	**–**	**–**
C	273	1.5131×10^5	$0.015\,\mathrm{m}^3$	1	**–**	**–**
D	273	3.1291×10^5	7.2537×10^{-3}	1	**–**	**–**
Process	$\mathbf{Q}\,(\mathrm{J})$	$\mathbf{W}_{int}\,(\mathrm{J})$	$\mathbf{\Delta E}_{int}\,(\mathrm{J})$	$\mathbf{W}_{eng}\,(\mathrm{J})$	$\mathbf{Q}_h$	$\mathbf{Q}_c$
CD	-1649.0	1649.0	0	0	0	1649.0

Process $D \to A$ adiabatic compression

The gas is again placed against a thermally nonconducting wall so no heat is exchanged with the surroundings. The temperature of the gas increases from T_c to T_h.The work done on the gas is W_{DA}



We know $Q = 0$ and that $\Delta E_{int} = W$. Again we can use

$$\Delta E_{int} = nC_V \Delta T$$

$$
\begin{aligned}
\Delta E_{int} &= (1\,\mathrm{mol})\,\frac{3}{2}\left(8.314\frac{\mathrm{J}}{\mathrm{mol\,K}}\right)(1023\,\mathrm{K}-273\,\mathrm{K}) \\
&= 9353.3\,\mathrm{J}
\end{aligned}
$$

so then

State	**T** (K)	**P** (Pa)	$\not{V}$ (m^3)	**n** (mol)	**–**	**–**
D	273	3.1291×10^{5}	7.2537×10^{-3}	1	**–**	**–**
A	1023	3.1×10^{6}	0.001	1	**–**	**–**
Process	**Q** (J)	$\mathbf{W}_{int}$ (J)	$\mathbf{\Delta E}_{int}$ (J)	$\mathbf{W}_{eng}$ (J)	$\mathbf{Q}_h$	$\mathbf{Q}_c$
DA	0	9353.3	9353.3	−9353.3	0	0

All together we have

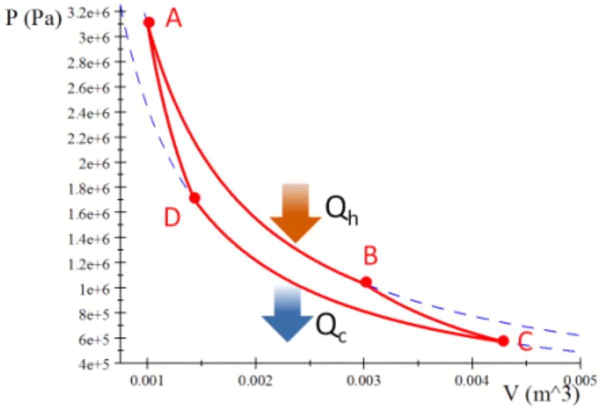
State	**T** (K)	**P** (Pa)	$\not{V}$ (m^3)	**n** (mol)
A	1023	8.5052×10^{6}	0.001	1
B	1023	4.1130×10^{6}	2.0679×10^{-3}	1
C	273	1.5131×10^{5}	0.015	1
D	273	3.1291×10^{5}	7.2537×10^{-3}	1

and our internal energy, heat, and work are

Process	**Q** (J)	$\mathbf{W}_{int}$ (J)	$\mathbf{\Delta E}_{int}$ (J)	$\mathbf{W}_{eng}$ (J)	$\mathbf{Q}_h$	$\mathbf{Q}_c$
AB	6179.3	−6179.3	0	6179.3	6179.3	0
BC	0	−9353.3	−9353.3	9353.3	0	0
CD	−1649.0	1649.0	0	−1649.0	0	1649.0
DA	0	9353.3	9353.3	−9353.3	0	0
Total	4530.3	−4530.3	0	4530.3	6179.3	1649.0

Question 123.19.2

We can see that process $A \rightarrow B$ provides Q_h and process $C \rightarrow D$ provides Q_C.Let's label this in our PV diagram.



We can also find the efficiency of the Carnot cycle. The net work is W_{ABCDA} and the efficiency is

$$\eta = \frac{W_{eng}}{|Q_h|} = 1 - \frac{|Q_c|}{|Q_h|}$$

No heat is transferred during the adiabatic processes, so the only heat transfer is only during the isothermal processes, we have

$$\begin{aligned} \eta &= 1 - \frac{\left|nRT_c \ln\left(\frac{\not{V}_C}{\not{V}_D}\right)\right|}{\left|nRT_h \ln\left(\frac{\not{V}_A}{\not{V}_B}\right)\right|} \\ &= 1 - \frac{\left|T_c \ln\left(\frac{\not{V}_C}{\not{V}_D}\right)\right|}{\left|T_h \ln\left(\frac{\not{V}_A}{\not{V}_B}\right)\right|} \end{aligned}$$

and using the adiabatic temperature-volume relationships

$$T_h \not{V}_A^{\gamma-1} = T_c \not{V}_D^{\gamma-1}$$

and

$$T_h \not{V}_B^{\gamma-1} = T_c \not{V}_C^{\gamma-1}$$

$$\frac{T_h \not{V}_A^{\gamma-1}}{T_h \not{V}_B^{\gamma-1}} = \frac{T_c \not{V}_D^{\gamma-1}}{T_c \not{V}_C^{\gamma-1}}$$

or

$$\frac{\not{V}_A^{\gamma-1}}{\not{V}_B^{\gamma-1}} = \frac{\not{V}_C^{\gamma-1}}{\not{V}_D^{\gamma-1}}$$

which gives us

$$\sqrt[\gamma-1]{\frac{\not{V}_A^{\gamma-1}}{\not{V}_B^{\gamma-1}}} = \sqrt[\gamma-1]{\frac{\not{V}_C^{\gamma-1}}{\not{V}_D^{\gamma-1}}}$$

then

$$\frac{\not{V}_A}{\not{V}_B} = \frac{\not{V}_D}{\not{V}_C}$$

using this we find

$$\begin{aligned} \eta &= 1 - \frac{\left|T_c \ln\left(\frac{\not{V}_C}{\not{V}_D}\right)\right|}{\left|T_h \ln\left(\frac{\not{V}_D}{\not{V}_C}\right)\right|} \\ &= 1 - \frac{\left|T_c \ln\left(\frac{\not{V}_C}{\not{V}_D}\right)\right|}{\left|-T_h \ln\left(\frac{\not{V}_C}{\not{V}_D}\right)\right|} \end{aligned}$$

or, removing the minus sign because of the absolute values

$$\eta = 1 - \frac{T_c}{T_h}$$

where the temperatures, of course, is in Kelvin. This means that for this process

$$\frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h}$$

and

$$\eta_C = 1 - \frac{T_c}{T_h} \qquad (35.5)$$

This is the general form for a Carnot cycle. Then for our example

$$\begin{aligned}\eta_C &= 1-\frac{273\,\mathrm{K}}{1023\,\mathrm{K}}\\ &= 0.733\,14\end{aligned}$$

This is a great review of how to use our thermodynamic equations for special process, even though we can't build a Carnot engine. But more importantly, we can judge the efficacy of engine designs. For example, let's take our Otto cycle from last time. Our design gave

$$\begin{aligned}\eta_{otto} &= 1-\frac{1}{\left(\frac{0.000\,5}{6.\,25\times 10^{-5}}\right)^{1.4-1}}\\ &= 0.564\,72\end{aligned}$$

So the Otto cycle example might be achievable. Of course, we can tell that our Otto cycle is still an idealization. We recognize that we neglected all friction, and assumed perfect adiabatic processes. But we can see that there is some hope of making our Otto cycle-based engines more efficient. But no Otto cycle that operates between the same $T_c = 273\,\mathrm{K}$ and $T_h = 1023\,\mathrm{K}$ can never be better than 73% effective because that is the maximum efficiency for a heat engine opperating between these two temperatures.

We are done with our introductory study of thermodynamics. Many of you will have junior level thermodynamics classes, so you are likely to see more of this. But we can stop for now. For physics majors the next physics class is likely to be PH220, electricity and magnetism. I would prefer to call this class "introduction to field theory" because we really study electric and magnetic fields for the first time. We will take up the topic of wave motion at the end of PH220 to mathematically describe light as electromagnetic waves. And then we will describe everything using waves in PH279, "modern physics," as our first introduction to quantum mechanics.

I hope you have enjoyed learning about waves, optics, and thermodynamics. I hope you will see the world a little differently (and think about the processing of seeing the world!). For physics majors, there is still so many cool things to learn. If you are leaving physics at this point, you can continue to learn about the world and the universe by reading and watching physics related material. If you are innovating new techniques or products, you might even want to hire the services of a physicist to help with the underlying principles for your new venture!

Experiment 1

Introduction

We have learned that

$$\omega^2 = \frac{k}{m}$$

and we know that

$$\omega = 2\pi f$$

and

$$T = \frac{1}{f}$$

so we would predict that

$$\begin{aligned} T &= \frac{1}{\frac{\omega}{2\pi}} = \frac{2\pi}{\omega} \\ &= 2\pi\sqrt{\frac{m}{k}} \end{aligned} \tag{A.1}$$

We would like to verify that this is true experimentally.

Unfortunately, springs don't come marked with their spring constants. You will need to measure your spring constant. A way to do this is to hang your spring motionless. When the mass is not moving Newton's second law tells us

$$\sum_y F = ma_y = F_S - F_g$$

With no change in velocity, the acceleration must be zero

$$0 = F_S - F_g$$

so

$$\begin{aligned} F_S &= F_g \\ k\Delta y &= mg \end{aligned}$$

or

$$k = \frac{mg}{\Delta y} \tag{A.2}$$

We believe that the equation of motion for our mass-spring system would be

$$\begin{aligned} y(t) &= y_{\max}\cos(\omega t + \phi_o) \\ &= y_{\max}\cos\left(\frac{2\pi}{T}t + \phi_o\right) \end{aligned} \tag{A.3}$$

Equipment

We have springs, masses, and motion detectors. Your motion detector should be placed below the mass as was demonstrated in class. Be careful to not drop the mass on the motion detector.

The motion detectors connect to our laptop computers. Use the LoggerPro program to collect data. Your lab instructor will help you set up the program and show you how to collect data.

Assignment

1. Find your spring constant
2. Using equation () predict the period of oscillation of your mass and spring.
3. Suspend your mass from the spring and capture several cycles of the motion using the motion detector.
4. Verify that equation () is a good model for our physical situation by fitting a sinusoidal curve to the data (ask instructor for help).
5. Using your curve fit equation, compare your predicted period to your measured period. Do this by matching the fit equation to our theoretical equation of motion

   $$\begin{aligned} Y &= A * \cos(Bx + C) + D \\ y(t) &= y_{\max}\cos\left(\frac{2\pi}{T}t + \phi_o\right) \end{aligned}$$

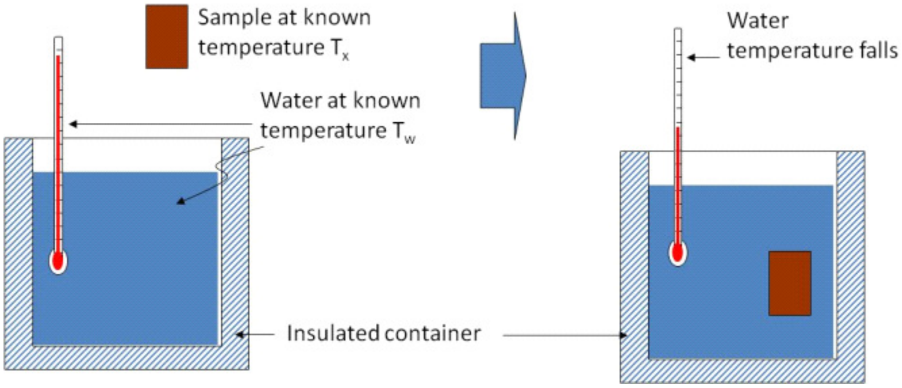
   we can see that

   $$B = \frac{2\pi}{T}$$

   and solve for T.
6. Do your predicted and measured periods match?
7. If they don't match, consider your inputs in equation () and try again.

Experiment 2

Introduction



We studied calorimetry earlier in the semester. The idea behind calorimetry is just conservation of energy. A calorimeter is supposed to be an insulated system, so energy can't leave by heat.

Your job today is to measure the latent heat of fusion of ice/water.

Some helpful reminders are the specific heat equation

$$Q = m_i c_i \Delta T$$

and the latent heat equation.

$$\begin{aligned} Q &= \pm L_f M && \text{melt/freeze} \\ Q &= \pm L_v M && \text{boil/condense} \end{aligned}$$

From our table we used in class, we know what the value should be.

Substance	$T_m\ (^\circ\mathrm{C})$	$L_f \left(\frac{\mathrm{J}}{\mathrm{kg}}\right)$	$T_m\ (^\circ\mathrm{C})$	$L_v \left(\frac{\mathrm{J}}{\mathrm{kg}}\right)$
Nitrogen (N_2)	-210	0.26×10^5	-196	1.99×10^5
Ethyl alcohol	-114	1.09×10^5	78	8.79×10^5
Mercury	-39	0.11×10^5	357	2.96×10^5
Water	0	3.33×10^5	100	22.6×10^5
Lead	328	0.25×10^5	1750	8.58×10^5

Find the latent heat of fusion for water/ice and calculate a percent error to see how well you did

$$\%Error = \frac{measured - acutal}{actual}$$

Equipment

I will have ice water available in a tub. I will also have calorimeter cups, thermometers, and triple beam balances for finding mass. Water will be available through our store room. Let me know if you need something else.

Assignment

Measure the latent heat of fusion of water.

1. Place some ice in water to form a slush (this will give you ice at $0\,^{\circ}\mathrm{C}$).
2. Place some water into the calorimeter cup, and measure its temperature and mass.
3. Place a piece of ice into the water in the calorimeter. You will need to find a way to measure its mass.
4. Measure the temperature change of the water after the ice has melted.
5. Calculate the latent heat, and compare to the accepted value.
6. Turn in your answer, $\%Error$, and all your group names on a single sheet of paper (the lab instructions sheet is fine)

Experiment 3

Introduction

A converging lens is one which is thicker at the center than at the edge, and which converges incident parallel rays to a real focus on the opposite side of the lens from the object. A diverging lens is thinner at the center than at the edge and diverges the light from a virtual focus on the same side of the lens as the object. The optic axis of a lens is a line drawn through the center of the lens perpendicular to the face of the lens. The principal focal point is a point on the principal axis through which incident rays parallel to the principal axis pass, or appear to pass, after refraction by the lens. The focal length, f, of the lens is the distance from the optical center of the lens to the principal focal point. The reciprocal of the focal length in meters is called the power of the lens and is expressed in diopters (d).

$$P = \frac{1}{f\,(\mathrm{m})}$$

If s is the distance of the object from the optical center of the lens, s' the distance of the image from the lens, and f the focal length, the thin lens equation is

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \tag{B.1}$$

The magnification is given by

$$m = -\frac{s'}{s} \tag{B.2}$$

For the case where more than one lens is present, Equation () is applied successively to each lens, treating the image formed by one lens as the object for the next lens, keeping in mind that the distances in Equation () are measured with respect to each individual lens, and not a single fixed point.

Equipment

We have $2\,\mathrm{m}$ sticks, lens and holders, paper screens, and light sources.

Assignment

Verify the lens equation (), and the image magnification equation (), for the lens with the longer focal length. I suggest you use at least three object distances greater than two focal lengths. I suggest you measure the image distance and the image height. I suggest you also measure at least three object distances between one and two focal lengths. Measure the image distance and the image height.

Plot your magnification results. Interpret the results, does the lens equation work?

Compute the magnification in each case and verify that equation () holds.

The lens equation is supposed to be good for virtual images as well as real images. We need to verify the equation for this case as well. I suggest you do this by measuring at least one object distance closer to the lens than one focal length, locate and measure the image distance. Measure the image height. Verify that equations () and () are valid for this case. Include this case on your graph.

Problem Types

Simple Harmonic Motion (SHM)

$$\begin{aligned} x(t) &= x_{\max}\cos(\omega t+\phi_o) \\ v(t) &= -\omega x_{\max}\sin(\omega t+\phi_o) \\ a(t) &= -\omega^2 x_{\max}\cos(\omega t+\phi_o) \end{aligned}$$

$$\omega = 2\pi f$$

$$f = \frac{1}{T}$$

$$v_{\max} = \omega x_{\max}$$

$$a_{\max} = \omega^2 x_{\max}$$

Energy in SHM

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}kx_{\max}^2\sin^2(\omega t+\phi) \end{aligned}$$

$$\begin{aligned} U_s &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}kx_{\max}^2\cos^2(\omega t+\phi) \end{aligned}$$

Angular Frequency and oscillation

For a mass sprig system

$$\omega = \sqrt{\frac{k}{m}}$$

for a pendulum

$$\omega = \sqrt{\frac{g}{L}}$$

for a physical pendulum

$$\omega = \sqrt{\frac{mgd}{\mathbb{I}}}$$

Damped oscillation

$$x\left(t\right)=x_{\max}e^{-\frac{b}{2m}t}\cos\left(\omega t+\phi_o\right)$$
$$\omega=\sqrt{\frac{k}{m}-\left(\frac{b}{2m}\right)^2}$$
$$x\left(t\right)=x_{\max}e^{-\frac{b}{2m}t}\cos\left(\left(\sqrt{\frac{k}{m}-\left(\frac{b}{2m}\right)^2}\right)t+\phi_o\right)$$

Forced oscillation

$$\mathbf{F}_d=-b\mathbf{v}$$
$$F\left(t\right)=F_o\sin\left(\omega_f t\right)$$
$$\Sigma F=F_o\sin\left(\omega_f t\right)-kx-bv_x=ma$$

$$x\left(t\right)=A\cos\left(\omega_f t+\phi\right)$$
$$A=\frac{\frac{F_o}{m}}{\sqrt{\left(\omega_f^2-\omega_o^2\right)^2+\left(\frac{b\omega_f}{m}\right)^2}}$$
$$\omega_o=\sqrt{\frac{k}{m}}$$

Waves

$$y\left(t\right)=A\cos\left(ax-\omega t+\phi\right)$$
$$k=\frac{2\pi}{\lambda}$$
$$v=\frac{\lambda}{T}$$
$$v=\lambda f$$

Waves on strings

$$v=\sqrt{\frac{T_s}{\mu}}$$
$$\mu=\frac{m}{L}$$

Sound waves

$$\begin{aligned} v_{sound} &= v_o\sqrt{1+\frac{T_c}{T_o}} \\ v_o &= 331\frac{\mathrm{m}}{\mathrm{s}} \\ T_o &= 273 \end{aligned}$$

$$P \equiv \frac{F}{A}$$

$$s\left(x,t\right) = s_{\max}\cos\left(kx-\omega t+\phi_o\right)$$

Spherical waves

$$y = A\left(r\right)\sin\left(kr-\omega t+\phi_o\right)$$

Light waves

$$n = \frac{c}{v}$$

$$c = 299792458\frac{\mathrm{m}}{\mathrm{s}}$$

Power and Intensity

$$\mathcal{P} = \frac{\Delta E}{\Delta t}$$

$$\mathcal{I} \equiv \frac{\mathcal{P}}{A}$$

$$A = 4\pi r^2$$

Sound Intensity Level

$$\beta = 10\log_{10}\left(\frac{I}{I_o}\right)$$

$$I_o = 1\times10^{-12}\frac{\mathrm{W}}{\mathrm{m}^2}$$

Doppler Effect

$$f' = \frac{v+v_d}{v}f \qquad \text{observer moving toward the source}$$

$$f' = \frac{v-v_d}{v}f \qquad \text{observer moving away from the source}$$

$$f' = \frac{v}{v-v_s}f \qquad \text{Source moving toward observer}$$

$$f' = \frac{v}{v + v_s} f$$ Source moving away from observer

$$f' = \frac{v \pm v_d}{v \mp v_s} f$$

Superposition

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

$$\begin{aligned} y_r &= y_{\max}\sin(kx - \omega t) + A\sin(kx - \omega t + \phi_o) \\ &= 2y_{\max}\cos\left(\frac{\phi_o}{2}\right)\sin\left(kx - \omega t + \frac{\phi_o}{2}\right) \end{aligned}$$

Standing waves

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\begin{aligned} y_r &= y_{\max}\sin(kx - \omega t) + y_{\max}\sin(kx + \omega t) \\ &= (2y_{\max}\sin(kx))\cos(\omega t) \end{aligned}$$

$$x = n\frac{\lambda}{4}$$

$$\begin{aligned} f_n &= \frac{v}{\lambda_n} = \\ &= \frac{n}{2L}v \\ &= \frac{n}{2L}\sqrt{\frac{T}{\mu}} \end{aligned}$$

Standing waves in pipes

Open on both ends

$$\begin{aligned} \lambda_n &= \frac{2}{n}L \\ f_n &= n\frac{v}{2L} \qquad n = 1, 2, 3, 4\ldots \end{aligned}$$

Closed on one end

$$\begin{aligned} \lambda_n &= \frac{4}{n}L \\ f_n &= n\frac{v}{4L} \qquad n = 1, 3, 5\ldots \end{aligned}$$

References

This appendix contains the references.

Index

This appendix contains the index.